Representations of Braid group from Branched covers of the Sphere

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10 April 2026

Branched Covers

Fixing n distinct points $a_1, \dots, a_n \in \mathbb{C}$ we have a Riemann surface

$$X_a^\circ = \{(x, y) \in \mathbb{C}^2 \mid y^d = (x - a_1) \cdots (x - a_n)\}.$$

$a = (a_1, \dots, a_n)$.

The map $\pi : X_a^\circ \to \mathbb{C}$ given by

$$\pi(x, y) = x$$

is a degree d branched covering.

Moreover π has cyclic deck group $\mathbb{Z}/d\mathbb{Z}$ generated by

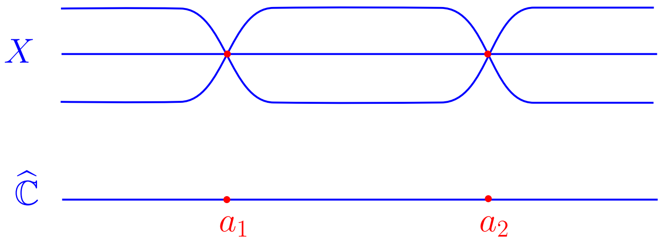
$$T(x, y) = (x, \zeta_d y).$$

Branched Covers

The surface X_a° can be completed to a compact Riemann surface X_a and π extends to a branched covering

$$\pi : X_a \to \widehat{\mathbb{C}}$$

branched over $a_1, \ldots, a_n$ and ∞.



Family of surfaces

Now if we vary the branch points $a_1, \ldots, a_n$ in $\mathbb{C}$ we get a family of Riemann surfaces X_a.

Let $\mathrm{Conf}_n(\mathbb{C})$ be the space of unordered n distinct points in the complex plane

$$\mathrm{Conf}_n(\mathbb{C}) := \frac{\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\}}{S_n}$$

then we get a surface X_a for each $a \in \mathrm{Conf}_n(\mathbb{C})$.

As we move points in $\mathrm{Conf}_n(\mathbb{C})$ the surfaces X_a change. Going around a loop in $\mathrm{Conf}_n(\mathbb{C})$ based at a we then get a homeomorphism of the surface X_a.

Monodromy representation

Loops in $\mathrm{Conf}_n(\mathbb{C})$, based at a, give rise to homeomorphisms of the surface X_a. Let us fix $a \in \mathrm{Conf}_a(\mathbb{C})$ and write $X = X_a$.

$$\text{loop in } \mathrm{Conf}_n(\mathbb{C}) \rightsquigarrow \text{homeo of } X \rightsquigarrow \text{linear map on } H_1(X)$$

Thus we get a representation of $\Pi_1(\mathrm{Conf}_n(\mathbb{C}))$ on $H_1(X)$. This is called the Monodromy representation.

$$H_1(X) := H_1(X, \mathbb{C}) = \Pi_1(X)^{\mathrm{ab}} \otimes \mathbb{C}.$$

Braid Groups

The braid group on n strands is the fundamental group

$$\mathcal{B}_n := \Pi_1(\mathrm{Conf}_n(\mathbb{C})).$$

A loop in $\mathrm{Conf}_n(\mathbb{C})$ at $a = (a_1, \ldots, a_n)$ lifts to a path

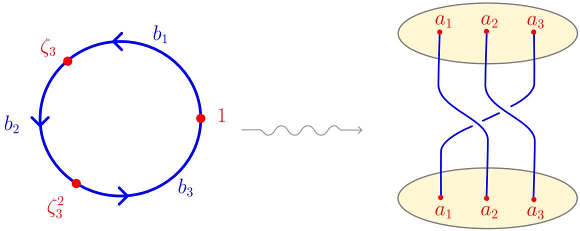
$$b : [0,1] \to \mathbb{C}^n,$$

such that

$b_i(t) \neq b_j(t)$,
$b(0) = (a_1, \ldots, a_n)$,
$b(1) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ for some permutation $\sigma \in S_n$.

Intuition! Braids can be thought of as n particles moving in the plane. They start and end at the same positions with possible exchange of places.

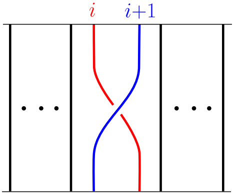
For example the braid $b(t) = (e^{2\pi it/3}, \zeta_3 e^{2\pi it/3}, \zeta_3^2 e^{2\pi it/3})$ in $\mathcal{B}_3$ is pictured below ($a = (1, \zeta_3, \zeta_3^2)$)



See the video *Braids: the movie* on youtube
https://youtube.com/playlist?list=PLyxHTRWELFBoijzAvXTDOLIF5yfkwN4wZ&si=q5wOLPQPk6bJa-VJ

Artin's presentation

The group $\mathcal{B}_n$ is generated by the simple braids $\sigma_1, \ldots, \sigma_{n-1}$.



σ_i

They satisfy the following relations:

- braid relation: $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$,
- far commutativity: $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i - j| > 1$.

Braids as Mapping classes

There is an isomorphism of the braid group with the Compactly supported mapping class group of $\mathbb{C}$ relative to a

$$\Phi : \mathcal{B}_n \to \mathrm{Mod}_c(\mathbb{C}, a) \qquad a = \{a_1, \ldots, a_n\}$$

where

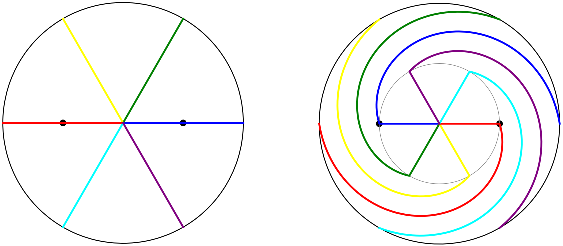
$$\mathrm{Mod}_c(\mathbb{C}, a) = \frac{\left\{ \begin{array}{l} \text{diffeomorphisms } f : \mathbb{C} \to \mathbb{C},\ f(a) = a \\ f \text{ identity ouside some compact set} \end{array} \right\}}{\text{homotopy}}$$

Intuition! We can think of $\mathbb{C}$ as a rubber sheet and the points $a_1, \ldots, a_n$ as pins on this rubber sheet. Now if the pins move they drag the rubber sheet along and create stretches and deformations.

For the generator σ_i, the mapping class $\Phi(\sigma_i)$ is represented by a half twist of a disk containing only a_i, a_{i+1}.

Suppose, $a_i = 1$, $a_{i+1} = -1$ and $|a_j| > 2$ for $j \neq i, i+1$ then

$$\Phi(\sigma_i)(z) = \begin{cases} -z & z| \leq 1, \\ z & 2 \leq |z|, \\ z\, e^{\pi i (2-|z|)} & 1 \leq |z| \leq 2. \end{cases}$$



Back to Monodromy representation

The diffeomorphisms in $\mathcal{B}_n \cong \mathrm{Mod}_c(\mathbb{C}, a)$ lift to the cover X.

By choosing the unique lift $\widetilde{b}$ of $b \in \mathcal{B}_n$ which is identity in a neighbourhood of $\pi^{-1}(\infty)$ we get a homomorphism

$$\mathcal{B}_n \to \mathrm{Mod}(X), \qquad b \mapsto \widetilde{b}.$$

We get an action of $\mathcal{B}_n$ on $H_1(X)$ by

$$b \cdot \xi = \widetilde{b}_* \xi, \qquad b \in \mathcal{B}_n,\ \xi \in H_1(X).$$

This is another more straightforward way of arriving at the monodromy representation.

Eigenspace decomposition

Recall the deck group of $\pi : X \to \widehat{\mathbb{C}}$ is generated by

$$T(x, y) = (x, \zeta_d y).$$

It turn out that the lifts $\widetilde{b}$ commute with T.

Hence the $\mathcal{B}_n$ action preserves the eigenspaces T-eigenspaces

$$H_1(X)_q := \{\xi \in H_1(X) \mid T_* \xi = q\xi\}.$$

Consequently we get a representation

$$\rho_q : \mathcal{B}_n \to \mathrm{GL}(H_1(X)_q).$$

Generalisations

McMullen (2013) investigates this monodromy representation of $\mathcal{B}_n$ for the cyclic covers X and gives several applications.

We attempt two natural generalisations (with partial success):

- Go from cyclic to abelian covers of $\widehat{\mathbb{C}}$ (with Saswati).
- Consider covers of higher genus surfaces rather than $\widehat{\mathbb{C}}$ (with Pranav and Saswati).

The first is easier and better explained so we will stick to that one in this presentation.

Abelian Covers

Let $\Lambda = (n_1, \ldots, n_m)$ be a partition of n and $d_1, \ldots, d_m$ be positive integers. Set $h_j = n_1 + \ldots + n_j$.

Let X be the completion of

$$\left\{(x, y_1, \ldots, y_m) \in \mathbb{C}^{m+1} \mid y_j^{d_j} = (x - a_{h_{j-1}+1}) \cdots (x - a_{h_j}),\ 1 \le j \le m\right\}.$$

Then $\pi : X \to \widehat{\mathbb{C}}$ has deck group $Z/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_m\mathbb{Z}$, generated by $T_1, \ldots, T_m$ where

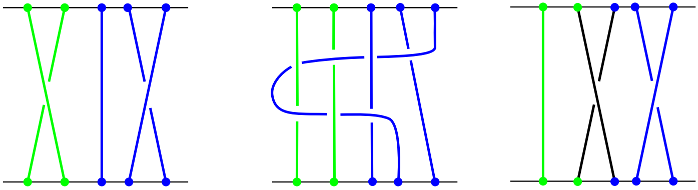
$$T_j(x, y_1, \ldots, y_m) = (x, y_1, \ldots, \zeta_{d_j} y_j, \ldots, y_m).$$

In this presentation let us stick to partitions of size $m \le 3$.

Mixed Braid Groups

For the Abelian cover X, the full braid group $\mathcal{B}_n$ does not lift to X.

The braids that lift are the ones that preserve the partition Λ. We denote by $\mathcal{B}_\Lambda$ the subgroup of $\mathcal{B}_n$ which preserves Λ.



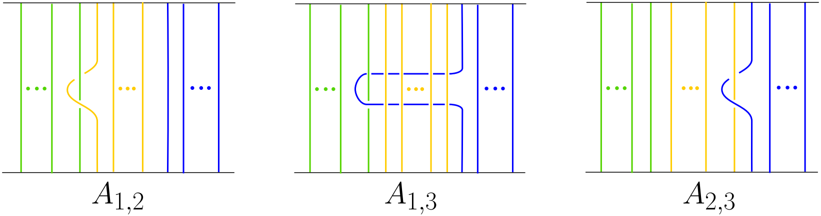
$\in \mathcal{B}_{(2,3)}$ $\in \mathcal{B}_{(2,3)}$ $\notin \mathcal{B}_{(2,3)}$

Formally the mixed braid group $\mathcal{B}_\Lambda$ is the pre-image of $S_{n_1} \times \cdots \times S_{n_m}$ under the natural homomorphism $\mathcal{B}_n \to S_n$.

Mixed Braid Groups

The mixed braid group $\mathcal{B}_\Lambda$ for $\Lambda = (n_1, n_2, n_3)$ is generated by

$$\sigma_1 \ldots \sigma_{h_1-1}, \quad \sigma_{h_1+1}, \ldots, \sigma_{h_2-1}, \quad \sigma_{h_2+1}, \ldots, \sigma_{n-1},$$
$$A_{1,2} := \sigma_{h_1}^2, \quad A_{2,3} := \sigma_{h_2}^2,$$
$$A_{1,3} := (\sigma_{h_2} \cdots \sigma_{h_1+1}) \sigma_{h_1}^2 (\sigma_{h_2} \cdots \sigma_{h_1+1})^{-1} \ .$$



Monodromy Action on Eigenspaces

Any element $b \in \mathcal{B}_\Lambda$ has a unique lift $\widetilde{b}$ to X which is identity in a neighbourhood of $\pi^{-1}(\infty)$.

Hence we have a monodromy action of $\mathcal{B}_\Lambda$ on $H_1(X)$ by

$$b \cdot \xi = \widetilde{b}_* \xi.$$

The lifts commute with the deck transformations $T_1, \ldots, T_m$.

Thus the action preserves the common eigenspaces

$$H_1(X)_q := \{\xi \in H_1(X) \mid (T_j)_* \xi = q_j \xi\}, \qquad q = (q_1, \ldots, q_m).$$

Consequently we now have a representation

$$\rho_q : \mathcal{B}_\Lambda \to \mathrm{GL}(H_1(X)_q).$$

Populating the Eigenspaces

For $q = (q_1, \ldots, q_m)$ such that $q_j \neq 1$ for all j we have

$$\dim H_1(X)_q = \begin{cases} n-1, & q_1^{n_1} \cdots q_m^{n_m} \neq 1, \\ n-2, & q_1^{n_1} \cdots q_m^{n_m} = 1. \end{cases}$$

We consider some sub-surfaces:

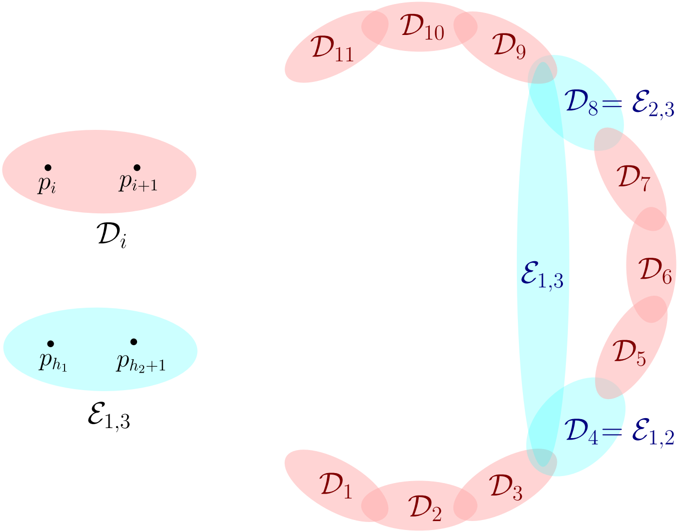
- let $\mathcal{D}_i$ be a disk containing p_i, p_{i+1}, and $\widetilde{\mathcal{D}}_i := \pi^{-1}\mathcal{D}_i$,
- let $\mathcal{E}_{1,3}$ be a disk containing p_{h_i}, p_{h_2+1}, and $\widetilde{\mathcal{E}}_{1,3} := \pi^{-1}\mathcal{E}_{1,3}$.

It turns out that

$$\dim H_1(\widetilde{\mathcal{D}}_i)_q = \dim H_1(\widetilde{\mathcal{E}}_{1,3})_q = 1$$

when $q_j \neq 1$ for all j.

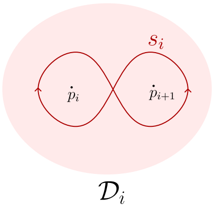
Disks

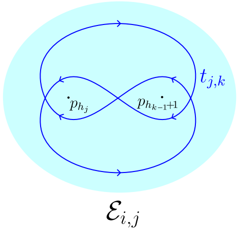


p_i
p_{i+1}
$\mathcal{D}_i$
p_{h_1}
p_{h_2+1}
$\mathcal{E}_{1,3}$
$\mathcal{D}_{11}$
$\mathcal{D}_{10}$
$\mathcal{D}_9$
$\mathcal{D}_8 = \mathcal{E}_{2,3}$
$\mathcal{D}_7$
$\mathcal{E}_{1,3}$
$\mathcal{D}_6$
$\mathcal{D}_5$
$\mathcal{D}_4 = \mathcal{E}_{1,2}$
$\mathcal{D}_1$
$\mathcal{D}_2$
$\mathcal{D}_3$

Generating curves

To find a basis of $H_1(\widetilde{\mathcal{D}}_i)_q$ and $H_1(\widetilde{\mathcal{E}}_{j,k})_q$ we look at the curves:





The lifts of the curves s_i and $t_{j,k}$ are loops.

Let G be the deck group.Then we can index the lifts as

$$s_i^\gamma, \quad t_{j,k}^\gamma, \qquad \text{for} \qquad \gamma \in G \qquad \text{so that}$$

$$\kappa_* s_i^\gamma = s_i^{\gamma+\kappa} \qquad \text{and} \qquad \kappa_* t_{j,k}^\gamma = t_{j,k}^{\gamma+\kappa} \qquad \kappa, \gamma \in G.$$

Eigen-basis

We define the eigenvectors

$$\omega_i = \text{constant} \times \sum_{\gamma \in G} q^{-\gamma} \; s_i^\gamma \qquad \in H_1(\widetilde{\mathcal{D}}_i)_q \qquad \text{and}$$

$$\phi_{j,k} = \text{constant} \times \sum_{\gamma \in G} q^{-\gamma} \; t_{j,k}^\gamma \qquad \in H_1(\widetilde{\mathcal{E}}_{j,k})_q \; .$$

For convenience of notation we also set $\omega_{h_1} = \phi_{1,2}$ and $\omega_{h_2} = \phi_{2,3}$.

Theorem

The classes $\omega_1, \ldots, \omega_{n-1}$ span $H_1(X)_q$. When $q_1^{n_1} q_2^{n_2} q_3^{n_3} \neq 1$ they are linearly independent, otherwise they satisfy one linear relation. Moreover,

$$\phi_{1,3} = \omega_{h_1} + \ldots + \omega_{h_2} \; .$$

Monodromy Action

Since σ_i has support $\mathcal{D}_i$ and $H_1(\widetilde{\mathcal{D}}_i)_q$ is one dimensional spanned by ω_i, the class ω_i has to be an eigenvector of σ_i. In fact

$$\sigma_i \cdot \omega_i = -q_j\, \omega_i \qquad \text{if} \qquad h_{j-1} < i < h_j.$$

Also σ_i is identity outside $\mathcal{D}_i$ and as a consequence σ_i is identity on a subspace complementary to $\mathrm{span}(\omega_i)$.

Similarly $\phi_{j,k}$ is an eigenvector of $A_{j,k}$ and

$$A_{j,k} \cdot \phi_{j,k} = q_j q_k\, \phi_{j,k} \qquad 1 \leq j < k \leq 3.$$

In fact, σ_i and $A_{j,k}$ act by complex reflections on $H_1(X)_q$.

There is a hermitian form on $H_1(X)$ which we denote by $\langle \cdot, \cdot \rangle$.

Let $q = (q_1, \ldots, q_m)$ such that $q_j \neq 1$ and $q_j q_k \neq 1$ for any $j, k \in \{1, \ldots, m\}$.

Theorem

The action of $\mathcal{B}_\Lambda$ on $H_1(X)_q$ is given as follows:

$$\sigma_i \cdot \omega_i = -q_j \, \omega_i, \qquad \text{and } \sigma_i \text{ is identity on } \mathrm{span}(\omega_i)^\perp \ ,$$
$$A_{j,k} \cdot \phi_{j,k} = q_j q_k \, \phi_{j,k} \qquad \text{and } A_{j,k} \text{ is identity on } \mathrm{span}(\phi_{j,k})^\perp \ .$$

where $\phi_{1,2} = \omega_{h_1}$, $\phi_{2,3} = \omega_{h_2}$ and $\phi_{1,3} = \omega_{h_1} + \ldots + \omega_{h_2}$.

For a subspace $W \subset H_1(X)_q$

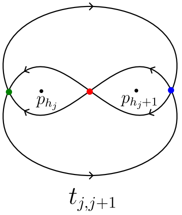
$$W^\perp = \{\eta \in H_1(X)_q \mid \langle \eta, \xi \rangle = 0 \ \forall \ \xi \in W\}.$$

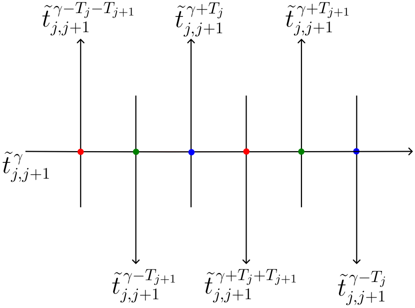
Intersection form

The previous theorem is incomplete without the knowledge of the Hermitian form.

$$
\begin{aligned}
\langle \omega_i, \omega_i \rangle &= \sqrt{-1}\left(\frac{1+q_j}{1-q_j}\right) & h_{j-1} < i < h_j, \\
\left\langle \omega_{h_j}, \omega_{h_j} \right\rangle &= \sqrt{-1}\left(\frac{1-q_j q_{j+1}}{(1-q_j)(1-q_{j+1})}\right) & 1 \leq j < m, \\
\langle \omega_{i-1}, \omega_i \rangle &= \sqrt{-1}\left(\frac{-q_j}{1-q_j}\right) & h_{j-1} < i \leq h_j
\end{aligned}
$$

Sample intersection computation





κ	$\tilde{t}_{j,j+1}^{\gamma} \cdot \tilde{t}_{j,j+1}^{\gamma+\kappa}$
T_j , T_{j+1} , $-T_j - T_{j+1}$	1
$-T_j$, $-T_{j+1}$, $T_j + T_{j+1}$	-1

Intersection form

For the partition $(2,3,2)$ the matrix $\langle \cdot, \cdot \rangle$ in term of the basis $\omega_1, \dots, \omega_{n-1}$ is:

$$\sqrt{-1}\begin{pmatrix} \frac{q_1+1}{-q_1+1} & \frac{q_1}{q_1-1} & 0 & 0 & 0 & 0 \\ \frac{1}{q_1-1} & \frac{-q_1 q_2+1}{q_1 q_2-q_1-q_2+1} & \frac{q_2}{q_2-1} & 0 & 0 & 0 \\ 0 & \frac{1}{q_2-1} & \frac{q_2+1}{-q_2+1} & \frac{q_2}{q_2-1} & 0 & 0 \\ 0 & 0 & \frac{1}{q_2-1} & \frac{q_2+1}{-q_2+1} & \frac{q_2}{q_2-1} & 0 \\ 0 & 0 & 0 & \frac{1}{q_2-1} & \frac{-q_2 q_3+1}{q_2 q_3-q_2-q_3+1} & \frac{q_3}{q_3-1} \\ 0 & 0 & 0 & 0 & \frac{1}{q_3-1} & \frac{q_3+1}{-q_3+1} \end{pmatrix}$$

Matrices of ρ_q

$$\rho_q(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & q_2 & -q_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_q(A_{1,2}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -q_1 q_2 + q_1 & q_1 q_2 & -q_1 + 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Matrices of ρ_q

$$\rho_q(A_{2,3}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q_2q_3 + q_2 & q_2q_3 & -q_2 + 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_q(A_{1,3}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -q_1q_3 + q_1 & q_3 & 0 & 0 & q_1q_3 - q_3 & -q_1 + 1 \\ -q_1q_3 + q_1 & q_3 - 1 & 1 & 0 & q_1q_3 - q_3 & -q_1 + 1 \\ -q_1q_3 + q_1 & q_3 - 1 & 0 & 1 & q_1q_3 - q_3 & -q_1 + 1 \\ -q_1q_3 + q_1 & q_3 - 1 & 0 & 0 & q_1q_3 - q_3 + 1 & -q_1 + 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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