Liquid Time-Constant Networks

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Formulation

- Let the hidden state flow of a network be declared by a system of linear ODEs of the form:

$$d\mathbf{x}(t)/dt = -\mathbf{x}(t)/\tau + \mathbf{S}(t),$$

and let $\mathbf{S}(t) \in \mathbb{R}^M$ represent the following nonlinearity: $\mathbf{S}(t) = f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)(A - \mathbf{x}(t))$, with parameters θ and A.

- Then, the Liquid-Time Constant (LTC) Network models the following continous-time dynamical system:

$$\boxed{\frac{d\mathbf{x}(t)}{dt} = -\left[\frac{1}{\tau} + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)\right]\mathbf{x}(t) + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)A}$$

- Here, τ defines the system's **time-constant**. LTCs represent ODEs that **vary their time-constants in an input-dependent manner** → "**liquid**".

Motivation

- Instead of modeling implicit nonlinearities, LTCs model linear first-order dynamical systems modulated via nonlinear interlinked gates.
- Inspired by the computational models of neural dynamics in small species.
- The LTC update is also similar to that of bilinear-approximated Dynamic Causal Models (DCMs), that are useful in learning on complex fMRI time-series signals.
- The expressivity of the LTC formulation can be studied via trajectory length analysis.
- **The goal is to capture complex non-linear interactions in potentially irregular time-series data.**

New Semi-implicit Fused ODE Solver

Algorithm LTC update by fused ODE Solver

Parameters: $\theta = \{\tau^{(N\times 1)}$ = time-constant, $\gamma^{(M\times N)}$ = weights, $\gamma_r^{(N\times N)}$ = recurrent weights, $\mu^{(N\times 1)}$ = biases$\}$, $A^{(N\times 1)}$ = bias vector, L = Number of unfolding steps, Δt = step size, N = Number of neurons,
Inputs: M-dimensional Input $\mathbf{I}(t)$ of length T, $\mathbf{x}(0)$
Output: Next LTC neural state $\mathbf{x}_{t+\Delta t}$
Function: FusedStep($\mathbf{x}(t)$, $\mathbf{I}(t)$, Δt, θ)
$\mathbf{x}(t+\Delta t)^{(N\times T)} = \frac{\mathbf{x}(t) + \Delta t f(\mathbf{x}(t), \mathbf{I}(t), t, \theta) \odot A}{1 + \Delta t\left(1/\tau + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)\right)}$
▷ $f(.)$, and all divisions are applied element-wise.
▷ $\odot$ is the Hadamard product.
end Function
$\mathbf{x}_{t+\Delta t} = \mathbf{x}(t)$
for $i = 1 \ldots L$ **do**
$\mathbf{x}_{t+\Delta t}$ = FusedStep($\mathbf{x}(t)$, $\mathbf{I}(t)$, Δt, θ)
end for
return $\mathbf{x}_{t+\Delta t}$

Recursively Folding Solver Output and Training via BPTT

Algorithm Training LTC by BPTT - Vanilla SGD

Inputs: Dataset of traces $[I(t), y(t)]$ of length T, RNNcell $= f(I, x)$
Parameter: Loss func $L(\theta)$, initial param θ_0, learning rate α, Output w $= W_{out}$, and bias $= b_{out}$
for $i = 1 \ldots$ number of training steps **do**
(I_b, y_b) = Sample training batch, $x := x_{t_0} \sim p(x_{t_0})$
for $j = 1 \ldots T$ **do**
$x = f(I(t), x)$, $\hat{y}(t) = W_{out}.x + b_{out}$, $L_{total} = \sum_{j=1}^{T} L(y_j(t), \hat{y}_j(t))$, $\nabla L(\theta) = \frac{\partial L_{tot}}{\partial \theta}$
$\theta = \theta - \alpha \nabla L(\theta)$
end for
end for
return θ

Complexity comparison for a single layer NN

	Vanilla BPTT	Adjoint
Time	$O(L \times T \times 2)$	$O((L_f + L_b) \times T)$
Memory	$O(L \times T)$	$\mathbf{O(1)}$
Depth	$O(L)$	$O(L_b)$
FWD acc	High	High
BWD acc	**High**	Low

Note: L = number of discretization steps, L_f = L during forward-pass. L_b = L during backward-pass. T = length of sequence, Depth = computational graph depth.

Summary of Main Contributions

- A new paradigm in continuous-time neural networks, effective in learning on irregularly-sampled data.
- LTC presents a novel approach for forward and backward passes, that balances accuracy and computation time.
- Trajectory length analysis shows that LTCs are significantly more expressive as compared to Neural Ordinary Differential Equations (NODE) and established sequence models.
- LTC time-constant and neural states are provably stable for unbounded inputs.
- LTCs are **universal approximators**.
- Architectures created using LTCs **can greatly reduce model size, while aiding explainability**.
- **LTCs can vary their behavior even post-training.**

Expressivity Measure – Trajectory Length

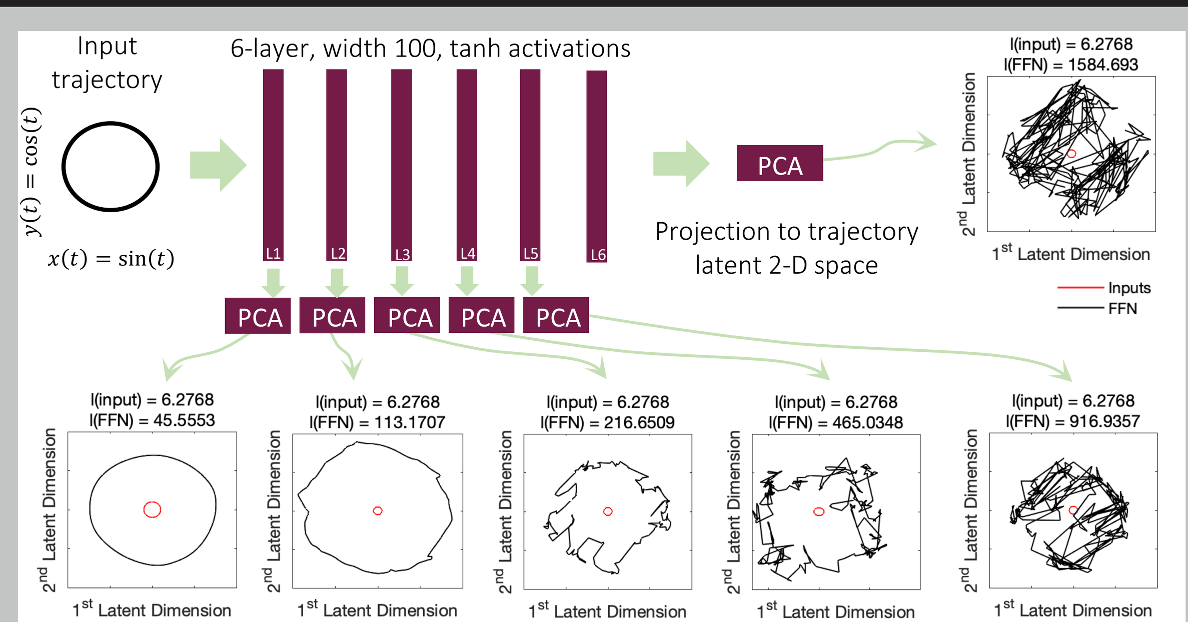


Figure: LTC's trajectory latent space becomes more complex with depth

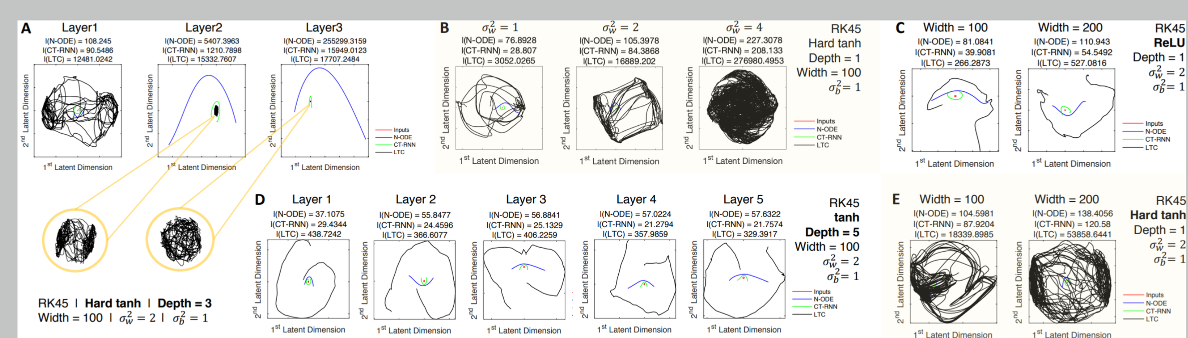


Figure: Trajectory latent w.r.t. different activations

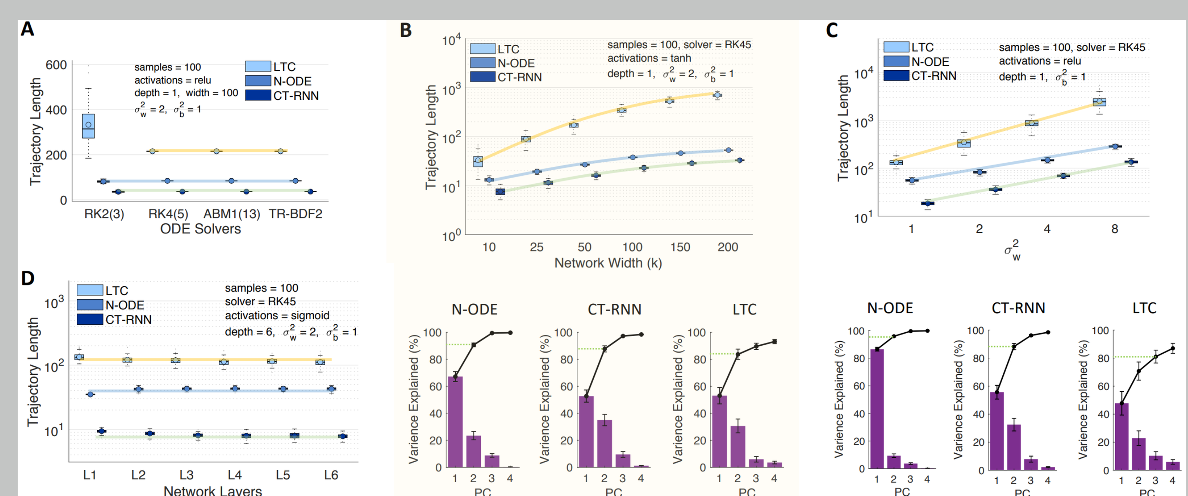
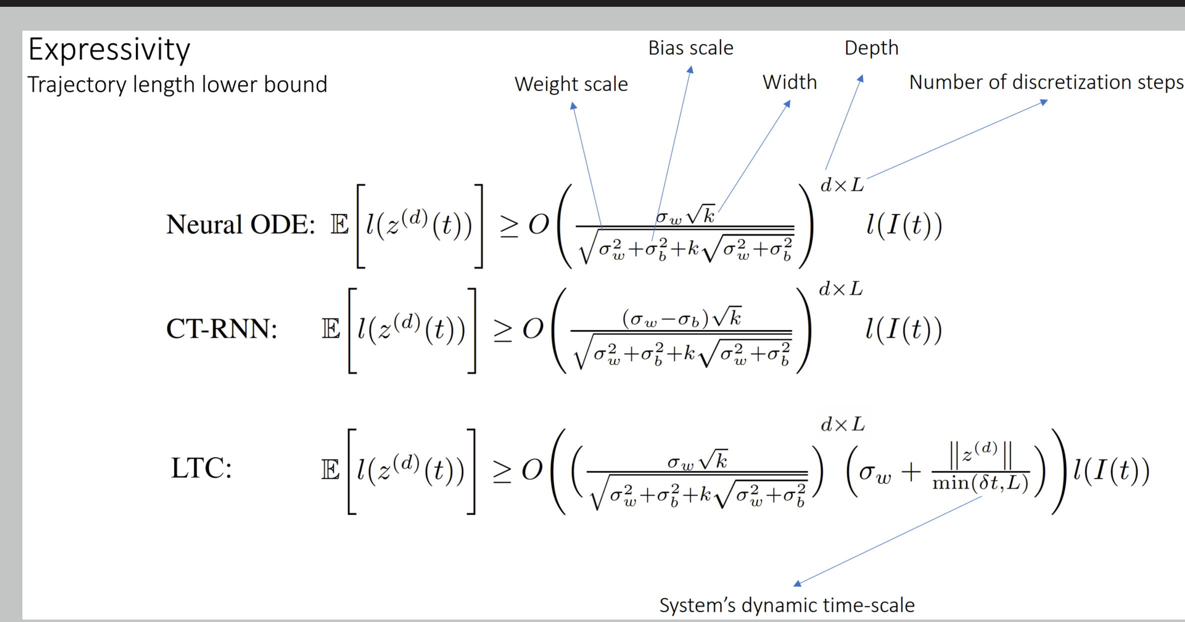


Figure: Trajectory latent w.r.t. different solvers vis-à-vis baselines

Expressivity Measure – Trajectory Length Lower Bounds



Practical Application — Lane Following - Network size comparison

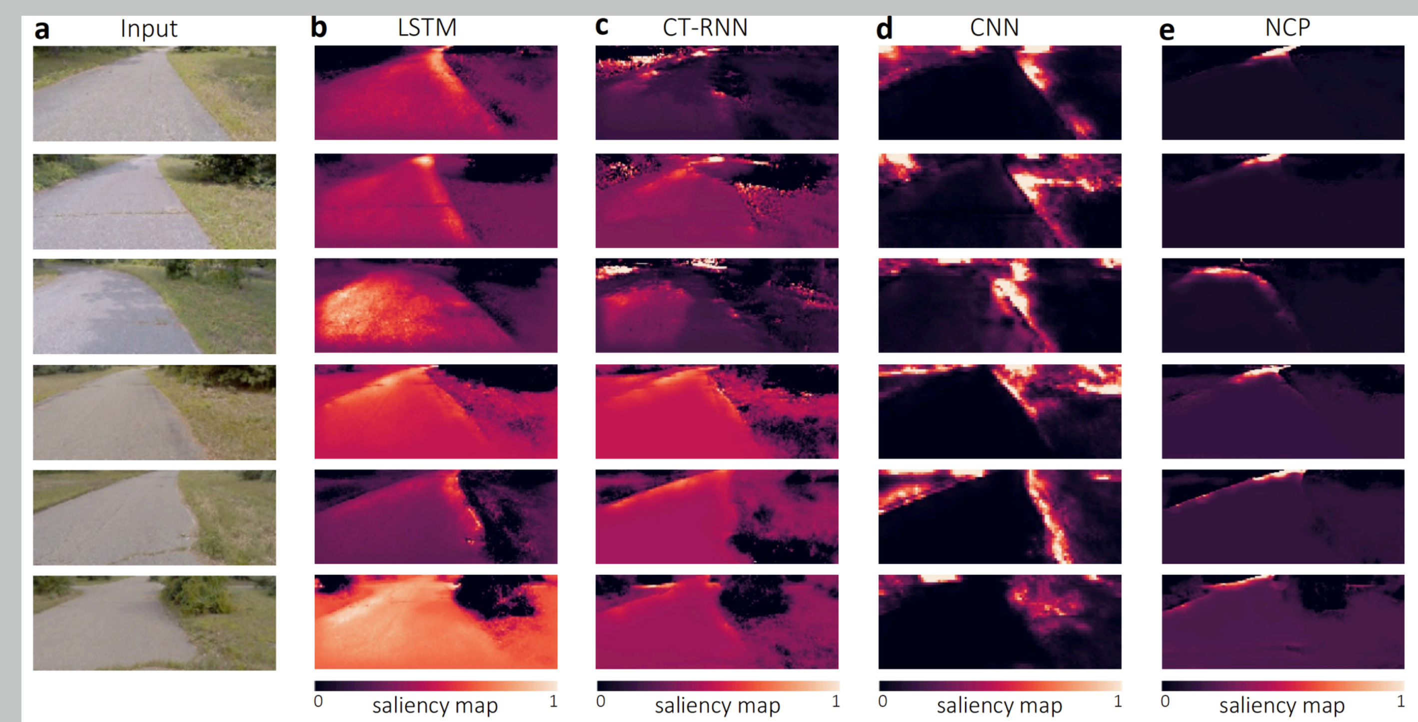


Figure: Saliency Maps - Where each network learns to attend while driving

Table: Network size comparison

Model	CNN parameters	RNN neurons	RNN synapses	RNN trainable parameters
CNN	5,068,900	-	-	-
CT-RNN	79,420	64	6,112	6,273
LSTM	79,420	64	24,640	24,897
NCP	79,420	19	253	1,065

Limitations

- Vanishing gradient phenomenon limiting applicability to learning long-term dependencies.
- Performance is tied to ODE solver used.
- Highly expressive but at an added time and memory cost.

References & Further Reading

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