Clausius-Mossotti relation: In linear dielectric, the polarization is $\vec{P} = \epsilon_0 \chi_e \vec{E}$ when the electric field $\vec{E}$ is weak. However, this electric field is the *total* field *i.e.* $\vec{E}_{\text{tot}} = \vec{E}_{\text{ext}} + \vec{E}_{\text{pol}}$, external and that due to polarization. For non-polar material, the induced dipole moment is $\vec{p} = \alpha \vec{E}_{\text{ext}}$, where α is the atomic polarizability. If we are working with weak fields, then we may approximate

$$\vec{P} = N\vec{p} \Rightarrow \epsilon_0 \chi_e \approx N\alpha \quad \rightarrow \quad \chi_e = \frac{N\alpha}{\epsilon_0}, \tag{29}$$

where N is the number of molecules per unit volume and is small. If we consider the non-polar molecule be an uniformly charged sphere of radius R, then $N = 1/(\frac{4}{3}\pi R^3)$.
Next, let us calculate $\vec{E}_{\text{pol}}$ when the molecule is placed in $\vec{E}_{\text{ext}}$, using Gauss's law and the way atomic polarizability is calculated in Example 4.1 (Griffiths),

$$E_{\text{pol}}\, 4\pi r^2 = \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0} \quad \Rightarrow \quad \vec{E}_{\text{pol}} = \frac{\rho r \hat{r}}{3\epsilon_0} = \frac{1}{4\pi\epsilon_0}\frac{Q\vec{r}}{R^3} = \frac{\vec{p}}{4\pi\epsilon_0 R^3}.$$

The $\vec{E}_{\text{pol}}$ is acting opposite to $\vec{E}_{\text{ext}}$, hence

$$\vec{E}_{\text{pol}} = \frac{-\alpha\, \vec{E}_{\text{ext}}}{4\pi\epsilon_0 R^3} \quad \Rightarrow \quad \vec{E}_{\text{tot}} = \left(1 - \frac{\alpha}{4\pi\epsilon_0 R^3}\right)\vec{E}_{\text{ext}} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\vec{E}_{\text{ext}}. \tag{30}$$

The susceptibility and polarizability get connected by,

$$\vec{P} = N\vec{p} = N\alpha\vec{E}_{\text{ext}} \quad \Rightarrow \quad \frac{N\alpha\vec{E}_{\text{tot}}}{1 - \frac{N\alpha}{3\epsilon_0}} = \epsilon_0\chi_e\vec{E}_{\text{tot}} \quad \Rightarrow \quad \chi_e = \frac{\frac{N\alpha}{\epsilon_0}}{1 - \frac{N\alpha}{3\epsilon_0}}. \tag{31}$$

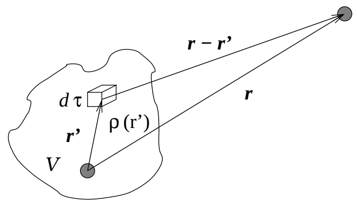
Solving for α, we obtain Clausius-Mossotti relation for non-polar molecule

$$\alpha = \frac{3\epsilon_0}{N}\frac{\chi_e}{3 + \chi_e} = \frac{3\epsilon_0}{N}\frac{K - 1}{K + 2} \tag{32}$$

where, $K = \chi_e + 1$.

Multipole expansion: (*Not in syllabus*) We use *multipole expansion* to determine potential at large distances away from a localized charge distribution, which may or may not be neutral. It is done by expansion of the potential of the charge distribution in powers of $1/r$. Let us first do it in a coordinate independent way, refering to the figure below, we write the potential at $\vec{r}$ as,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'. \tag{33}$$



Since $|\vec{r} - \vec{r}'| = (r^2 + r'^2 - 2\vec{r}\cdot\vec{r}')^{1/2}$ and $|\vec{r}| \gg |\vec{r}'|$, we have,

$$\begin{aligned}
|\vec{r} - \vec{r}'|^{-1} &= \left(r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'\right)^{-1/2} \\
&= \frac{1}{r}\left[1 - \left(\frac{2\vec{r}\cdot\vec{r}'}{r^2} - \frac{r'^2}{r^2}\right)\right]^{-1/2} \\
&= \frac{1}{r}\left[1 + \frac{1}{2}\left(\frac{2\vec{r}\cdot\vec{r}'}{r^2} - \frac{r'^2}{r^2}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)\left(\frac{2\vec{r}\cdot\vec{r}'}{r^2} - \frac{r'^2}{r^2}\right)^2 + \cdots\right]
\end{aligned}$$

Collecting the terms with same power of r',

$$|\vec{r}-\vec{r}'|^{-1} \;=\; \frac{1}{r}+\frac{\vec{r}\cdot\vec{r}'}{r^3}+\frac{1}{2}\left(\frac{3(\vec{r}\cdot\vec{r}')^2}{r^5}-\frac{r'^2}{r^3}\right)+\cdots \tag{34}$$

Therefore, the potential in Eqn. (33) becomes,

$$V(\vec{r}) \;=\; \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\left[\frac{1}{r}+\frac{\vec{r}\cdot\vec{r}'}{r^3}+\frac{1}{2}\left(\frac{3(\vec{r}\cdot\vec{r}')^2}{r^5}-\frac{r'^2}{r^3}\right)+\cdots\right]\rho(\vec{r}')d\tau' \tag{35}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int\rho(\vec{r}')d\tau'+\sum_i\frac{r_i}{r^3}\int r_i'\,\rho(\vec{r}')d\tau'+\sum_{i,j}\frac{1}{2}\frac{r_ir_j}{r^5}\int\left[3r_i'r_j'-\delta_{ij}r'^2\right]\rho(\vec{r}')d\tau'+\cdots\right]. \tag{36}$$

The above equation (36) is the desired expansion of potential in $1/r$. To gain some physical insight of the expansion, we define the followings:

$$\text{Monopole}: \quad q \;=\; \int_{\mathcal{V}}\rho(\vec{r}')d\tau' \tag{37}$$

$$\text{Dipole moment}: \quad p_i \;=\; \int_{\mathcal{V}}r_i'\rho(\vec{r}')d\tau' \tag{38}$$

$$\text{Quadrupole moment}: \quad Q_{ij} \;=\; \int_{\mathcal{V}}\left(3r_i'r_j'-\delta_{ij}r'^2\right)\rho(\vec{r}')d\tau' \tag{39}$$

In the expression for dipole moment above, Eqn. (38), it is written in component form which, when written in vector notaiton, is same as before viz.

$$\vec{p} = \int_{\mathcal{V}}\vec{r}'\,\rho(\vec{r}')d\tau'$$

In polar coordinate, the expression (36) is given by,

$$V(\vec{r}) \;=\; \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{\mathcal{V}}(r')^n\,P_n(\cos\theta)\,\rho(\vec{r}')d\tau' \tag{40}$$

the same as Eqn. 3.95 of Griffiths. This explains the origin of higher moments in potential, Eqn. (15) of a general localized charge distribution. Just for exercise, try to calculate the octople moment by considering the terms r'^3 in the equations (34, 35).