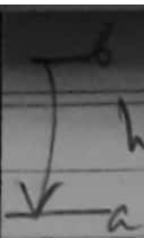
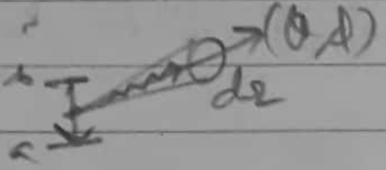
Spontaneous emission in (θ, ϕ) direction w r t $\vec{r}_{ab}$ in elemental solid angle $d\Omega$



$$W^{sp}_{ab} = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{\pi 2\hbar\,|M_{ba}|^2}{Vm^2\omega_{ba}}/sp$$ rate of emission with ω due to sp transition

with angular frequency in range ω to $\omega + d\omega$

What is the rate of emission of a photon to be emitted in a given (θ, ϕ) direction within a solid angle element $d\Omega$

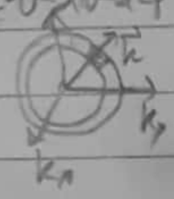


Need to calculate density of photon states $\varphi(\omega)$ along (θ, ϕ) and W provides the rate of occupancy to the states.

Within a cube of side L the available photon states:

$$(k_x, k_y, k_z) = \left(n_x \frac{2\pi}{L}, n_y \frac{2\pi}{L}, n_z \frac{2\pi}{L}\right); \quad n_x, n_y, n_z \in (-\infty, \dots -1, 0, 1, \dots \infty)$$

$\therefore$ # of state within wavevector element: $\dfrac{k^2\, dk\, d\Omega}{\left(\frac{2\pi}{L}\right)^3}$; $d\Omega = \sin\theta\, d\theta\, d\phi$



$\therefore$ " " " " range ω to $\omega + d\omega$: $\left(\dfrac{\omega}{c}\right)^2 \dfrac{d\omega}{c}\, d\Omega\, \dfrac{V}{(2\pi)^3}$

$$\therefore \quad \varphi(\omega)\, d\omega\, d\Omega = \left(\frac{\omega}{c}\right)^2 \frac{V}{c(2\pi)^3}\, d\omega\, d\Omega \rightarrow \text{\# of state in range } \omega \text{ to } \omega + d\omega \text{ within } d\Omega \text{ along } (\theta, \phi)$$

$\therefore$ rate of emission of photon in the range ω to $\omega + d\omega$ within $d\Omega$ along (θ, ϕ): $W^{sp}_{ab}(\omega, \Omega)\, \varphi(\omega)\, d\omega\, d\Omega$

$$\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{\pi 2\hbar\,|M_{ba}|^2}{Vm^2\omega_{ba}}\,\delta(\omega - \omega_{ba})\,\varphi(\omega)\, d\omega\, d\Omega$$

$$\therefore \text{Total}: \int_\omega W^{sp}_{ab}(\omega, \Omega)\varphi(\omega)\frac{d\omega}{d\Omega} = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{\pi^2\hbar\,|M_{ba}|^2}{Vm^2\omega_{ba}}\left(\frac{\omega_{ba}}{c}\right)^2\frac{V}{c(2\pi)^3}\, d\Omega$$

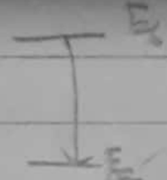
$$= \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{\hbar\,\omega_{ba}}{8\pi m^2 c^3}|M_{ba}|^2\, d\Omega$$

$$= W^{sp}_{ab}(\Omega)\, d\Omega$$

Oscillator strength
Thomas Reichi sum rule

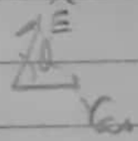
$$\therefore W^{sp}_{ab}(\Omega) = \left(\frac{e^2}{4\pi\epsilon_0}\right) \frac{\hbar\omega_{ba}}{8\pi m^2 c^3} |M_{ba}|^2$$

$$M_{ba} = m\omega_{ba} \frac{\hat{E}\cdot\vec{r}_{ba}}{\hbar}$$



$$\int W^{sp}_{ab}(\Omega)\, d\Omega$$

$$= \left(\frac{e^2}{4\pi\epsilon_0}\right) \frac{\hbar\omega_{ba}}{8\pi m^2 c^3} \times \left(\frac{m^2\omega_{ba}^2}{\hbar^2} |r_{ba}|^2 \underbrace{\int \cos^2\theta \, d\Omega}_{\frac{4\pi}{3}} \right)$$



$$= \left(\frac{e^2}{4\pi\epsilon_0}\right) \frac{\omega_{ba}^2}{2mc^3} \frac{1}{2} \left[\frac{2m\,\omega_{ba}\,|r_{ba}|^2}{3\hbar} \right]$$

dimensionless quantity

Define "oscillator strength"

$$\boxed{f_{ba} = \frac{2m\omega_{ba}|r_{ba}|^2}{3\hbar}}$$

Note $f_{ba} > 0$ for absorption since $\omega_{ba} > 0$; $f_{ba} < 0$ for emission $\because \omega_{ba} < 0$

$$f^x_{ba} = \frac{2m\omega_{ba}}{3\hbar} |x_{ba}|^2 = \frac{2m\omega_{ba}}{3\hbar} \langle b|x|a\rangle\langle a|x|b\rangle$$

Recall $\hat{p}_x = m(i\hbar)^{-1}[x, H_0]$; $H_0|a\rangle = E_a|a\rangle$, $H_0|b\rangle = E_b|b\rangle$

$$\Rightarrow \langle b|p_x|a\rangle = m(i\hbar)^{-1}\langle b|xH_0 - H_0 x|a\rangle = m(i\hbar)^{-1}(E_a - E_b)\langle b|x|a\rangle$$

$$\Rightarrow \langle b|x|a\rangle = -i\frac{\langle b|p_x|a\rangle}{m\omega_{ba}}$$

Similarly $\langle a|x|b\rangle = -i\langle a|p_x|b\rangle / m\omega_{ab}$

$\omega_{ba} = -\omega_{ab}$

$= i\langle a|p_x|b\rangle / m\omega_{ba}$

$$\therefore f^x_{ba} = \frac{m\omega_{ba}}{3\hbar}\Big[\underbrace{\langle b|x|a\rangle}_{-i\frac{\langle b|p_x|a\rangle}{m\omega_{ba}}}\langle a|x|b\rangle + \langle b|x|a\rangle\underbrace{\langle a|x|b\rangle}_{i\frac{\langle a|p_x|b\rangle}{m\omega_{ba}}}\Big]$$

$$\frac{1}{3\hbar}\, i\Big[-\langle b|p_x|a\rangle\langle a|x|b\rangle + \langle b|x|a\rangle\langle a|p_x|b\rangle\Big]$$

Average oscillator strength

$$\sum_b f^x_{ba} = \frac{i}{3\hbar}\Big[\langle a|p_x \underbrace{\sum_l |l\rangle\langle l|}_{I} x|a\rangle - \langle a|x \underbrace{\sum_l |l\rangle\langle l|}_{I} p_x|a\rangle\Big]$$

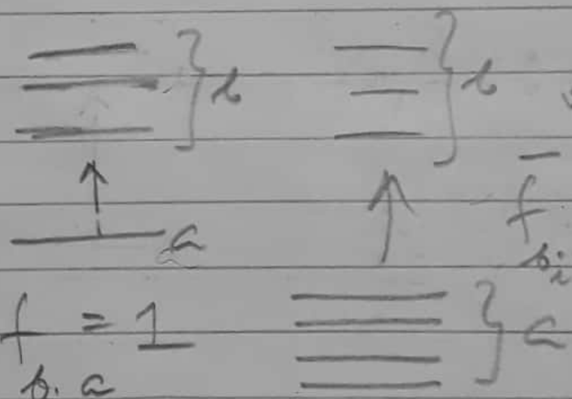
CONS $\{l\}$

$$= \frac{i}{3\hbar}\Big[\langle a|\,[p_x, x]\,|a\rangle\Big].$$

Recall $[x, p_x] = i\hbar \Rightarrow \sum_b f^x_{ba} = \frac{i}{3\hbar}(-i\hbar)\langle a|a\rangle = \frac{1}{3}$

Similarly $\sum_b f^y_{ba} = \sum_b f^z_{ba} = \frac{1}{3} \longrightarrow$ Thomas Reiche Sum Rule

$$\Rightarrow \sum_b f_{ba} = 1$$



Average $\bar{f}_{b_i a}$

$$\bar{f}_{b_i a} = \frac{1}{4}\sum_j^4 f_{b_i a_j}$$

$$\sum_{i=1}^{3} f_{b_i a} = 1$$

Since

$$\sum_{i=1}^{3} f_{b_i a_j} = 1$$

4 $\}N_4$
3 $\}N_3$
2 $\}N_2$
1 $\}N_1$

$$\bar{f}_{21} = \frac{1}{N_2}\sum_{i=1}^{N_2}\sum_{j=1}^{N_1} f_{2_i 1_j}$$

so that

$$\sum_{n=2} \bar{f}_{n1} = 1$$