TEACHERS TRAINING PROGRAM, KSOM

CONTENTS

1. LECTURE 1 (28 NOVEMBER 2019)

This lecture[1] will mostly deal with families and sequences of holomorphic function on a fixed domain.

1.1. **Montel's theorem.** The first result that we shall see is a theorem that broadly deals with sequences of holomorphic functions on an open subset of the complex plane. It gives a criterion for such a sequence to have a convergent subsequence.

Definition 1 (Normal family). *Let $\mathfrak{F}$ be a set of holomorphic functions on Ω an open subset of $\mathbb{C}$, it is called* normal *if any sequence $\{f_n\}_{n=0}^{\infty}$ of functions from $\mathfrak{F}$ has a subsequence $\{f_{n_k}\}_{k=0}^{\infty}$ which converges uniformly to a function $f : \Omega \to \mathbb{C}$ on compact subsets of Ω. That is for any $K \subset \Omega$ compact and $\epsilon > 0$ there is a positive integer L such that*

$$|f(z) - f_{n_k}(z)| < \epsilon \quad \text{for all } z \in K \text{ and } k > L.$$

Definition 2 (Uniform boundedness). *A set of complex valued functions $\mathfrak{F}$ on a $\Omega \subset \mathbb{C}$ is said to be* uniformly bounded *on $E \subset \Omega$ if there is $M > 0$ such that*

$$|f(z)| < M \quad \text{for all } f \in \mathfrak{F} \text{ and } z \in E.$$

The number M is called the uniform bound for $\mathfrak{F}$ on E.

Theorem 3 (Montel's theorem). *A set $\mathfrak{F}$ of holomorphic functions on $\Omega \subset \mathbb{C}$ open, is normal, if it is uniformly bounded on any compact subset of Ω.*

The proof has three parts, which I break up into two lemmas and an exercise. The proof of the first lemma requires complex analysis in the form of Cauchy's integral formula. Where as the next two parts are proven using just topological arguments and hold in much more generality than our situation.

Lemma 4. *If F is uniformly bounded on a compact subset $K \subset \Omega$, then $\mathfrak{F}$ is equicontinuous on K.*

Proof. Recall that $\mathfrak{F}$ is equicontinuous on K if for any $\epsilon > 0$ there is a $\delta > 0$ such that for any $z, w \in K$ and any $f \in \mathfrak{F}$ if $|z - w| < \delta$ then $|f(z) - f(w)| < \epsilon$.

Let $r > 0$ be such that $|z_1 - z_2| > 3r$ for any $z_1 \in K$ and $z_2 \in \mathbb{C} - \Omega$. Let M be the uniform bound for $\mathfrak{F}$ on K. Let $z, w \in K$ such that $|z - w| < r$ and γ be the circle $\{\zeta \in \mathbb{C} \mid |\zeta - w| = 2r\}$. Then $\gamma \subset \Omega$ and for any $\zeta \in \gamma$

$$|\zeta - z| \geq |\zeta - w| - |w - z| > r.$$

Now using the Cauchy integral formula

$$f(z) - f(w) = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) d\zeta.$$

We have the following bound

$$\left| \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right| = \left| \frac{w - z}{(\zeta - z)(\zeta - w)} \right| \leq \frac{|z - w|}{r^2}.$$

Hence we get

$$|f(z) - f(w)| \leq \frac{1}{2\pi} \frac{2\pi r M}{r^2} |z - w|.$$

---

[1]Prerequisites: Cauchy Integral formula, uniform convergence, Schwarz lemma and some basic notions from point set topology.

Hence for $|z - w| < r$ we have $|f(z) - f(w)| \leq \frac{M}{r}|z - w|$ on K.

For $\epsilon > 0$ choose $\delta < r$ such that $\frac{M}{r}\delta < \epsilon$, then for $z, w \in K$ and $|z - w| < \delta$ we clearly have $|f(z) - f(w)| < \epsilon$. $\qquad\square$

This result is very special for holomorphic functions. On the other hand consider the sequence of functions $\{x \mapsto x^n\}$ on the interval $[0, 1]$. These are of course uniformly bounded but are not equicontinuous (why?).

Now we show that uniform boundedness and equicontinuity on a compact set together gives a uniformly convergent subsequence. This result does not require the functions to be holomorphic or even that the co-domain is $\mathbb{C}$. It only uses the fact that the domain is a compact metric space.

Lemma 5 (Arzela-Ascoli theorem). *If $\mathfrak{F}$ is uniformly bounded and equicontinuous on a compact subset $K \subset \Omega$, then any sequence of functions from $\mathfrak{F}$ has a subsequence which is uniformly convergent on K.*

Proof. Let $\{f_n\}$ be a sequence of functions from $\mathfrak{F}$. Choose a countable dense subset of K,

$$A = \{a_1, a_2, a_3, \ldots\}.$$

Since $\{f_n(a_1)\}$ is bounded, it has a convergent subsequence, $\{f_{n,1}(a_1)\}$. Again since $\{f_{n,1}(a_2)\}$ is bounded it has a convergent subsequence $\{f_{n,2}(a_2)\}$. Continuing in this manner we can assume $\{f_{n,k}(a_i)\}$ are all convergent for $i \leq k$ and choose a subsequence $f_{n,k+1}$ of the sequence $f_{n,k}$ such that $\{f_{n,k+1}(a_{k+1})\}$ converges. It is then easy to see that the diagonal subsequence $\{f_{n,n}\}$ converges on all of A.

We shall now show that $\{g_n = f_{n,n}\}$ is a uniformly Cauchy sequence on K, hence it converges uniformly on K.

Let $\epsilon > 0$, by equicontinuity of $\mathfrak{F}$ we can find $\delta > 0$ such that $z, w \in K$ and $|z - w| < \delta$ implies $|f(z) - f(w)| < \epsilon/3$. Since K is compact, there is a positive integer N such that

$$\bigcup_{i=1}^{N} B_\delta(a_i) \supset K.$$

Let M be a large enough positive integer so that $|g_n(a_i) - g_m(a_i)| < \epsilon/3$ for $n, m > M$ and $i = 1, \ldots, N$. Now for any $z \in K$, $|z - a_i| < \delta$ for some $i \in \{1, \ldots, N\}$ so

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(a_i)| + |g_n(a_i) - g_m(a_i)| + |g_m(a_i) - g_m(z)| < \epsilon.$$

This implies $g_n(z)$ is Cauchy and hence converges for each $z \in K$, let us call the limit $g(z)$. This shows g_n converges to g point-wise. To show uniform convergence let $\epsilon > 0$, choose a positive integer M large enough so that $|g_n(z) - g_m(z)| < \epsilon/2$ for $n, m \geq M$ and any $z \in K$. For any $z \in K$ there is an $N_z > M$ such that $|g_{N_z}(z) - g(z)| < \epsilon/2$,. Thus for any $z \in K$ and $n > M$,

$$|g_n(z) - g(z)| \leq |g_n(z) - g_{N_z}(z)| + |g(z) - g_{N_z}(z)| < \epsilon.$$

$\qquad\square$

Now we can complete the proof of Montel's theorem.

Exercise 1. *Show that $K_n = \{z \in \Omega : |z| \leq n$ and $|z - w| \geq \frac{1}{n}$ for all $w \in \mathbb{C} - \Omega\}$ is a compact subset of Ω, $K_n \subset K_{n+1}$ and*

$$\bigcup_{n=1}^{\infty} K_n = \Omega.$$

Exercise 2. *Complete the proof of Theorem 3 using the previous exercise.*

1.2. **Uniform limits of Holomorphic functions.** We have the following important results about uniformly convergent sequence of holomorphic functions. The first is left as an exercise.

Exercise 3. *Let $\{f_n\}$ be a sequence of holomorohic functions on Ω such that it converges uniformly on any compact subset of Ω to a function $f : \Omega \to \mathbb{C}$. Show that f is also holomorphic on Ω and f_n' converges to f' uniformly on compact subsets of Ω.*

Theorem 6 (Hurwitz). *Let Ω be a connected open set in $\mathbb{C}$ and $\{f_n\}$ a sequence of holomorphic functions on Ω which converges to a holomorphic function $f : \Omega \to \mathbb{C}$ uniformly on compact sets. Suppose all the f_n are injective, then f is either injective or constant.*

Proof. Assume f is neither injective nor constant. Pick $z_1, z_2 \in \Omega$ such that $f(z_1) = f(z_2)$. Let $g(z) = f(z) - f(z_1)$ and $g_n(z) = f_n(z) - f_n(z_1)$, then $\{g_n\}$ converges to g uniformly on compact subsets of Ω.

Since g is non-constant hence z_2 is an isolated zero of g. Choose r small enough so that $\overline{B}_r(z_2) \subset \Omega$ and g does not vanish on $\overline{B}_r(z_2)$ except at z_2, thus

$$\frac{1}{2\pi i} \int_{\partial \overline{B}_r(z_2)} \frac{g'(z)dz}{g(z)} = 1.$$

On the other hand g_n does not vanish on $\overline{B}_r(z_2)$ for any n since f_n is injective. Hence

$$\frac{1}{2\pi i} \int_{\partial \overline{B}_r(z_2)} \frac{g_n'(z)dz}{g_n(z)} = 0.$$

However by uniform convergence the later integrals should converge to the former, which is a contradiction. $\qquad\square$

Exercise 4. *Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function and $f(a) = 0$ show that there is $r > 0$ such that $\overline{B}_r(a) \subset \Omega$ and a is the only zero of f in this closed ball. Show that if n is the order of vanishing of f at a then*

$$\frac{1}{2\pi i} \int_{\partial \overline{B}_r(a)} \frac{f'(z)dz}{f(z)} = n.$$

1.3. **Biholomorphisms of the unit disc and the upper half plane.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane. We want to determine all the biholomorphisms $f : \mathbb{D} \to \mathbb{D}$. This can be done using the Scharz lemma. First let us recall what a biholomorphism is.

Definition 7. *Let U and V be open subsets of $\mathbb{C}$. A holomorphic function $f : U \to V$ is called a biholomorphism if it has a holomorphic inverse $g : V \to U$. If such a biholomorphism exists, U and V are said to be biholomorphic.*

For complex analysis two open sets that are biholomorphic are identical in every sense. In particular a biholomorphism is also a homeomorphism so the open sets are topologically also the same.

Exercise 5. *Give a biholomorphism $f : \mathbb{D} \to \mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ is the upper half plane. What does f do to the boundary of $\mathbb{D}$?*

Let $\alpha \in \mathbb{D}$, and consider the meromorphic function
$$\phi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$
This has a pole outside the closed unit disc and is holomorphic on $\mathbb{D}$. Note that
$$\phi_\alpha(\alpha) = 0 \quad \text{and} \quad \phi(0) = \alpha.$$

We shall show that ϕ_α is a biholomorphism of the unit disk. If $|z| = 1$, then
$$|\phi_\alpha(z)| = \left| \frac{\alpha - z}{z(\overline{z} - \overline{\alpha})} \right| = 1.$$
Hence by maximum modulus principle ϕ_α maps $\mathbb{D}$ to $\mathbb{D}$. Finally a simple calculation shows that $\phi_\alpha \circ \phi_\alpha(z) = z$, that is ϕ_α is its own inverse.

We shall show that these are all the biholomorphisms of $\mathbb{D}$ upto rotations. We shall need the Scharz lemma for that so let us recall the statement.

Theorem 8 (Schwarz lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic such that $f(0) = 0$, then*

(1) $|f(z)| \leq |z|$ *for all $z \in \mathbb{D}$ and if equality holds for some $z \neq 0$ then f is a rotation, that is $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$,*
(2) $|f'(0)| \leq 1$ *and if equality holds then f is a rotation.*

The proof is a straightforward application of the maximum modulus principle and can be found for instance in *Complex Analysis*, Stein and Shakarachi, Chapter 8, Section 2. Using this we can easily determine all the biholomorphisms of $\mathbb{D}$.

Proposition 9. *Let $f : \mathbb{D} \to \mathbb{D}$ be a bilolomorphism, then $f = e^{i\theta} \phi_\alpha$ for some $\alpha \in \mathbb{D}$ and $\theta \in \mathbb{R}$.*

Proof. This is obtained by applying the Schwarz lemma twice. Let $\alpha = f^{-1}(0)$, then $g(z) = f \circ \phi_\alpha(z)$ fixes 0. By Schwarz lemma $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. By taking $w = g(z)$ we have $|g^{-1}(w)| \leq |w|$, which shows $|z| \leq |g(z)|$ so we must have $|z| = |g(z)|$ for all $z \in \mathbb{D}$ and $g(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$ which completes the proof of the proposition. $\square$

Exercise 6. *Show that biholomorphisms $f : \mathbb{H} \to \mathbb{H}$ are of the form*
$$f(z) = \frac{az + b}{cz + d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

2. Lecture 2 (29 November 2019)

In this lecture[2] we shall determine which open subsets of $\mathbb{C}$ are biholomorphic to the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. But first we start with some definitions.

2.1. Homotopy version of Cauchy's theorem.

We start this section with a very important theorem in complex analysis which allows us to define anti-derivatives of holomorphic functions on certain open subsets of the complex plane which are called simply connected. As a consequence we shall define an inverse to the exponential function called the logarithm on simply connected open sets.

Definition 10 (Curve). *Let $\Omega \subset \mathbb{C}$ be open. A curve γ in Ω is a continuous function $\gamma : [0,1] \to \Omega$. A closed curve has the same starting and ending points that is $\gamma(0) = \gamma(1)$.*

We shall only deal with piecewise smooth curves so that we can do complex line integral on them.

Definition 11 (Homotopy). *Two curves γ_1 and γ_2 in Ω are said to be homotopic if $\gamma_1(0) = \gamma_2(0) = a$ and $\gamma_1(1) = \gamma_2(1) = b$ and there is a continuous function $H : [0,1] \times [0,1] \to \Omega$ such that $H(s,0) = a$, $H(s,1) = b$, $H(0,t) = \gamma_1(t)$ and $H(1,t) = \gamma_2(t)$. Such a function H is called a homotopy between γ_1 and γ_2. A closed curve γ in Ω is called null homotopic if it is homotopic to the constant curve $c(t) = \gamma(0)$.*

Theorem 12 (Cauchy's theorem). *If γ_1 and γ_2 are two piecewise smooth curves in Ω that are homotopic and $f : \Omega \to \mathbb{C}$ is holomorphic then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

This is an extremely powerful theorem and a detailed proof can be found in Stein and Shakarachi, Chapter 3, Section 5. As a consequence of this theorem we find that if γ is a null-homotopic closed curve in Ω then

$$\int_{\gamma} f(z)dz = 0.$$

This motivates the following definition.

Definition 13. *An open set $\Omega \subset \mathbb{C}$ is called simply connected if it is connected and every closed curve is null homotopic.*

Intuitively this just means that Ω does not have any holes because the interior of any closed curve can be filled in. The following theorem gives an equivalent condition which is often easier to check.

Theorem 14. *A connected open set $U \subset \mathbb{C}$ is simply connected if and only if for any closed curve on γ in U and any holomorphic function $f : U \to \mathbb{C}$*

$$\int_{\gamma} f(z)dz = 0.$$

Proof. Give reference. □

Exercise 7. *Show that the annulus $A = \{z \in \mathbb{C} \mid 0 < r < |z| < R\}$ is not simply connected.*

---

[2]Prerequisites: Homotopy version of Cauchy's theorem, Simple connectivity

Exercise 8. *Show that any convex open set is simply connected and infer that $\mathbb{D}$ is simply connected.*

Let us now define a logarithm of a non-vanishing holomorphic function on a simply connected open set.

Let $\Omega \subset \mathbb{C}$ be open and simply connected and $f : \Omega \to \mathbb{C}$ be a holomorphic function such that $f(z) \neq 0$ for any $z \in \Omega$. Then there is a function $g : \Omega \to \mathbb{C}$ holomorphic such that

$$\exp(g(z)) = f(z).$$

Such a function is called a logarithm of f.

This follows easily from the Cauchy's theorem because it allows us to define an anti-derivative of a holomorphic function. Notice that if g does exist then differentiating $f(z) = \exp(g(z))$ gives us

$$f'(z) = \exp(g(z))g'(z) \Rightarrow g'(z) = \frac{f'(z)}{f(z)}.$$

Since f does not vanish on Ω the function $f'(z)/f(z)$ is holomorphic on Ω. Fix $z_0 \in \Omega$ and choose a complex number c_0 such that $\exp(c_0) = f(z_0)$. For any $z \in \Omega$ we can always find a piecewise smooth curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = z_0$ and $\gamma(1) = z$ since Ω is path connected. We define

$$g(z) = \int_{\gamma} \frac{f'(z)}{f(z)} dz.$$

Exercise 9. *Using Cauchy's theorem show that $g(z)$ does not depend on the choice of γ. Moreover g is holomorhic on Ω and $g'(z) = f'(z)/f(z)$.*

Now consider the function $F(z) = f(z)\exp(-g(z))$, then F is clearly holomorphic on Ω and

$$F'(z) = f'(z)\exp(-g(z)) - f(z)f'(z)\exp(-g(z))/f(z) = 0$$

Since Ω is connected this implies that F is constant, moreover $F(z_0) = f(z_0)/\exp(c_0) = 1$. Thus $f(z) = \exp(g(z))$ just as we wanted.

Note that this is only one branch of the logarithm of f, in fact $h(z) = g(z) + 2\pi i$ would also work fine. So g is not unique (but two such functions will only differ by an integer multiple of $2\pi i$).

Exercise 10. *Let Ω be a simply connected open subset of $\mathbb{C}$ and f a non-vanishing holomorphic function on Ω. Let n be a positive integer then show that there is a holomorhic function g on Ω such that $g^n(z) = f(z)$ for all $z \in \Omega$.*

Exercise 11. *Let Ω be a simply connected open set in $\mathbb{C}$ and f a hon-vanishing holomorphic function on Ω. Let g be a logarithm of f. Show that if f is injective then so is g, and $g(z)+2\pi i \notin g(\Omega)$ for any $z \in \Omega$. On the other hand give an example where g is injective but f is not.*

2.2. Riemann Mapping theorem.
The theorem of this sub-section although first formulated by Riemann was first proven by Koebe according to the book of Stein and Shakarachi.

Exercise 12. *Show that if $\Omega \subset \mathbb{C}$ is biholomorphic to $\mathbb{D}$ then Ω is simply-commected and $\Omega \neq \mathbb{C}$.*

Theorem 15 (Riemann Mapping)**.** *Any simply-connected, proper open subset of $\mathbb{C}$ is biholomorphic to $\mathbb{D}$.*

Let Ω be a proper simply connected open subset of $\mathbb{C}$. Fix a point $z_0 \in \Omega$. We shall look at the set of injective, holomorphic functions

$$\mathfrak{F} = \{f : \Omega \to \mathbb{D} \mid f(z_0) = 0\}.$$

The proof can be broken up into 3 parts. Firstly we shall show that $\mathfrak{F}$ is non-empty. Secondly we shall show that there is a function f in $\mathfrak{F}$ whose derivative at z_0 has the maximum modulus among the all the functions in $\mathfrak{F}$. Finally we shall show that the function f is surjective, completing the proof.

Step 1. Since Ω is proper choose $a \in \mathbb{C} - \Omega$. The $z \mapsto z - a$ is non-zero on Ω and since Ω is simply connected we can define a logarithm for $z - a$, that is there is a holomorphic function

$$h : \Omega \to \mathbb{C}$$

such that $\exp(h(z)) = z - a$. Clearly this h is then injective.

Since h is non-constant and holomorphic it is an open mapping. Hence, we know that $h(\Omega)$ contains a small closed ball of some radius r around $h(z_0)$. Let $w_0 = h(z_0) + 2\pi i$, since $\overline{B}_r(h(z_0)) \subset h(\Omega)$ we must have

$$\overline{B}_r(w_0) \cap h(\Omega) = \emptyset.$$

This is because if $h(\zeta_1) \in \overline{B}_r(w_0)$, then $h(\zeta_1) = h(\zeta_2) + 2\pi i$ where $h(\zeta_2) \in \overline{B}_r(h(z_0))$, then by applying exp we see that $\zeta_1 = \zeta_2$ which is a contradiction.

We thus have $|h(z) - w_0| > r$ for any $z \in \Omega$. The function $f : \Omega \to \mathbb{C}$ given by

$$f(z) = \frac{1}{h(z) - w_0}$$

is thus bounded, $|f(z)| < 1/r$ for all $z \in \Omega$. By translating and scaling we get a function in $\mathfrak{F}$

$$g(z) = \frac{f(z) - f(z_0)}{1/r + |f(z_0)|}.$$

The function g is clearly injective, $|g(z)| < 1$ for any $z \in \Omega$ and $g(z_0) = 0$. Proving that $\mathfrak{F}$ is non-empty. $\qquad\square$

Step 2. The family $\mathfrak{F}$ is uniformly bounded hence normal. Let $r > 0$ be small enough so that $\overline{B}_r(z_0) \subset \Omega$ and let $\gamma = \partial\overline{B}_r(z_0)$ be the boundary circle. Let $f \in \mathfrak{F}$ then by the Cauchy's integral formula for derivatives

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta \right| \leq \frac{1}{r}.$$

Let $B = \sup\{|f'(z_0)| : f \in \mathfrak{F}\}$ which is clearly finite. Moreover, $B > 0$ since there is atleast one injective function in $\mathfrak{F}$, (a holomorphic function is not injective if it's derivative vanishes at some point). There is a sequence of functions $\{f_n\}$ from $\mathfrak{F}$ such that $f_n(z_0) \to B$ as $n \to \infty$. By Montel's theorem this sequence has a subsequence $g_k = f_{n_k}$ which converges to a function $f : \Omega \to \mathbb{C}$ uniformly on compact subsets.

The function f is holomorphic being locally uniform limit of holomorphic functions. Since $|f'(z_0)| = B > 0$ f is non-constant, hence injective being a limit of injective functions. Since $g_k(z) \to f(z)$ for all z and $|g_k(z)| < 1$ we must have $|f(z)| \leq 1$. Since f is an open mapping we must have

$$f(\Omega) \subset \mathbb{D}.$$

$\qquad\square$

Step 3. We shall show that if f is not surjective then there is a function $g \in \mathfrak{F}$ such that $|g'(z_0)| > |f'(z_0)|$ thus arriving at a contradiction. Assume that the $f : \Omega \to \mathbb{D}$ obtained in Step 2 is not surjective and choose $a \in \mathbb{D} - f(\Omega)$. We have a biholomorphism of the unit disc ϕ_a that takes a to 0

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}.$$

Notice that $\phi_a \circ f : \Omega \to \mathbb{D}$ does not vanish. Hence we have a holomorphic function $h : \Omega \to \mathbb{D}$ such that $h^2(z) = \phi_a \circ f(z)$, in particular we can take

$$h(z) = \exp\left(\frac{\log(\phi_a(f(z)))}{2}\right).$$

Let $b = h(z_0)$ then clearly $b^2 = \phi_a(0) = a$. Let $g = \phi_b \circ h$. Then a calculation using chain rule shows that

$$g'(z_0) = \frac{1 + |b|^2}{2b} f'(z_0).$$

Thus $|g'(z_0)| > |f'(z_0)|$. $\qquad\square$

This completes the proof of the Riemann mapping theorem.

Exercise 13. *Let Ω be a proper, simply connected, open subset of $\mathbb{C}$ and $z_0 \in \Omega$. Show that there is a unique biholomorphism $f : \Omega \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0)$ is a positive real number*

Exercise 14. *Find biholomorphisms from $\mathbb{D}$ to the following open subsets of $\mathbb{C}$:*

(1) $B = \{z \in \mathbb{C} \mid -\pi/2 < \operatorname{Im} z < \pi/2\}$,
(2) $Q = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0, \operatorname{Re} z > 0\}$,
(3) $S = \mathbb{C} - [-\infty, 0]$,
(4) $T = \{z \in \mathbb{C} \mid 0 \leq \arg z \leq A < 2\pi\}$,
(*) $A = \{z \in \mathbb{C} : |z| < 1 \text{ and } |z - 1| < 1\}$.

The previous theorem is very special for proper simply connected open sets and fails for other types of open sets. For example let $A(r) = \{z \in \mathbb{C} \mid 1 < |z| < r\}$ be an annulus then clearly $A(r)$ is homeomorphic to $A(r')$ but if $r \neq r'$ then they are not biholomorphic. See Theorem 14.22 of Rudin, *Real and Complex Analysis*.

There is the following extension of the Riemann mapping theorem.

Theorem 16. *Let U be a simply connected open set in $\mathbb{C}$ such that the boundary ∂U is a simple closed curve then any biholomorphism $\phi : U \to \mathbb{D}$ extends to a homeomorphism $\widetilde{\phi} : \overline{U} \to \overline{\mathbb{D}}$.*

A proof of this can found in Rudin, *Real and Complex Analysis*, Chapter 14, Theorem 14.19.

3. Lecture 3 (30 November 2019)

In this lecture we shall see a generalisation of analytic continuation along curves and analytic coverings.

3.1. **Analytic continuation.** An analytic function element at $z \in \mathbb{C}$ is a pair (f, D) where D is a disc containing z and $f : D \to \mathbb{C}$ is holomorphic. If (f_1, D_1) and (f_2, D_2) are analytic function elements at z then we say $(f_1, D_1) \sim (f_2, D_2)$ if

$$z \in D_1 \cap D_2 \text{ and } f_1 = f_2 \text{ on } D_1 \cap D_2.$$

Exercise 15. *Show that $\sim$ is an equivalence relation.*

Definition 17. *Let $\Omega \subset \mathbb{C}$ be a connected open set and γ be a curve in Ω. Let (f, D) be a function element at $a = \gamma(1)$. Then a function element (g, D') at $\gamma(1)$ is called an analytic continuation of (f, D) along γ if there is a partition of $[0, 1]$, $0 = s_0 < s_1 < \ldots < s_n = 1$ and a collection of function elements*

$$\mathscr{C} = \{(f_1, D_1) \ldots (f_n, D_n)\}$$

such that $\gamma([s_{i-1}, s_i]) \subset D_i$, $(f_{i-1}, D_{i-1}) \sim (f_i, D_i)$ as function elements at $\gamma(s_{i-1})$, $(f_1, D_1) = (f, D)$ and $(f_n, D_n) = (g, D')$. Such a collection $\mathscr{C}$ will be called a holomorphic chain along γ starting at (f, D) and ending at (g, D').

Exercise 16. *Let $z_0 \in \mathbb{C}$ and (f, D) be a function element at z_0. Let γ be a curve in $\mathbb{C}$ starting at z_0. Suppose there is $\Omega \subset \mathbb{C}$ open, containing γ and a holomorphic function $\widetilde{f} : \Omega \to \mathbb{C}$ which agrees with f on D, then show that (f, D) can be analytically continued along γ.*

The next theorem says that analytic continuations along curves are unique if they exist.

Theorem 18. *Let γ be a curve in $\mathbb{C}$. Let (f, D) and (f', D') be function elements at $\gamma(0)$. Let (g, B) be an analytic continuation of (f, D) along γ and (g', B') an analytic continuation of (f', D') along γ. If $(f, D) \sim (f', D')$ at $\gamma(0)$ then $(g, B) \sim (g', B')$ at $\gamma(1)$.*

Proof. Let $\mathscr{C} = \{(f_1, D_1), \ldots, (f_n, D_n)\}$ be a holomorphic chains along γ, starting at (f, D) and ending at (g, B). Let $\mathscr{B} = \{(f'_1, D'_1), \ldots, (f'_m, D'_m)$ be another chain starting at (f', D') and ending at (g', B').

Without loss of generality we may assume $n = m$ and both chains correspond to the same partition $0 = s_0 < \ldots < sn = 1$. (Otherwise we may just refine the partitions by taking their union and repeat the function elements as required).

Assume that $(g, B) \nsim (g', B')$. Let i be the smallest integer such that $(f_i, D_i) \nsim (f'_i, D'_i)$. Clearly $i > 1$ and $(f_{i-1}, D_{i-1}) \sim (f'_{i-1}, D'_{i-1})$ at $\gamma(s_{i-1})$. We also have

$$\gamma(s_{i-1}) \in D_{i-1} \cap D_i \cap D'_{i-1} \cap D'_i.$$

Further $f_i = f_{i-1}$ on $D_{i-1} \cap D_i$ and $f'_i = f_{i-1}$ on $D'_i \cap D'_{i-1}$. Hence, $f_i = f'_i$ on $D_i \cap D'_i$ because of the connectedness of the intersection. Hence we have a contradiction. $\square$

Exercise 17. *Show that if γ is a closed curve and (g, D') is an analytic continuation of a function element (f, D) at $\gamma(0)$, then it may not be true that $(f, D) \sim (g, D')$. (Hint. Use logarithm or square root.)*

Theorem 19. *Let $\Omega \subset \mathbb{C}$ be a connected open set. Let $z_0 \in Omega$ and (f, D) be a function element at z_0 such that it can be analytically continued along any curve in Ω. If γ_0 and γ_1 are curves in Ω such that $\gamma_0(0) = \gamma_1(0) = z_0$ and $\gamma_0(1) = \gamma_1(1) = z_1$ and are homotopic then analytic continuations at z_1 along γ_0 and γ_1 are equivalent.*

Proof. Consider a homotopy $H : [0,1] \times [0,1] \to \Omega$ between γ_i. Fix $s \in [0,1]$ and look at the curve $\gamma_s(t) = H(t,s)$. This is a curve which starts at z_0 and ends at z_1. Let (g_s, D_s) be the function element at z_1 obtained by analytic continuation of (f,D) along γ_s. We shall show that there is a $\delta > 0$ such that $(g_s, D_s) \sim (g_{s'}, D_{s'})$ for $|s - s'| < \delta$.

Let $\mathscr{C} = \{(f_1, D_1), \ldots, (f_n, D_n)\}$ be a holomorphic chain along γ_s starting at (f,D) and ending at (g_s, D_s), and let $0 = t_0 < \ldots < t_n = 1$ be the corresponding partition. Since $\gamma_s([t_{i-1}, t_i])$ is a compact subset of D_i,

$$\epsilon_i = \inf\{|z - w| : z \notin D_i, w \in \gamma_s([t_{i-1}, t_i])\} > 0$$

Choose $\epsilon > 0$ such that $\epsilon < \min\{\epsilon_1, \ldots, \epsilon_n\}$, then by uniform continuity of H on the compact set $[0,1] \times [0,1]$ we have $\delta > 0$ such that $|\gamma_s(t) - \gamma_{s'}(t)| < \epsilon$ for $|s - s'| < \delta$. Hence $\gamma_{s'}([t_{i-1}, t_i]) \subset D_i$. It follows that $\mathscr{C}$ is a chain along $\gamma_{s'}$, and (g_s, D_s) is an analytic continuation of (f,D) along $\gamma_{s'}$ too. Thus by uniqueness of analytic continuation $(g_s, D_s) \sim (g_{s'}, D_{s'})$.

Let $U = \{s \in [0,1] \mid (g_s, D_s) \sim (g_0, D_0)\}$, then from the above result we conclude that U and $[0,1] - U$ are both open subsets of $[0,1]$ hence one of them must be empty. Since U is non-empty and in particular $0 \in U$ we have $U = [0,1]$ which completes the proof of the theorem. $\qquad\square$

Corollary 20 (Monodromy theorem). *If $\Omega \subset \mathbb{C}$ is open and simply connected and if (f,D) is a function element at $z_0 \in \Omega$ which can be analytically continued along any curve in Ω, then there is a holomorphic function $\widetilde{f} : \Omega \to \mathbb{C}$ such that $\widetilde{f} = f$ on D.*

Proof. For any $z \in \Omega$ we have a path $\gamma : [0,1] \to \Omega$ from z_0 to z. Let (g, B) be an analytic continuation of (f,D) along γ. If γ' is another path in Ω from z_0 to z then it is homotopic to γ because Ω is simply connected. Let (g', B') be an analytic continuation of (f,D) along γ'. Then by the above theorem we have $(g, D) \sim (g', D')$ and in particular $g(z) = g'(z)$.

Hence the function $\widetilde{f}(z) = g(z)$ is well defined on Ω. To see that it is holomorphic at z note that $\widetilde{f} = g$ on $B \cap \Omega$. $\qquad\square$

3.2. Analytic coverings.

These are an important class of holomorphic functions which are local homeomorphisms with a lifting property. The primary example as we shall see is the exponential function.

Definition 21. *Let U, V be open subsets of $\mathbb{C}$. A function $f : U \to V$ is called a holomorphic covering map if f is holomorphic and each $z \in V$ has an open neighborhood W such that*

$$f^{-1}(W) = \bigsqcup_{i \in I} W_i$$

such that $f|_{W_i}$ is a biholomorphism onto U for each i, ie. there is a holomorphic function $g_i : W \to W_i$ such that $f \circ g_i(z) = z$. Such W is called a uniformly covered neighborhood.

The exponential function $\exp : \mathbb{C} \to \mathbb{C} - \{0\}$ is the prime example for a holomorphic covering map. To see that it is a covering let $z_0 = re^{i\theta}$ be any point in $\mathbb{C} - \{0\}$. Consider

$$U = \mathbb{C} - \{\rho e^{-i\theta} \mid \rho \in [0, \infty)\}.$$

This is an open set containing z_0. Then

$$\exp^{-1}(U) = \bigsqcup_{n \in \mathbb{Z}} V_n \quad \text{where} \quad V_n = \{z \in \mathbb{C} \mid 2\pi n - \theta < \operatorname{Im} z < 2\pi(n+1) - \theta\}$$

and $\exp : V_n \to U$ is a holomorphic bijection, which implies that it is a biholomorphism, showing that U is a uniformly covered neighbourhood of z_0.

Exercise 18. *Show that $p_n : \mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$ given by $p_n(z) = z^n$ is a holomorphic covering map.*

Exercise 19. *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial function and let*
$$S = \{z \in \mathbb{C} \mid P'(z) = 0\}.$$
Let $V = \mathbb{C} - P(S)$ and $U = \mathbb{C} - P^{-1}(P(S))$ then show that $P : U \to V$ is a covering.

Theorem 22. *Let U be a simply connected open set and $h : W \to V$ a holomorphic covering, then for any holomorphic function $f : U \to V$ there is $g : U \to W$ such that $f = h \circ g$.*

Such a g is called a lift of f through h. The previous theorem says that for a holomorphic covering such lifts always exist. The function g is not unique though.

Proof. Let $u_0 \in U$ and pick any point $w_0 \in W$ such that $h(w_0) = f(u_0)$. Let A be a uniformly covered neighborhood of v_0 and let A' be the component of $f^{-1}(A)$ that contains w_0. There is a holomoprhic function $k : A \to A'$ such that $h(k(z)) = z$ for all $z \in A$. Let $B_0 \subset f^{-1}(A)$ be an open disc around u_0. Define $g_0 : B_0 \to W$ by $g_0 = k \circ f$. Clearly $h \circ g_0 = f$ on B_0.

We shall show that the function element (g_0, B_0) at u_0 can be analytically continued along any curve γ in U starting at u_0. For any $s \in [0,1]$ let A_s be a uniformly covered neighborhood of $f(\gamma(s))$, and by continuity of f there is a disc $D_s \subset f^{-1}(A_s)$ centered at γ_s. Let $D_0 = B_0$. Choose $\delta_s > 0$ be small enough so that for $s \in (0,1)$, $\gamma((s - \delta_s, s + \delta_s)) \subset D_s$, if $s = 0$ let $\gamma([0, \delta_0)) \subset D_0$ and $\gamma((1 - \delta_1, 1]) \subset D_1$.

By compactness of $[0,1]$ there is a finite set $0 = s_0 < s_1 < \ldots < s_{n-1} < s_n = 1$ such that $[0, \delta_0), (s_1 - \delta_{s_1}, s_1 + \delta_{s_1}), \ldots, (1 - \delta_1, 1]$ covers $[0,1]$. Moreover by throwing out a few intervals and shortening some if necessary we may even assume that $s_{i-1} < s_i - \delta_{s_i} < s_{i-1} + \delta_{s_{i-1}} < s_i$.

Let $t_i \in (s_i - \delta_{s_i}, s_{i-1} + \delta_{s_{i-1}})$, $t_0 = 0$ and $t_{n+1} = 1$, then $\gamma([t_{i-1}, t_i]) \subset D_{i-1} = D_{s_{i-1}}$ and $f(D_i) \subset A_i = A_{s_i}$.

Note that $\gamma(t_1) \in D_0 \cap D_1$ and let A'_i be the component of $h^{-1}(A_1)$ which contains $k(\gamma(t_1))$ and let $k_1 = h^{-1} : A_1 \to A'_1$ then clearly $g_1 = f \circ k_1 : D_1 \to W$ agrees with g_0 on $D_0 \cap D_1$ and we get a function element (g_1, D_1). Continuing in this manner we get function elements (g_i, D_i) giving an analytic continuation of (g_0, D_0) along γ.

By the monodromy theorem we have thus a holomorphic function $g : U \to W$. Moreover $f = h \circ g$ on D_0 hence on the entire U since U is connected. $\qquad \square$

Exercise 20. *Show that if $h : U \to V$ is a holomorphic covering where U is connected and V is simply connected then h is a biholomorphism.*

Exercise 21. *Define logarithm of a non-vanishing holomorphic function on a simply connected open set using the fact that the exponential map is a holomorphic covering.*

4. LECTURE 4 (1 DECEMBER 2019)

In this lecture we shall prove the Little Picard theorem. We shall give a proof of this theorem using a covering map to $\mathbb{C} - \{0, 1\}$ from the upper half plane $\mathbb{H}$. The covering map is obtained using the Riemann mapping theorem and Schwarz refelction principle.

4.1. **Reflection principle.** This gives a way of extending an analytic functions through reflection along lines and circles.

Theorem 23. *Let V be any domain which is symetric about the line $L = \{z \in \mathbb{C} \mid \operatorname{Im} z = 0\} = \mathbb{R}$ and let $A = V \cap L$. Let $V^+ = V \cap \mathbb{H}$ and $f : V^+ \cup A \to \mathbb{C}$ be a continuous function which is holomorphic on V^+, and such that $f(A) \subset \mathbb{R}$ then there is an extension $\widetilde{f} : V \to \mathbb{C}$ holomorphic which agrees with f on V^+.*

For a proof we refer to Stein and Shakarachi, *Complex Analysis*, Theorem 5.6, Chapter 2. We can easily define $\widetilde{f}$ using the reflection along L given by $\overline{z}$, namely define

$$\widetilde{f}(z) = \overline{f(\overline{z})} \quad \text{for} \quad z \in V - \mathbb{H}.$$

One can then show that this is holomorphic on the entire V using Morera's theorem. We can do the same for any line L if V is symmetric about L and f maps the portion of L that intersects V to $\mathbb{R}$.

Similarly we have a reflection ρ about the unit circle $C = \{z \in \mathbb{C} \mid |z| = 1\}$ given by $\rho(z) = 1/\overline{z}$. Let V be an open set such that $\rho(V) = V$. Let $V^+ = V \cap \mathbb{D}$ and $A = V \cap C$. Suppose $f : V^+ \cap A \to \mathbb{C}$ is continuous and holomorphic on V^+ such that $f(A) \subset \mathbb{R}$ then f extends to a holomorphic function $\widetilde{f} : V \to \mathbb{C}$ that agrees with f on V^+. In particular we define

$$\widetilde{f}(z) = \overline{f(1/\overline{z})} \quad \text{for} \quad z \in V - \mathbb{D}.$$

Again this generalises to any circle C is the complex plane.

4.2. **Little Picard Theorem.** The group $SL(2/\mathbb{Z})$ acts on $\mathbb{H}$ by möbius transformations

$$\gamma(z) = \frac{az + b}{cz + d} \quad \text{where} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Clearly $\gamma(z) = \tau(z)$ if $\tau = -\gamma$. Thus in fact the group $\Gamma = SL(2/\mathbb{Z})/\{\pm I\}$ acts on $\mathbb{H}$.

There is a homomorphism $SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/2\mathbb{Z})$ and let Γ' be the kernel. Clearly $-I \in \Gamma'$ and take $\Gamma(2) = \Gamma'/\{\pm I\}$. It can be shown that $\Gamma(2)$ is generated by the classes of

$$\sigma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The function $\tau(z) = z + 2$ is just a translation and fixes ∞. Where as $\sigma(z) = \dfrac{z}{2z + 1}$ fixes 0.

Let Q be the region in the upper half plane given by

$$Q = \{z \in \mathbb{H} \mid -1 \le \operatorname{Re} z < 1, |2z + 1| \ge 1, |2z - 1| > 1\}.$$

We shall prove that Q is a fundamental domain for the action of $\Gamma(2)$, that is any $\Gamma(2)$ orbit of $\mathbb{H}$ intersects Q at exactly 1 point.

Proposition 24. *The region Q satisfies the following:*

(1) *If $\gamma_1, \gamma_2 \in \Gamma(2)$ are distinct elements then $\gamma_1(Q) \cap \gamma_2(Q) = \emptyset$.*

(2) $\mathbb{H} = \cup_{\gamma \in \Gamma(2)} \gamma(Q)$

Statement (1) is equivalent to $\gamma(Q) \cap Q = \emptyset$ for any element of $\Gamma(2)$ other than the identity. It is easy to see that $\tau(Q) \cap Q = \emptyset$ and $\sigma(Q) \cap Q = \emptyset$. We outline the proof of this fact in the following exercise.

Exercise 22. *Show that for $\gamma \in \Gamma(2)$ which is not the identity $\gamma(Q) \cap Q = \emptyset$ using the following steps:*

(1) *Show this first for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $c = 0$.*

(2) *If $c = 2d$ then show that $a = d = \pm 1$ prove it in this case.*

(3) *Finally if $c \neq 0$ and $c \neq 2d$ then first show that γ^{-1} also satisfies this. Show that $|cz + d| > 1$ for all $z \in Q$. It then follows that $\operatorname{Im} \gamma(z) < \operatorname{Im} z$. If there is $z \in Q$ such that $\phi(z) \in Q$ then use ϕ^{-1} to get a contradiction.*

We outline the proof of statement (2) in the following exercise.

Exercise 23. *Show that $\mathbb{H} = \bigsqcup_{\gamma \in \Gamma(2)} \gamma(Q)$ through the following steps:*

(1) *Fix z in $\mathbb{H}$. Show that there is a $\gamma_0 \in \Gamma(2)$ such that $\operatorname{Im} \gamma(z) \leq \operatorname{Im} \gamma_0(z)$*

(2) *Use $\gamma = \tau^k \sigma^{\pm 1} \tau^{-n}(\gamma_0(z))$ and part 1 to show that for some n and k, $\gamma^{-1}(z) \in Q$.*

Now we shall construct a function $\lambda : \mathbb{H} \to \mathbb{C} - \{0, 1\}$ and show that it is a holomorphic covering. Let $Q^+ = \{z \in Q \mid \operatorname{Re} z > 0\}$ be the right half of Q. By the Riemann mapping theorem there is a biholomorphism $\lambda : Q^+ \to \mathbb{H}$ since Q^+ is simply connected. By Theorem 16 this map extends to a homeomorphism $\lambda : \overline{Q^+} \to \overline{\mathbb{H}}$. Without loss of generality we may assume that $\lambda(0) = 0$, $\lambda(1) = 1$ and $\lambda(\infty) = \infty$ (otherwise replace λ by λ' where $\lambda'(z) = \frac{(\lambda(z)-\lambda(0))(\lambda(1)-\lambda(\infty))}{(\lambda(z)-\lambda(\infty))(\lambda(1)-\lambda(0))}$).

By the reflection principle λ extends to all of Q through the formula $\lambda(z) = \overline{\lambda(-\overline{z})}$ for $z \in Q - Q^+$. This defines a continuous bijection $\lambda : Q \to \mathbb{C} - \{0, 1\}$ which is holomorphic on the interior of Q. Figure 1 shows this map on Q, it maps Q^+ to the upper half plane, the boundary of Q^+ to the real line and the rest of the interior of Q to the lower half plane.

Now extend λ to the entire upper half plane as follows. For any $z \in \mathbb{H}$, there is a unique element $\gamma \in \Gamma(2)$ such that $z \in \gamma(Q)$, hence $\gamma^{-1}(z) \in Q$. We define

$$\lambda(z) = \lambda(\gamma^{-1}(z)).$$

Exercise 24. *Show that this definition makes λ holomorphic on the entire $\mathbb{H}$.*

Finally to see that λ is a covering we have to demonstrate uniformly covered neighbourhoods whose union is the entire $\mathbb{C} - \{0, 1\}$. Let $A_1 = (-\infty, 0]$, $A_2 = [0, 1]$ and $A_3 = [1, \infty)$, then define

$$U_1 = \mathbb{C} - (A_2 \cup A_3), \quad U_2 = \mathbb{C} - (A_1 \cup A_3), \quad U_3 = \mathbb{C} - (A_1 \cup A_2).$$

Clearly $U_1 \cup U_2 \cup U_3 = \mathbb{C} - \{0, 1\}$ and we claim that U_1 are uniformly covered. I shall show this only for U_1 and leave the rest as exercise. If Q° is the interior of Q then

$$\lambda^{-1}(U_1) = \bigsqcup_{\gamma \in \Gamma(2)} \gamma(Q^\circ),$$

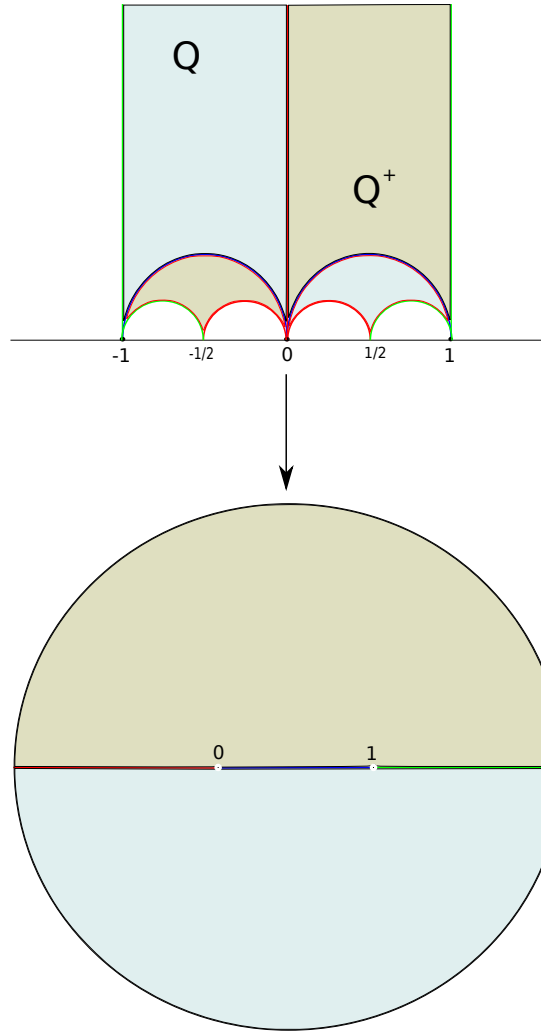and by the definition of λ it is a biholomorphism from $\gamma(Q^\circ)$ to U_1.

FIGURE 1. Modular Function

Theorem 25 (Little Picard). *Let $a, b \in \mathbb{C}$ be two distinct points and $f : \mathbb{C} \to \mathbb{C} - \{a, b\}$ be holomorphic, then f is constant.*

Proof. Let $g(z) = \dfrac{f(z) - a}{b - a}$, then $g : \mathbb{C} \to \mathbb{C} - \{0, 1\}$ is holomorphic. Since $\lambda : \mathbb{H} \to \mathbb{C} - \{0, 1\}$ is a holomorphic covering map, there is a holomorphic function $h : \mathbb{C} \to \mathbb{H}$ such that $g = \lambda \circ h$.

Then $k(z) = (h(z) - i)/(h(z) + i)$ is a holomorphic map from $\mathbb{C}$ to $\mathbb{D}$ hence constant. Thus h is also a constant and therefore so is g. Finally that would imply f is constant. $\square$