Arakelov Geometry of Modular Curves $X_0(p^2)$

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Bogomolov's Conjecture

Let X be a smooth projective geometrically connected curve over a number field K with genus $g > 1$. Let J be the Jacobian of X. Fix an embedding $\phi : X \to J$ for some degree 1 divisor of X.

Bogomolov's conjecture states that there is an $\epsilon > 0$ such that

$$\{x \in X(\overline{K}) \mid h_{\mathrm{NT}}(\phi(x)) < \epsilon\}$$

is finite; h_{NT} denotes the Neron-Tate height on J.

This conjecture was proved by Ullmo (1998) and Zhang(1998), Our goal is to prove an effective version of Bogomolov's conjecture for modular curves $X_0(p^2)$.

The approach will be to use the work of *Shou-Wu Zhang (1993)* which in turn uses Arakelov intersection pairing.

To work in Zhang's setting we need a *minimal regular and semistable model* of $X_0(p^2)$ over some number field.

In this talk I shall focus on the construction of this model. Towards the end I shall briefly explain how we get to effective Bogomolov.

Modular Curves

We are interested in *congruence subgroups* of the modular group $\mathrm{SL}(2,\mathbb{Z})$ of the form

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Such a group Γ acts on the upper half plane by Möbius transformations and the quotient $Y(\Gamma) = \mathbb{H}/\Gamma$ is a Riemann surface. Usually $Y(\Gamma)$ is *non-compact* but of finite type.

By adding finitely many points $p_1, \ldots, p_m \in \mathbb{P}^1(\mathbb{Q})$ we can compactify $Y(\Gamma)$ and get a compact Riemann surface:

$$X(\Gamma) = Y(\Gamma) \cup \{p_1, \ldots, p_m\}.$$

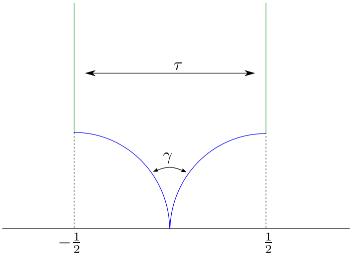
The points $p_1, \ldots, p_m$ are known as the *cusps* of Γ.

$X(\Gamma)$ is a projective algebraic curve over $\mathbb{C}$ and its genus is denoted by g_Γ.

Toy Example

The group $\Gamma_0(2)$ is generated by $\tau = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$.

The fundamental domain in $\mathbb{H}$ for $\Gamma_0(2)$ is:



$$\tau(z) = z + 1$$

$$\gamma(z) = \frac{z}{2z+1}$$

$\Gamma_0(2)$ has 2 cusps corresponding to 0 and ∞ and

$$X_0(2) = X(\Gamma_0(2)) \cong \mathbb{P}^1.$$

Modular Interpretation

In fact $X_0(N)$ is an algebraic curve over $\mathbb{Q}$, and hence defined over any number field K.

$X_0(N)(K)$, the K points of $X_0(N)$, are in bijection with isomorphism classes of pairs

$$(E, C)$$

where E is an elliptic curve over K, and $C \subset E(K)$ is a cyclic subgroup of order N.

Hence these objects are important in arithmetic geometry.

We shall denote by g_N the genus of $X_0(N)$. This can be calculated using the Riemann Hurwitz formula.

Models of Algebraic Curves

Let X be an algebraic curve over a number field K and let $\mathcal{O}_K$ be the ring of integers of K.

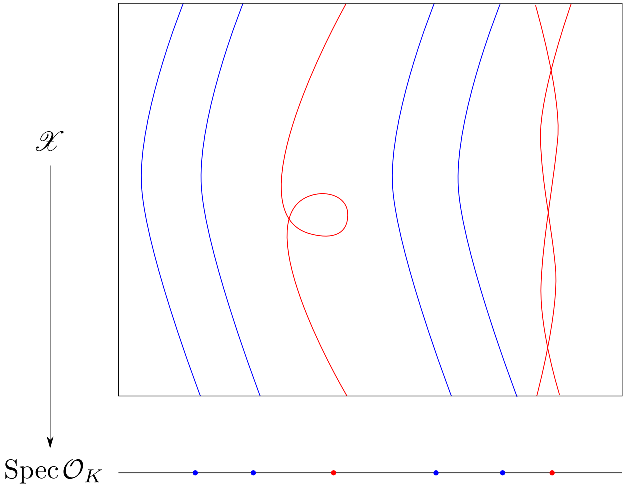
Definition

A *model* $\mathscr{X}$ of X over $\mathcal{O}_K$ is a normal scheme with proper, flat morphism $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_K$ with 1 dimensional fibers, such that, the generic fiber $\mathscr{X}_0$, which is a curve over K, is isomorphic to X.

For any prime $\mathfrak{p}$ of $\mathcal{O}_K$, the fiber $\mathscr{X}_\mathfrak{p}$ *is a curve over the residue field* $\kappa(\mathfrak{p}) = Q(\mathcal{O}_K/\mathfrak{p})$.

A fiber $\mathscr{X}_\mathfrak{p}$ is called *special*, if it is a singular curve.

Models of Algebraic Curves



Minimal Regular Model

A model $\mathscr{X}$ of X over $\mathcal{O}_K$ is called *regular* if $\mathscr{X}$ is a regular scheme. In this case $\mathscr{X}$ is an *arithmetic surface*.

If the genus $g(X) > 1$, there is a *unique regular model, $\mathscr{X}$, which is minimal in the sense of birational morphisms*.

That is any proper birational morphism from $\mathscr{X}$ is necessarily an isomorphism.

Equivalently $\mathscr{X}$ does not have any prime vertical divisor that can be blown down without introducing a singularity.

Semi-stable Model

A model $\mathscr{X}$ of X over $\mathcal{O}_K$, is called *semi-stable* if

- all fibers are reduced,
- special fibers only have nodal singularities.

Stable reduction theorem (Deligne-Mumford) says that semi-stable models always exist after base change.

That is if $\mathscr{X} \to \operatorname{Spec} K$ is not semi-stable, there is a finite extension K'/K so that

$$\mathscr{X}' = \mathscr{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_{K'}$$

is semi-stable.

Minimal regular and semistable model for $X_0(p^2)$

We prove the following theorem:

Theorem (Banerjee, Borah, C.)

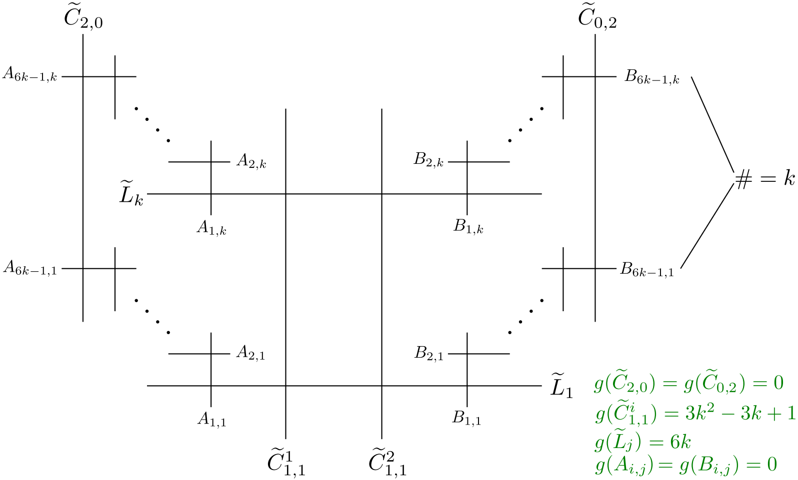
There is a minimal regular and semistable model for $X_0(p^2)$ over the ring of integers of $K = \mathbb{Q}(\sqrt[r]{p}, \zeta_{p+1})$ where $r = (p^2 - 1)/2$.

Let us denote this model by $\mathscr{X}_0(p^2)$.

The special fibers of $\mathscr{X}_0(p^2)$ are precisely the fibers over primes $\mathfrak{q}$ of $\mathcal{O}_K$ that lie above $(p) \in \operatorname{Spec} \mathbb{Z}$.

The geometry of the special fibers, crucial for our calculations, depend on the residue of p modulo 12.

When $p = 12k + 1$ the special fibers of $\mathscr{X}_0(p^2)$ are as follows:



$g(\widetilde{C}_{2,0}) = g(\widetilde{C}_{0,2}) = 0$
$g(\widetilde{C}^i_{1,1}) = 3k^2 - 3k + 1$
$g(\widetilde{L}_j) = 6k$
$g(A_{i,j}) = g(B_{i,j}) = 0$

Edixhoven's model

Edixhoven described a regular model $\mathscr{X}_{\mathbb{Z}}$ of $X_0(p^2)$ over $\mathbb{Z}$. However, this model is not minimal or semi-stable.

Our starting point is this model. To this model we apply the procedure that we shall elaborate on presently.

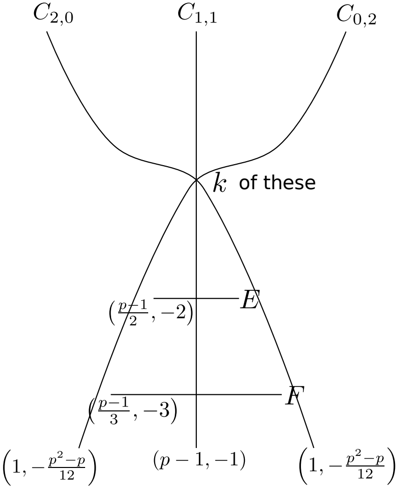
The calculations are slightly different for different values of p modulo 12, let us restrict ourselves to $p = 12k + 1$.

The regular model $\mathscr{X}_{\mathbb{Z}}$ has only one special fiber over $(p) \in \operatorname{Spec} \mathbb{Z}$:

$p = 12k + 1$

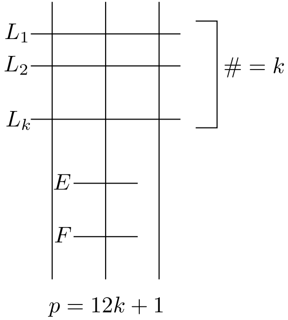
Each component is a $\mathbb{P}^1$

For each pair (n, m) adjacent to a componet, n = multiplicity, m = self-intersection



Clearly $\mathscr{X}_{\mathbb{Z}}$ is not semistable as the special fiber is not reduced and has triple intersections. We do the following:

1. Blow up all the triple intersection points to obtain $\mathscr{X}_0^{\sharp}(p^2)_{\mathbb{Z}}$.



Special fiber of $\mathscr{X}_0^{\sharp}(p^2)_{\mathbb{Z}}$. Multiplicity of each L_i is $p + 1$.

2. Base change to $\mathcal{O}_K$ to obtain $\mathscr{X}_0^\sharp(p^2)_{\mathcal{O}_K}$

$$\mathscr{X}_0^\sharp(p^2)_{\mathcal{O}_K} = \mathscr{X}_0^\sharp(p^2)_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathcal{O}_K$$

with $K = \mathbb{Q}(\sqrt[r]{p}, \zeta_{p+1})$ where $r = (p^2-1)/2$

The ideal $p\mathcal{O}_K$ has the following prime factorization

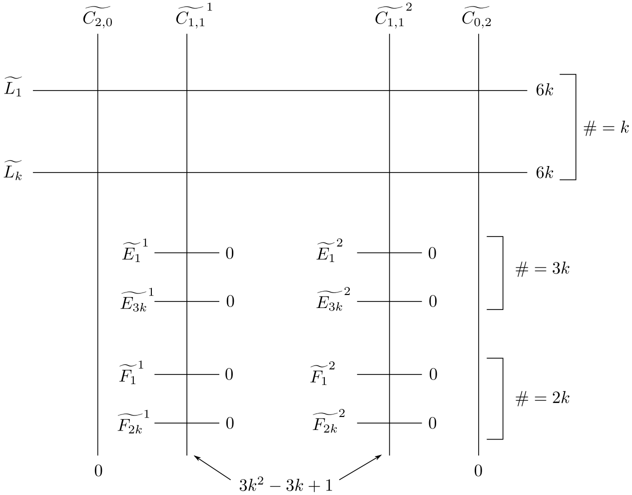
$$p\mathcal{O}_K = \mathfrak{p}_1^r \cdots \mathfrak{p}_s^r, \qquad s = \varphi((p+1)/2),$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are distinct prime ideals of $\mathcal{O}_K$ and $\mathcal{O}_K/\mathfrak{p}_i \cong \mathbb{F}_{p^2}$.

Hence, special fibers of $\mathscr{X}_0^\sharp(p^2)_{\mathcal{O}_K}$ are $(\mathscr{X}_0^\sharp(p^2)_{\mathcal{O}_K})_{\mathfrak{p}_i}$ and are isomorphic to $(\mathscr{X}_0^\sharp(p^2)_{\mathbb{Z}})_{(p)} \times \operatorname{Spec}\mathbb{F}_{p^2}$.

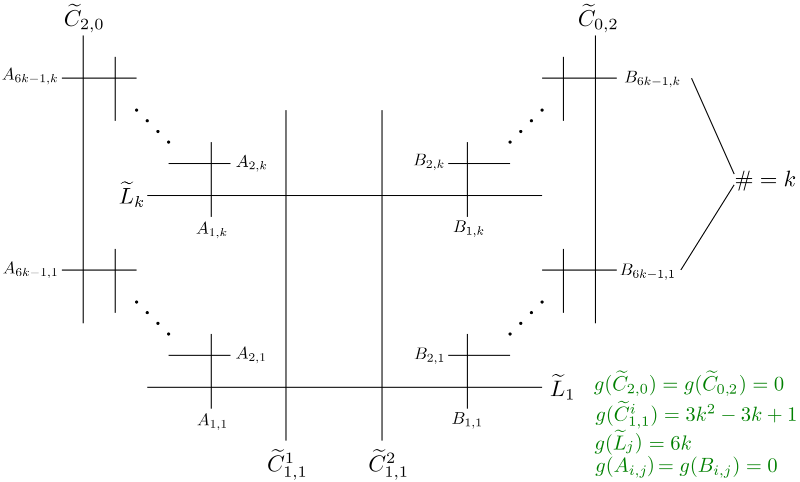
Unfortunately $\mathscr{X}_0^\sharp(p^2)_{\mathcal{O}_K}$ is not normal.

3. *Normalise* $\mathscr{X}_0^\sharp(p^2)_{\mathcal{O}_K}$ *to get* $\mathscr{X}_0^\flat(p^2)_{\mathcal{O}_K}$.



Special fiber of $\mathscr{X}_0^\flat(p^2)_{\mathcal{O}_K}$.

4. Finally desingularise $\mathscr{X}_0^\flat(p^2)_{\mathcal{O}_K}$ and blow down the rational components with self-intersection -1. This yields the minimal regular model $\mathscr{X}_0(p^2)_{\mathcal{O}_K}$ which is also semistable as desired.



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Arakelov's Intersection Pairing

Let $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_K$ be an arithmetic surface with generic fiber $X = \mathscr{X}_K$.

For each embedding $\sigma : K \to \mathbb{C}$ we have a connected Riemann surface

$$X_\sigma = X \times_{\operatorname{Spec} K, \sigma} \operatorname{Spec} \mathbb{C}.$$

Collectively we denote

$$\mathscr{X}_\infty = \coprod_{\sigma : K \to \mathbb{C}} X_\sigma.$$

For two divisors C, D on $\mathscr{X}$ the Arakelov pairing is given by

$$\langle C, D \rangle = \langle C, D \rangle_{\mathrm{fin}} + \langle C, D \rangle_\infty.$$

Where

$$\langle C, D\rangle_{\mathrm{fin}} = \sum_{x\in\mathscr{X}^{(2)}} \log|\mathcal{O}_{\mathscr{X},x}/(C_x, D_x)|$$

is the usual algebraic intersection.

Moreover, in the special case when C and D have no common prime divisors

$$\langle C, D\rangle_{\infty} = -\sum_{\sigma:K\to\mathbb{C}}\sum_{\alpha,\beta} n_{\alpha,\sigma} m_{\beta,\sigma}\mathfrak{g}^{\sigma}_{\mathrm{can}}(P_{\alpha,\sigma}, Q_{\beta,\sigma}),$$

where $C_\sigma = \sum_\alpha n_{\alpha,\sigma}P_{\alpha,\sigma}$ and $D_\sigma = m_{\beta,\sigma}Q_{\beta,\sigma}$ and $\mathfrak{g}^{\sigma}_{\mathrm{can}}$ is the canonical Green's function on X_σ.

Canonical Sheaf

Let X be a smooth projective curve over K of genus $g_X > 1$ and $\mathscr{X}_{\mathcal{O}_K}$ the minimal regular model. Let $\omega_{\mathscr{X}}$ be the canonical sheaf on $\mathscr{X}$.

The quantity

$$\omega_X^2 = \frac{\langle \omega_{\mathscr{X}}, \omega_{\mathscr{X}} \rangle}{[K : \mathbb{Q}]}$$

is independent of K if $\mathscr{X}$ is semi-stable, so is an invariant of X.

We call this the *stable arithmetic self-intersection number* of X.

Main Result

We obtain an asymptotic expression for $\omega_{p^2} = \omega^2_{X_0(p^2)}$ for primes $p > 13$.

Theorem (Banerjee, Borah,—)

$$\omega_{p^2} = 2g_{p^2} \log p^2 + \frac{p \log p^2}{8} + o(p \log p^2).$$

In comparison Mayer shows that the asymptotic expression for ω^2 in the case of $X_1(N)$ is $3g_N \log N + o(g_N \log N)$.

We shall first mention applications of this result, then give an outline of our proof.

Effective Bogomolov

S.W. Zhang introduced the *admissible pairing*, for divisors on arithmetic surfaces, closely related to the Arakelov pairing.

Let ω_a^2 denote the admissible self-intersection of the canonical sheaf.

Using the geometry of the special fiber the admissible self intersection can be calculated from the Arakelov self intersection.

Zhang showed that the ϵ in Bogomolov's conjecture can be explicitly controlled in terms ω_a^2.

Using Zhang's work we can prove the following effective version of Bogomolov's conjecture.

Theorem

For a sufficiently large prime p, and any $\epsilon > 0$, the set

$$\left\{ x \in X_0(p^2)(\overline{\mathbb{Q}}) \mid h_{\mathrm{NT}}(\phi(x)) < \left(\frac{1}{2} - \epsilon\right) \log(p^2) \right\}$$

is finite whereas

$$\left\{ x \in X_0(p^2)(\overline{\mathbb{Q}}) \mid h_{\mathrm{NT}}(\phi(x)) \leq (1 + \epsilon) \log(p^2) \right\}$$

is infinite.

Certain Horizontal Divisors

Consider the cusps of $\Gamma_0(p^2)$ corresponding to 0 and ∞. These are points of $X_0(p^2)(\mathbb{Q})$.

Let H_0 and H_∞ be the horizontal divisors of $\mathscr{X}_0(p^2)_{\mathcal{O}_K}$ corresponding to these points of the generic fiber.

H_0 intersects exactly one of $\widetilde{C}_{0,2}$ and $\widetilde{C}_{2,0}$; we call that component of the special fiber $\widetilde{C}_0$.

H_∞ intersects the other component and we label that $\widetilde{C}_\infty$.

Certain Vertical Divisors

Define the vertical divisors

$$V_{0,p} = \left(12 - 12g_{p^2}\right) \widetilde{C}_0 + 7\widetilde{C}_{1,1}^1 + 7\widetilde{C}_{1,1}^2 + \sum_{i=1}^{k} \frac{p-1}{2} x\widetilde{L}_i$$
$$+ \sum_{i=1}^{k} \sum_{l=1}^{6k-1} \left[lx + \frac{p-1-2l}{p-1} \left(12 - 12g_{p^2}\right) \right] A_{l,i} + \sum_{i=1}^{k} \sum_{l=1}^{6k-1} (lx) B_{l,i},$$

and

$$V_{\infty,p} = \left(12 - 12g_{p^2}\right) \widetilde{C}_\infty + 7\widetilde{C}_{1,1}^1 + 7\widetilde{C}_{1,1}^2 + \sum_{i=1}^{k} \frac{p-1}{2} x\widetilde{L}_i$$
$$+ \sum_{i=1}^{k} \sum_{l=1}^{6k-1} \left[lx + \frac{p-1-2l}{p-1} \left(12 - 12g_{p^2}\right) \right] B_{l,i} + \sum_{i=1}^{k} \sum_{l=1}^{6k-1} (lx) A_{l,i}.$$

Main Lemma

With all these definitions in place it is now easy to see that for any canonical divisor $\mathcal{K}_{\mathscr{X}_0(p^2)}$ the divisor

$$D_{m,p} = \mathcal{K}_{\mathscr{X}_0(p^2)} - (2g_{p^2} - 2)H_m + V_{m,p}, \quad m = 0, \infty$$

is orthogonal to any vertical divisor with respect to the Arakelov pairing.

As a consequence we have:

Lemma

$$\begin{aligned}\omega^2_{\mathscr{X}_0(p^2)} =& - 4g_{p^2}(g_{p^2} - 1) \langle H_0, H_\infty \rangle \\ & + \frac{1}{g_{p^2} - 1} \left[g_{p^2} \langle V_{0,p}, V_{\infty,p} \rangle - \frac{V^2_{0,p} + V^2_{\infty,p}}{2} \right] + O(\log p).\end{aligned}$$

Asymptotic formula

In the proposition the first summand involves pairing between horizontal divisors. It can be calculated using the canonical Green's function

$$\langle H_0, H_\infty \rangle = [K : \mathbb{Q}]\mathfrak{g}_{\mathrm{can}}(0, \infty).$$

An estimate for the Green's function was obtained in our first paper.

Theorem (Banerjee, Borah, —)

$$\mathfrak{g}_{\mathrm{can}}(0, \infty) = -\frac{\log p}{p^2} + o\left(\frac{\log p}{p^2}\right).$$

The remaining terms $\frac{1}{g_{p^2}-1}\left[g_{p^2}\left\langle V_{0,p},V_{\infty,p}\right\rangle - \frac{V_{0,p}^2+V_{\infty,p}^2}{2}\right]$ involve intersections of vertical divisors only and can be explicitly computed.

Together they yield the asymptotic expression for the stable arithmetic self intersection

$$\omega_{p^2}^2 = \frac{\left\langle \omega_{\mathscr{X}_0(p^2)}, \omega_{\mathscr{X}_0(p^2)} \right\rangle}{(p^2-1)/2}.$$

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Thank You!

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