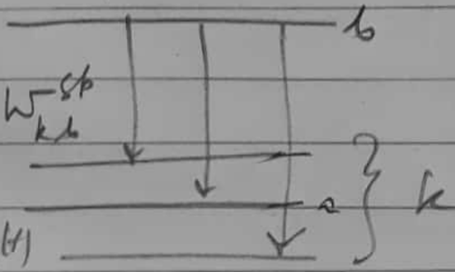
Emergence of natural line width due to spontaneous decay

Note that tau_b infinity would imply Cb(t)=1 for all t , means no decay: E uncertainty is 0. Therefore a finite tau_b suggest that Eb is not sharply defined. It has a width which leads to the Lorentzn. line shape we see next page.

Lifetime of excited states.



$$\dot N_k(t) = N_b(t)\, W^{sp}_{kb}$$

$$\dot N_b(t) = -\sum_k \dot N_k(t)$$

Recall introduction of spontaneous transition by Einstein

$$\therefore \dot N_b(t) = -N_b(t) \sum_k W^{sp}_{kb}$$

$$N(t) = N(t=0)\exp\left(-t/\tau_b\right)$$

This is the generic survival equation

$$\tau_b^{-1} = \sum_k W^{sp}_{kb}$$

$$|C_b(t)|^2 \propto N_b(t) \Rightarrow C_b(t) \propto \sqrt{N_b(t)} \Rightarrow C_b(t) \simeq \exp\left(-t/2\tau_b\right)$$

stimulated process

Recall $\dot C_a(t) = (i\hbar)^{-1} \sum_n \langle \phi_a | H' | \phi_n \rangle C_n(t)\, e^{i\omega_{an} t}$

we simplify this by assuming $C(t) \approx C_n(t \le 0) = \delta_{nb}$ in RHS

Now we improve on the assumption by letting

$$C_n(t) = \delta_{nb} \exp\left(-t/2\tau_b\right) \text{ for } t \geq 0$$

$$\Rightarrow C_a(t) = (i\hbar)^{-1} \int_0^t H'_{ab} \exp\left(i\omega_{ab} - i/2\tau_b\right) t'\, dt'$$

Putting in the expression of $\vec{A}$ in H'_{ab}:

$$C_a(\omega, t) = -\frac{e}{m} A_0(\omega) M_{ab}\, e^{i\delta_\omega} \int_0^t \exp\left[i(\omega - \omega_{ba}) - \frac{1}{2\tau_b}\right] t'\, dt'$$

$$= -\frac{e}{m} A_0(\omega) M_{ab}\, e^{-i\delta_\omega} \left[\frac{\exp\left[i(\omega - \omega_{ba})t - \frac{t}{2\tau_b}\right] - 1}{\left[i(\omega - \omega_{ba}) - \frac{1}{2\tau_b}\right]} \right]$$

$$C_a(\omega, t) \propto \left[\quad \right]$$

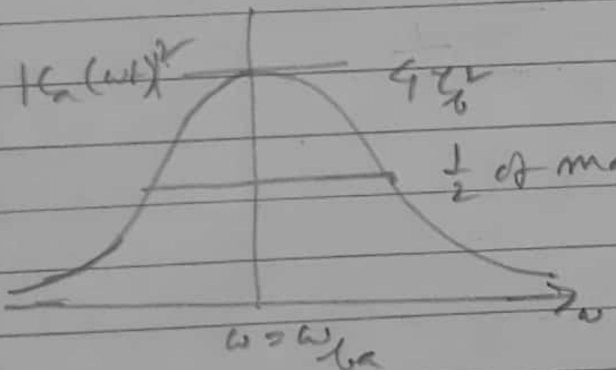
Note that we are assuming validity of survival eqation for Nb(t)

Derivation of spontaneous decay rate

$$|c_a(\omega,t)|^2 \propto \frac{\left(e^{-i(\omega-\omega_{ba})t}e^{-t/2\tau_b}-1\right)\left(e^{i(\omega-\omega_{ba})t}e^{-t/2\tau_b}-1\right)}{\frac{1}{4\tau_b^2}+(\omega-\omega_{ba})^2}$$

$$= \frac{e^{-t/\tau_b} - e^{-t/2\tau_b}\,2\cos(\omega-\omega_{ba})t + 1}{(1/4\tau_b^2)+(\omega-\omega_{ba})^2}$$

$$\therefore t \gg \tau_b : |c_a(\omega,t)|^2 \propto \frac{1}{\frac{1}{4\tau_b^2}+(\omega-\omega_{ba})^2}$$



$\frac{1}{2}$ of max at $\omega = \omega_{ba} \pm \frac{1}{2\tau_b}$

FWHM $= \frac{1}{\tau_b} = \Gamma$ natural line width

Note: $\Delta E \Delta t \approx \hbar \Rightarrow \Delta E \approx \hbar/\tau_b$

b ↑ a and we $\Rightarrow \Delta\omega \approx 1/\tau_b$

Now recall the st process a ↑ $c_a(t)$ derived above

in $\dot{c}_b(t)$ with $A_0(\omega) = \sqrt{\frac{n(\omega)\hbar}{2\epsilon_0\omega}} = \sqrt{\frac{\hbar}{2V\epsilon_0\omega}}$; (for 1 photon)

$\dot{c}_b(t) = (i\hbar)^{-1}\langle\phi_a|H'|\phi_b\rangle c_a(t)e^{i\omega_{ba}t}$, with $c_n(1)=\delta_{na}c_a(t)$

putting in $A(\omega,t)$

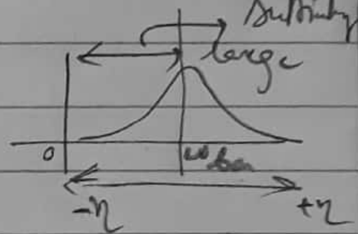
$$= \int -\frac{e}{2m}A_0(\omega)M_{ba}e^{+i\delta_\omega}e^{i(\omega_{ba}-\omega)t}c_a(t)\,d\omega;\quad \because e^{-i\omega t} \text{ of } A \text{ contributes to absorption}$$

$$= \int \frac{-e}{2m}\left(\frac{\hbar}{2V\epsilon_0\omega}\right)^{1/2}M_{ba}e^{i\delta_\omega}e^{i(\omega_{ba}-\omega)t}\,d\omega\left[\frac{\left(\frac{-e}{m}\right)A_0(\omega)M_{ab}e^{-i\delta_\omega}\left[\exp\left(i(\omega-\omega_{ba})t-\frac{t}{2\tau_b}\right)-1\right]}{\left[i(\omega-\omega_{ba})-1/2\tau_b\right]}\right]$$

Equivalence of spontaneous and stimulated processes

$$= \frac{1}{2}\left(\frac{e}{m}\right)^2 \left(\frac{\hbar}{2V\epsilon_0\omega}\right) \int d\omega\, |M_{ba}|^2 \frac{\left[e^{-t/2\tau} - e^{i(\omega_{ba}-\omega)t}\right]}{\left[i(\omega-\omega_{ba}) - 1/2\tau\right]},$$

Note: $$\lim_{\eta\to+\infty} \int_{-\eta}^{+\eta} \frac{1}{x-\alpha+i\beta}\, dx = -i\pi$$



$$\lim_{\eta\to+\infty} \int_{-\eta}^{+\eta} \frac{e^{-ixt}}{x-\alpha+i\beta}\, dx = 2\pi i e^{-i(\alpha-i\beta)t}$$

using the above results with $x=\omega, \alpha=\omega_{ba}; \beta=1/2\tau$ and assuming M_{ba} to be slowly varying with about ω_{ba}

$$\dot{C}_b(t) = \frac{1}{2}\left(\frac{e}{m}\right)^2 \frac{\hbar\pi}{2V\epsilon_0\omega_{ba}} |M_{ba}|^2 e^{-t/2\tau}$$

Now since $C_b(t) = e^{-t/2\tau}$ and $\dot{C}_b(t) = -\frac{1}{2\tau}e^{-t/2\tau}$

We can deduce that $$W^{sp}_{ab} = \frac{1}{\tau} = \frac{1}{2}\left(\frac{e}{m}\right)^2 \frac{\hbar\pi}{2V\epsilon_0\omega_{ba}} |M_{ba}|^2$$

sp b→a

$$= \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{\hbar\pi^2}{V m^2\omega_{ba}} |M_{ba}|^2$$

Recall stimulated transition rate:

$$W^{st}_{ab} = \frac{\pi^2}{m^2 c}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{I}{\omega_{ba}^2}|M_{ba}|^2 = \frac{\pi^2}{m^2 c}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{Nc\hbar\omega}{V\omega_{ba}^2}|M_{ba}|^2 = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{N\hbar\pi^2 |M_{ba}|^2}{V m^2\omega_{ba}}$$

(semiclassical)

Note the similarity of the W^{sp}_{ab} and W^{st}_{ab}.

QED: $$W_{ab} = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{(N+1)\hbar\pi^2}{V m^2\omega_{ba}}|M_{ba}|^2 \; ; \quad W^{st}_{ba} = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{N\hbar\pi^2}{V m^2\omega_{ba}}|M_{ba}|^2$$

(emission) (absorption)

$\Rightarrow$ N=0 leads to the W^{sp}_{ab} $\Rightarrow$ st and sp processes occur on same footing!

NOTE: The QED transition rate thus indeed be considered as a combination of stimulated and spontaneous transitions rates. Semiclassical derivation only gave the stimulated rate. We derived the spontaneous rate from the assumed validity of survival eqation (N(t)=N(0)exp(-t/Tau) for population of systems in a given states (Nb(t)).