

de Broglie's hypothesis: wave-particle duality

Light behaves as wave when it undergoes interference, diffraction etc. and is completely described by Maxwell's equations. But then, the wave nature of electromagnetic radiation is called into question when it is involved in blackbody radiation, photoelectric effect and such. Einstein forwarded his idea of photon, bundle of quantized radiant energy localized in a small volume, as a way to describe particle-like nature of light. The energy and momentum of such a photon was proposed to be,

$$E = h\nu \quad \text{and} \quad p = \frac{E}{c} = \frac{h}{\lambda}.$$

de Broglie (1924) made a great unifying, speculative hypothesis that just as radiation has particle-like properties, electrons and other material particles possess wave-like properties. For free material particles, de Broglie assumed that the associated wave also has a frequency ν and wavelength λ related to its energy E and momentum p ,

$$\nu = \frac{E}{h} \quad \text{and} \quad \lambda = \frac{h}{p}. \quad (60)$$

For non-relativistic particles having mass m and moving with a velocity v and kinetic energy $E_k = mv^2/2$, the de Broglie wavelength is

$$\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mE_k}} \quad (61)$$

For high energy particles, $E^2 = p^2c^2 + m_0^2c^4$, having kinetic energy $E_k = E - m_0c^2$, the momentum is $pc = \sqrt{E_k(E_k + 2m_0c^2)}$ and hence the de Broglie wavelength is,

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{E_k(E_k + 2m_0c^2)}}. \quad (62)$$

For instance, the de Broglie wavelength of an object of mass $m = 1.0\text{kg}$ and moving with a velocity $v = 10\text{m/s}$ is ($h = 6.6 \times 10^{-34}\text{J}\cdot\text{s}$)

$$\lambda = \frac{h}{mv} = \frac{6.6 \times 10^{-34}\text{J}\cdot\text{s}}{1.0 \times 10\text{kg} \cdot \text{m/s}} = 6.6 \times 10^{-35}\text{m} = 6.6 \times 10^{-25}\text{\AA}.$$

The de Broglie wavelengths of an electron ($m = 9.1 \times 10^{-31}\text{kg}$) at kinetic energy 100eV and 0.1MeV are ($1\text{eV} = 1.6 \times 10^{-19}\text{J}$),

$$\begin{aligned} \lambda &= \frac{h}{\sqrt{2mE}} = \frac{6.6 \times 10^{-34}}{(2 \times 9.1 \times 10^{-31} \times 100 \times 1.6 \times 10^{-19})^{1/2}} = 1.2\text{\AA} \\ \lambda_{\text{rel}} &= \frac{ch}{\sqrt{E(E + 2m_0c^2)}} = 0.037\text{\AA} \\ \lambda_{\text{nr}} &= \frac{h}{\sqrt{2mE}} = 0.038\text{\AA} \end{aligned}$$

Sure enough, de Broglie's hypothesis of wave-particle duality was confirmed by Davisson and Germer (1927) and G.P. Thomson (1927) [refer Eisberg & Resnick, pg 64 – 67].

The de Broglie hypothesis gives an interesting physical insight into Bohr's quantization rule (34),

$$mvr = pr = \frac{nh}{2\pi},$$

where p is the linear momentum of an electron in an allowed orbit of radius r . If we use equation (60), the expression for p in terms of de Broglie's wavelength $p = h/\lambda$, Bohr's quantization rule can be written as,

$$\frac{hr}{\lambda} = \frac{nh}{2\pi} \Rightarrow 2\pi r = n\lambda \quad n = 1, 2, 3, \dots \quad (63)$$

implying the allowed orbits are those in which the circumference of the orbit can contain exactly an integral number of de Broglie wavelengths.

Let us express the de Broglie wave of a free micro particle by a plane wave of constant amplitude A , which will represent a particle of energy $E = h\nu = \hbar\omega$ and momentum $p = h/\lambda = \hbar k$,

$$\psi(\vec{r}, t) = Ae^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad \text{in 1-dim} \Rightarrow \psi(x, t) = Ae^{i(kx-\omega t)} \quad (64)$$

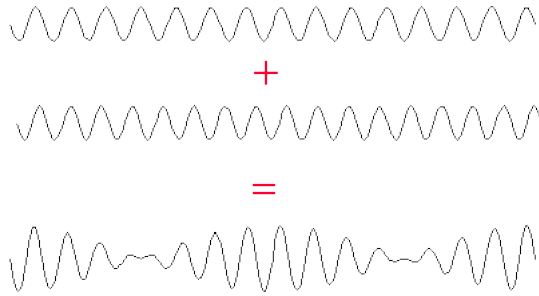
Assuming $|\psi(x, t)|^2$ gives the probability of finding the particle at space-time point (x, t) (according to Max Born's idea that came much later), it is constant A^2 everywhere. This implies that probability of finding our particle is same everywhere or, in other words, we do not know where it is. But our idea of particle is one having definite momentum and at the same time a specific position *i.e.* it is localized. If we know the position of the particle fairly accurately, then probability of finding it in different place must be confined to that space (say, Δx) outside which the probability is zero. Therefore, the matter wave must somehow be retracted in that space with the particle \Rightarrow we have a wave train of length Δx and wavelength of the wave train corresponds to particle momentum.

In order to manufacture a wave train we revoke the idea of group of moving waves of classical wave motion. Suppose we have two waves of (ω, k) and $(\omega + d\omega, k + dk)$ and for simplicity let them be represented as

$$\psi_1 = A \cos[kx - \omega t] \quad \text{and} \quad \psi_2 = A \cos[(k + dk)x - (\omega + d\omega)t]$$

We superpose these two waves, considering $d\omega \ll \omega$ and $dk \ll k$, to obtain,

$$\begin{aligned} \psi &= \psi_1 + \psi_2 \\ &= 2A \cos\left[\frac{dk}{2}x - \frac{d\omega}{2}t\right] \cos\left[\frac{2k + dk}{2}x - \frac{2\omega + d\omega}{2}t\right] \\ &\approx 2A \cos\left[\frac{dk}{2}x - \frac{d\omega}{2}t\right] \cos[kx - \omega t]. \end{aligned} \quad (65)$$



From the plot of above function ψ we see that two waves of slightly different frequency and wavelength, interfere and reinforce in such a way as to produce a series of groups. These groups, and the individual waves they contain, are both moving in the same direction. The velocity of the group, called *group velocity* v_g and velocity of the individual waves, called *phase velocity* v_p , are given by,

$$v_g = \frac{d\omega}{dk} \quad v_p = \frac{\omega}{k} \quad (66)$$

If the particle of mass m is moving with a velocity v so the kinetic energy $E = mv^2/2$ and momentum $p = mv$, then

$$\omega = \frac{E}{\hbar} \Rightarrow d\omega = \frac{dE}{\hbar} \quad \text{and} \quad k = \frac{p}{\hbar} \Rightarrow dk = \frac{dp}{\hbar}$$

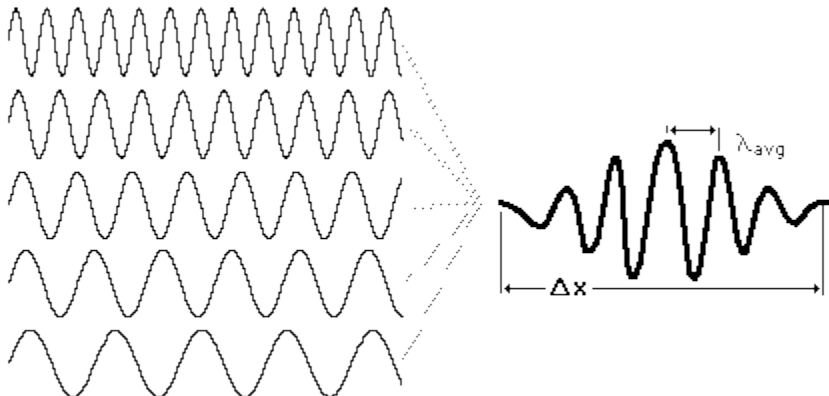
which leads to,

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp} = \frac{mv dv}{m dv} = v$$

i.e. the velocity of the particle is equal to velocity of the group of matter wave describing the particle. The same is true for relativistic particle.

The superposition of two matter waves of slightly varying frequency and wavenumber k manage to create succession of groups but not one group which is the wave train. To obtain a wave train or a *wave packet* of finite extent in space, we need superposition of large number waves with different but slightly varying frequency and wavenumber,

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) e^{i(kx - \omega(k)t)}.$$



Assuming $A(k)$ is nonzero and constant only in an interval, $k_0 - \Delta k/2 \leq k \leq k_0 + \Delta k/2$, if Δk is not too large, we can expand $\omega(k)$ about k_0 ,

$$\omega(k) = \omega(k_0) + (k - k_0) \left(\frac{d\omega}{dk} \right)_{k=k_0} + \dots$$

and write the above wave function as,

$$\begin{aligned} \psi(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{k_0 - \Delta k/2}^{k_0 + \Delta k/2} dk A e^{i(kx - \omega(k)t)} \\ &= \frac{A}{\sqrt{2\pi}} e^{i(k_0 x - \omega(k_0)t)} \int_{k_0 - \Delta k/2}^{k_0 + \Delta k/2} dk e^{i(k - k_0)[x - (d\omega/dk)_0 t]} \\ &= \sqrt{\frac{2}{\pi}} A \frac{\sin[\Delta k(x - (d\omega/dk)_0 t)/2]}{x - (d\omega/dk)_0 t} e^{i(k_0 x - \omega(k_0)t)} \end{aligned} \quad (67)$$

The equation (67) defines a wave packet of finite extent, $\Delta x \approx 2 \times 2\pi/\Delta k$, whose group velocity is $v_g = (d\omega/dk)_{k=k_0}$ and phase velocity is $v_p = \omega(k_0)/k_0$ as before.

A very interesting illustration of wave-nature of microscopic particle is Feynman's thought experiment – *double-slit experiment* with classical and quantum particles – for details of which see Feynman's Lecture vol 3.