# Teichmüller Theory

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***by***

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***to the***


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**DECLARATION**

I hereby declare that I am the sole author of this thesis in partial fulfillment of the requirements for a postgraduate degree from National Institute of Science Education and Research (NISER). I authorize NISER to lend this thesis to other institutions or individuals for the purpose of scholarly research.

Soham Samanta
Signature of the Student
Date: 30/04/2024

The thesis work reported in the thesis entitled Teichmüller Theory was carried out under my supervision, in the school of Mathematical Sciences at NISER, Bhubaneswar, India.

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Signature of the thesis supervisor
School: SMS
Date: 30/04/2024

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# ABSTRACT

Starting with the definition of Teichm uller space we will move on to the description of **Fricke coordinates**, which helps in the realization of the Teichm uller space as a subset of a Euclidean space. Subsequently, when provided with the length of its three boundaries, we will show the uniqueness and existence of a pair of pants. We will then decompose a closed Riemann surface of genus $g(\geq 2)$ into $2g-2$ pair of pants by cutting the surface, R, along $3g-3$ mutually disjoint simple closed geodesics with respect to the hyperbolic metric on $R$. Given that it's clear that $R$ can be reconstructed by gluing all the resulting pieces 'suitably', we naturally consider, as a system of coordinates for the Teichm uller space $T_g$, the pair of the set of lengths of all geodesics used in the above decomposition into pants and the set of the so-called twisting parameters used to glue the pieces. Such a system of coordinates is called **Fenchel-Nielsen coordinates** on $T_g$. The main aim of the first chapter will be to construct these coordinates using the Fricke Coordinates and then finally show how they are continuous. Specifically, we will construct a real analytic mapping from the **Fricke space** to $(R^+)^{3g-3}(S^1)^{3g-3}$. In the subsequent chapters we are going to deal with the complex analytic theory of Teichmüller Space. First we will represent the Teichmüller Space $T(\Gamma)$ by quasiconformal mappings of the Riemann sphere $\hat{\mathbf{C}}$ which are conformal on the lower half-plane $H^*$. Now, with the help of ***Schwarzian derivatives***, we will construct an embedding (***Ber's embedding***), the image ( $T_B(\Gamma)$ ) of which will inherit the complex manifold structure of $A_2(H^*, \Gamma)$. Then identifying $T(\Gamma)$ with $T_B(\Gamma)$, we get the intended complex structure. We will then move on to prove a relation between the ***Teichmüller Modular Group***, $\boldsymbol{Mod(\Gamma)}$ and the biholomorphic automorphism group, $Aut(T(\Gamma))$, given by Royden.

# Contents

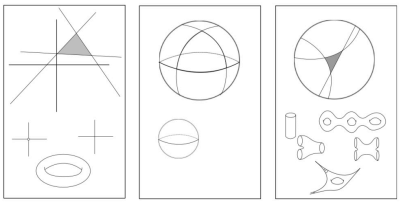
# Chapter 1

# Fenchel-Nielsen Coordinates

## 1.1 Characterization of Universal Coverings

■ A Riemann surface that is biholomorphic to $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} - \{0\}$ or a torus is said to be of **exceptional type**.

**Characterization Theorem of the Universal Coverings of Riemann Surfaces**. A Riemann surface has $\mathbb{H}$ as holomorphic universal covering if and only if it is not of exceptional type. The complex plane $\mathbb{C}$ universally covers and can only cover the surfaces of exceptional type, except $\hat{\mathbb{C}}$



## 1.2 Riemann surfaces via their group models

■ Most Riemann surfaces are a quotient of $\mathbb{H}$ by a subgroup of PSL(2, $\mathbb{R}$).

■ This opens the door to the study these subgroups and the exploration of Riemann surfaces via their group models.

**Definition:** PSL(2, $\mathbb{R}$) is a Lie group with the topology induced by the identification of the $2 \times 2$ real matrices with $\mathbb{R}^4$. We say that a discrete subgroup of PSL(2, $\mathbb{R}$) is a Fuchsian group.

**Theorem 1.1.** *Let $\Gamma$ be a Fuchsian group without elliptic elements. Then $\mathbb{H}/\Gamma$ is a connected Riemann surface and the projection $\mathbb{H} \to \mathbb{H}/\Gamma$ is a covering.*
*Moreover, if $\Gamma_1$ and $\Gamma_2$ are Fuchsian groups without elliptic elements, $\mathbb{H}/\Gamma_1$ is biholomorphic to $\mathbb{H}/\Gamma_2$ iff $\Gamma_1$ and $\Gamma_2$ are conjugate by an element of PSL(2, $\mathbb{R}$).*

*We have*

{ *Riemann surfaces not of execeptional type modulo biholomorphism* }$\leftrightarrow$ {*Conjugacy classes of elliptic-free discrete subgroups of PSL(2,* $\mathbb{R}$*)*}

## 1.3 Teichmüller Space of the Torus

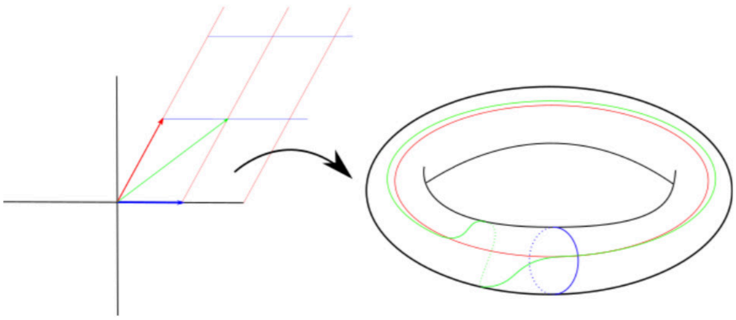
**Definition 1.2** (Lattice)**.** *: A group of the form* $z_1\mathbb{Z} \times z_2\mathbb{Z}$ *with* $z_1, z_2 \in \mathbb{C}$ *not zero and linearly independent over* $\mathbb{R}$ *is called a lattice.*

■ It is known that all the quotients of $\mathbb{R}^2$ by lattices are tori

■ In contrast to the fact that all lattices give rise to homeomorphic tori, not all yield **biholomorphic** tori.

■ Every torus is **biholomorphic** to a torus of the form $\mathbb{C}/\Gamma_\tau$, with $\Gamma_\tau$ a lattice group with basis 1 and $\tau$ so that $Im(\tau) > 0$.

■ For any two $\tau$ and $\tau^{'}$ in $\mathbb{H}$, two tori $\mathbb{T}_{\tau'}$, $\mathbb{T}_\tau$ are biholomorphically equivalent iff $\tau$ and $\tau^{'}$ are related by a unimodular transformation, that is, there exists an $h \in PSL(2,\mathbb{Z})$, such that $h(\tau) = \tau^{'}$.



Here $\tau$ is red, 1 is blue and $\tau^{'}$ is green. In the torus the green curve is a curve homotopic to the projection of the $\tau^{'}$ segment.

The torus $\mathbb{H}/\Gamma_\tau$ is equipped with a canonical choice of fundamental group generators:

■ As shown in the above figure, the line segments given by 1 and $\tau$ in $\mathbb{C}$ become the "meridian and equator" loops in $\mathbb{H}/\Gamma_\tau$

■ With this choice, we can regard 1 as (1,0) and $\tau$ as (0,1) in $\mathbb{Z}\times\mathbb{Z} \cong \Gamma_\tau \cong \pi(\mathbb{H}/\Gamma_\tau, [0])$.

■ Now take $\tau^{'} = \tau + 1$, we know that $\mathbb{H}/\Gamma_\tau$ and $\mathbb{H}/\Gamma_{\tau'}$ are biholomorphic.

■ However, it is clear that $\tau^{'}$ represents (1,1) in $\mathbb{Z}\times\mathbb{Z}$ hence $\tau^{'}$ and 1 from a different pair of generators of the fundamental group.

■ In the "Teichmüller Space", $\mathbb{H}/\Gamma_\tau$ and $\mathbb{H}/\Gamma_{\tau'}$ will be different torus.

■ That is, we mark Riemann surfaces by choosing certain generators of the fundamental group.

■ All this motivates the following definition.

## 1.4 Definition of Teichmüller Space

Let $R$ be a Riemann surface. A marking $\Sigma_p$ is a set of generators of the fundamental group of $R$ based at $p \in R$.

Two markings $\Sigma_p$ and $\Sigma_q^{'}$ are said to be equivalent if there is a continuous path in $R$ between $p$ and $q$ that induces an isomorphism $\pi(R,p) \to \pi(R,q)$ by conjugation that sends $\Sigma_p$ to $\Sigma_q^{'}$.

Finally, $(R, \Sigma_p)$ and $(S, \Sigma_q^{'})$ are equivalent if there is a biholomorphism $h$ such that $h_*(\Sigma_p)$ is equivalent $\Sigma_q^{'}$.

The class of $(R, \Sigma_p)$ is denoted by $[R, \Sigma_p]$ and it's called a marked Riemann surface.

The Teichmüller space of $F_g$ is defined as $\mathcal{T}_g = \{[R, \Sigma_p]\}$ with $R$ homeomorphic to $F_g$. In particular, the Teichmüller space $\mathcal{T}_1$ of the torus is the set of all marked tori.

For $\tau \in \mathbb{H}$, let $\Sigma(\tau) = \{A(\tau), B(\tau)\}$ be the marking on $\mathbb{T}_\tau$ at $[0]$ determined by the generators induced by 1 and $\tau$.

For instance, in the figure above $\Sigma(\tau)$ would be the blue and red loops and $\Sigma(\tau^{'})$ would be the blue and green loops.

The markings $\Sigma(\tau)$ and $\Sigma(\tau^{'})$ in the example are not equivalent.

**Example 1.3** (Teichmüller Space of the Torus).

**Theorem 1.4.** *We have that* $[\mathbb{T}_\tau, \Sigma(\tau)] = [\mathbb{T}_{\tau'}, \Sigma(\tau^{'})]$ *in* $\mathcal{T}_1$ *iff* $\tau = \tau^{'}$.

*Therefore, there is a bijection between* $\mathcal{T}_1$ *and* $\mathbb{H}$.

**Lemma 1.5.** *Every torus is biholomorphic to a torus of the form* $\mathbb{C}/\Gamma_\tau$*, with* $\Gamma_\tau$ *a lattice group with a basis 1 and* $\tau$ *so that* $Im(\tau) > 0$.

**Lemma 1.6.** *Let* $f : \mathbb{T}_{\tau'} \to \mathbb{T}_\tau$ *be holomorphic. Then there exists a holomorphic map*

*$\tilde{f} : \mathbb{C} \to \mathbb{C}$ such that $\tilde{f}(0) = 0$ and $f([z]) = [\tilde{f}(z)]$.*

*If $f$ is a biholomorphism, then $\tilde{f}(z) = \alpha z$ for $\alpha \in \mathbb{C}^*$.*

*Proof.* If $\tau = \tau^{'}$ the claim is trivial.

If $[\mathbb{T}_\tau, \Sigma(\tau)] = [\mathbb{T}_{\tau^{'}}, \Sigma(\tau^{'})]$ we have a biholomorphic $h : \mathbb{T}_{\tau'} \to \mathbb{T}_\tau$ such that $h_*(\Sigma(\tau^{'}))$ is equivalent to $\Sigma(\tau)$.

Note that both $h_*(\Sigma(\tau^{'}))$ and $\Sigma(\tau)$ are markings at $[0]$ given that $h([0]) = [\tilde{h}(0)]$.

Then, the hypothesis tells us that $h_*(A(\tau^{'})) = A(\tau)$ and $h_*(B(\tau^{'})) = B(\tau)$.

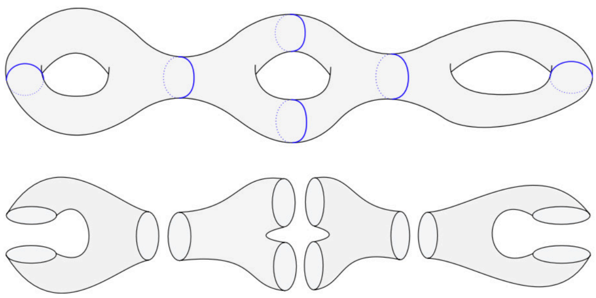
Therefore, $\tilde{h}$ sends the basis $1$, $\tau^{'}$ to $1$, $\tau$.

From the lemma 2 we have that $\tilde{h}(z) = \alpha z$ hence $\tilde{h}(\tau^{'}) = \alpha\tau^{'} = \tau$ and $\tilde{h}(1) = \alpha = 1$.

We conclude that $\tau = \tau^{'}$

From Lemma 1, the result follows. □

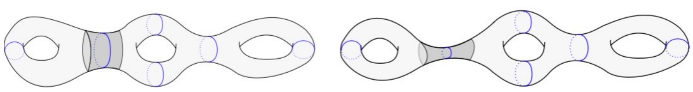
## 1.5 Fenchel-Nielsen Coordinates



■ This is a 3-holed torus.

■ The main idea is to decompose it into 6 pair of pants by cutting along geodesics like in the figure above.

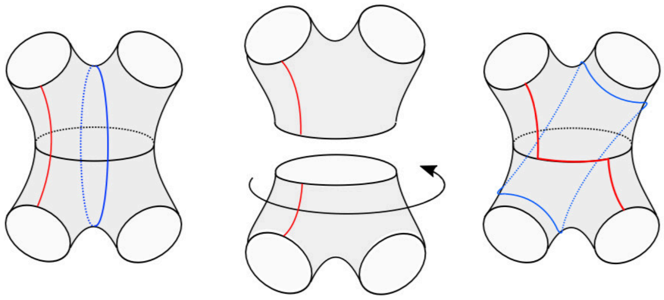
### 1.5.1 Length functions



We denote the length of the six decomposing geodesic by $l_1, l_2, l_3 ... l_6$. The wonderful thing is that varying the length of those geodesic we get different hyperbolic structures. For example, letting $l_1$ vary we get a path in $\mathcal{T}_g$.

■ Notice that this provides a good notion of closeness of hyperbolic structures. In the figure above we can see two different points on $\mathcal{T}_3$

### 1.5.2 Twist parameter



In the heuristic spirit, one may wonder what happens if we twist one of the sleeves before pasting them together? Intuitively this should change the lengths of curves in the resulting surface, should it not? and if so, should it not change the hyperbolic marking?

## 1.6 Existence and Uniqueness of Pants

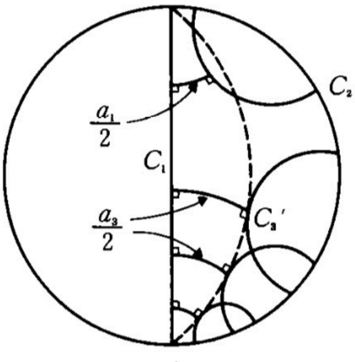
**Definition 1.7** (Pair of Pants). *Let $a, b, c \in (0, \infty)$. A pair of pants is a hyperbolic surface that is diffeomorphic to $\Sigma_{0,3,0}$ such that the boundary components have length **a, b** and **c**. respectively.*

Let $L_1, L_2, L_3$ be the boundary components, which are *simple closed geodesics*, of the pair of pants.

### 1.6.1 Existence of a pair pants with given lengths of the boundary components

**Theorem 1.8.** *For an arbitrarily given triple $(a_1, a_2, a_3)$ of positive numbers, there exists a Pair of pants $P$ such that*

$l(L_j) = a_j, j = 1, 2, 3$



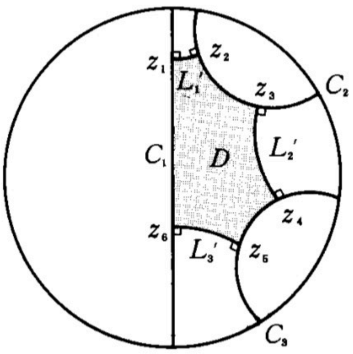

**Proof**

■ Let $C_1$ be the part of the imaginary axis in $\Delta$.

■ Fix another geodesic, say $C_2$, on $\Delta$ such that the Poincaré distance between $C_1$ and $C_2$ is equal to $a_1/2$.

■ On the other hand, geodesics on $\Delta$ from which the Poincaré distance to $C_1$ are equal to $a_3/2$ form a real one-parameter family (i.e., the family of circular arcs $C_3'$ tangent to the broken circular arc in Fig.)

■ **Conclusion:** There exists a geodesic, say $C_3$, in this family such that the Poincaré distance between $C_2$ and $C_3$ is equal to $a_2/2$.




■ let $z_1$ and $z_2$ be the points in $\Delta$ uniquely determined by the condition

$\rho(z_1, z_2) = a_1/2$, $z_1 \in C_1$, $z_2 \in C_2$.

Let $L_1'$ be the geodesic connecting $z_1$ and $z_2$.

■ Similarly, let $\{z_3, z_4\}$ and $\{z_5, z_6\}$ be the pairs of points uniquely determined by the conditions

$\rho(z_3, z_4) = a_2/2, z_3 \in C_2, z_4 \in C_3,$

$\rho(z_5, z_6) = a_3/2, z_5 \in C_3, z_6 \in C_1$, respectively.

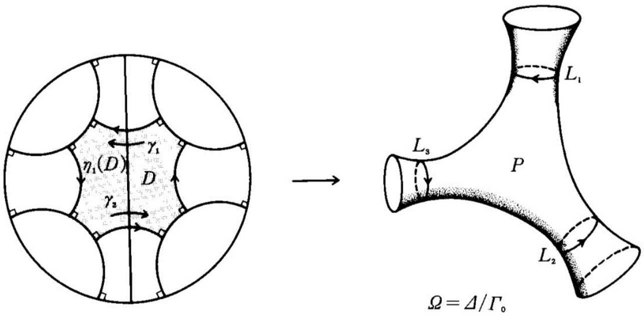
■ Denote by $L_2^{'}$ and $L_3^{'}$, respectively, the geodesics connecting $z_3$ and $z_4$, and $z_5$ and $z_6$ in the above figure.

■ Let $D$ be the closed hyperbolic hexagon bounded by $\{C_j, L_j^{'}\}_{j=1}^{3}$.

■ Let $\eta_j (j = 1, 2, 3)$ be the reflection w.r.t. $C_j$, i.e., the anti-holomorphic automorphism of $\hat{\mathbb{C}}$ preserving $C_j$ pointwise.

Set

$\gamma_1 = \eta_1 \circ \eta_2$, $\gamma_2 = \eta_3 \circ \eta_1$




$\Omega = \Delta / \Gamma_0$

Then $\gamma_1$ and $\gamma_2$ are hyperbolic elements of $Aut(\Delta)$.

■ Let $\Gamma_0$ be the group generated by these $\gamma_1$ and $\gamma_2$.

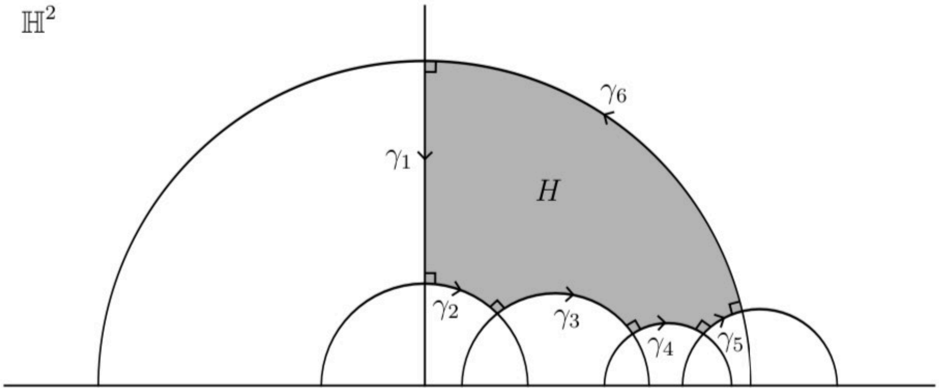
## 1.7 Uniqueness of Pair of Pants

**Theorem 1.9.** *Let $a, b, c \in (0, \infty)$ and let $P$ and $P'$ be pairs of pants with boundary curves of lengths $a, b$ and $c$. Then there exists an isometry*
$\phi : P \to P'$

**Lemma 1.10.** *Let $z \in \mathbb{H}$ and let $\gamma \subset \mathbb{H}^2$ be a geodesic so that $z \notin \gamma$. Then d(z, $\gamma$) := inf* $\{$*d(z, w); $w \in \gamma$*$\}$
*is realized by the intersection point of the perpendicular from $z$ to $\gamma$.*
*Likewise, any two geodesics that don't intersect and are not asymptotic to the same point in $\mathbb{R} \cup \{\infty\}$ have a unique common perpendicular.*
*Moreover, this perpendicular minimizes the distance between them.*

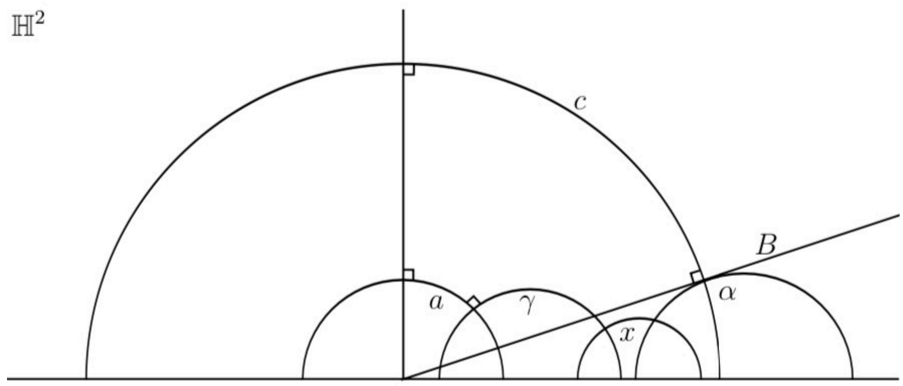
### 1.7.1 Right Angled Hexagon

A right-angled hexagon $H \subset \mathbb{H}^2$ is a compact simply connected closed subset whose boundary consists of 6 geodesic segments, that meet each other orthogonally.



**Lemma 1.11.** *Let $a, b, c \in (0, \infty)$. Then there exists a right angled hexagon $H \subset \mathbb{H}^2$ with three non-consecutive sides of length $a, b$ and $c$ respectively.*
*Moreover, if $H^{'}$ is another right angled hexagon with this property, then there exists a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that $A(H^{'}) = H^{'}$.*

*Proof.* Let us start with the existence.
Let $\gamma_{im}$ denote the positive imaginary axis and set $B = \{z \in \mathbb{H}^2; d(z, \gamma_{im}) = c\}$.
$B$ is a one-dimensional submanifold of $\mathbb{H}^2$.
Because the map $z \to \lambda z$ is an isometry that preserves $\gamma_{im}$ for every $\lambda > 0$, it must also preserve $B$.
This means $B$ is a straight line.
Now construct the following picture:



That is, we take the geodesic though the point $i \in \mathbb{H}^2$ perpendicular to $\gamma_{im}$ and at a distance $a$ draw a perpendicular geodesic $\gamma$.

Furthermore, for any $p \in B$, we draw the geodesic $\alpha$ that realizes a right angle with the perpendicular from $p$ to $\gamma_{im}$.

Now let $x = d(\alpha, \gamma) = inf\{d(z, w); z \in \gamma, w \in \alpha\}$.

■ Because of Lemma 1, $x$ is realized by the common perpendicular to $\alpha$ and $\gamma$.

By moving $p$ over $B$, we can realize any positive value for $x$ and hence obtain our hexagon $H(a, b, c)$.

We also obtain uniqueness from the picture above. Indeed, given any right angled hexagon $H^{'}$ with three non-consequtive sides of length $a, b$ and $c$, apply a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that the geodesic segment of length $a$ starts at $i$ and is orthogonal to the imaginary axis. This implies that the geodesic after $a$ gets mapped to the geodesic $\gamma$. Furthermore, one of the endpoints of the geodesic segment of length $c$ needs to lie on the line $B$. We now know that the geodesic $\alpha$ before that point needs to be tangent to $B$. Because $\alpha$ and $\beta$ have a unique common perpendicular. The tangency point of $\alpha$ to $B$ determines the picture entirely. Because the function that assigns the length $x$ of the common perpendicular to the tangency point is injective, we obtain that there is a unique solution. □

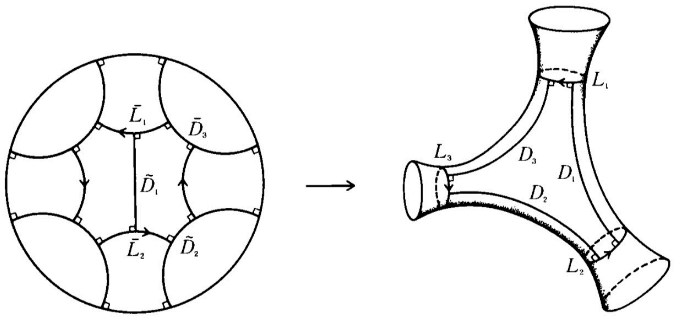
Now we prove the uniqueness of the pair of pants.

*Proof.* There exists a unique orthogonal geodesic between every pair of boundary components of $P$.

These three orthogonals decompose $P$ into right-angled hexagons out of which three non-consecutive sides are determined. Lemma 2 now tells us that this determines the hexagons up to isometry and this implies that $P$ is also determined up to isometry.

□

### 1.7.2 Reflection




Every pair of pants $P$ has an *anti-holomorphic* automorphism $J_p$ of order 2.

Moreover, the set $F_{J_P} = \{z \in P | J_P(z) = z\}$ of all fixed points of $J_P$ consists of three geodesics $\{D_j\}_{j=1}^3$ in $P$ satisfying the following condition:

For every $j(j = 1, 2, 3)$, $D_j$ has the endpoints on, and is orthogonal to, both $L_j$ and $L_{j+1}$, where $L_4 = L_1$

We call $J_P$ the **reflection** of $P$.

## 1.8 Normalized Fuchsian Model

If a universal covering surface $\tilde{R}$ of a Riemann surface ,$R$ is the upper half-plane $\mathbb{H}^2$, we call its universal covering transformation group $\Gamma$ a **Fuchsian model**.

■ Let $[R, \Sigma] \in T_g(\geq 2)$, where $\Sigma = \{[A_j], [B_j]\}_{j=1}^g$ is a marking on $R$, i.e., a canonical system of generators of the fundamental group $\pi_1(R, p_0)$ of a closed Riemann surface $R$ of genus $g$.

■ Under the isomorphism between $\pi_1(R, p_0)$ and a Fuchsian model $\Gamma$, denote by $\alpha_j$ and $\beta_j$ the elements of $\Gamma$ corresponding to $[A_j]$ and $[B_j]$ in $\pi_1(R, p_0)$, respectively, for each $j = 1, 2, 3, ...g$

■ Note: $\forall \delta \in Aut(\mathbb{H}^2)$, the group $\Gamma' = \delta \Gamma \delta^{-1}$ is a Fuchsian model of the same $R$.

■ Therefore in order to assign uniquely a Fuchsian model $\Gamma$ to a given marking $\Sigma$ on $R$, we impose the normalization conditions:

(i) $\beta_g$ has its repelling and attractive fixed points at 0 and $\infty$, respectively.

(ii) $\alpha_g$ has its attractive fixed point at 1.

**Remark:** For a given marking $\Sigma$ on a closed Riemann surface of a genus g, there always exists a Fuchsian model of $R$ which satisfies the normalization conditions.

**Reason:** 1) We have a result which says that:- Every element of a Fuchsian model of a closed Riemann surface of genus g ( $\geq 2$ ) consists only of the identity and the *hyperbolic elements.*

So, both $\alpha_g$ and $\beta_g$ are hyperbolic.

2) We have another result: Let $\gamma$ and $\delta$ be two elements of a Fuchsian group $\Gamma$. If $\gamma$ is *hyperbolic* and $\delta \neq id$, then one of te following holds:

(i) Fix($\gamma$)=Fix($\delta$). (ii) Fix($\gamma$) $\cap$ Fix($\delta$)= $\phi$

Now since $\alpha_g$ and $\beta_g$ are not commutative, by the above result we have have that Fix($\alpha_g$)$\cap$Fix($\beta_g$) $= \phi$. So, let $a$ (attractive fixed point), $b$ (repelling fixed point) $\in$ Fix($\beta_g$) and $c$ (attractive fixed point distinct from $a$ and $b$) $\in$ Fix($\alpha_g$). Then consider the fuchsian element $\delta$ such that $\delta(b) = 0, \delta(a) = \infty, \delta(c) = 1$. Then we can replace $\beta_g$ and $\alpha_g$ with $\delta\beta_g\delta^{-1}$ and $\delta\alpha_g\delta^{-1}$.

**Proposition 1.12.** *For a given marking $\Sigma$ on a closed Riemann surface $R$ of genus $g$ ($\geq 2$), a canonical system of generators $\{\alpha_j, \beta_j\}_{j=1}^{g}$ of a Fuchsian model $\Gamma$ of $R$ which satisfies the normalization conditions with respect to $\Sigma$ is* ***uniquely*** *determined by the point $[R, \Sigma]$ in $T_g$.*

We call this Fuchsian group $\Gamma$ the *normalized Fuchsian model* of a marked closed Riemann surface $[R, \Sigma]$.

The system of generators $\{\alpha_j, \beta_j\}_{j=1}^{g}$ is referred to as its *canonical system of generators*, which satisfies the sole fundamental relation:-

$\prod_{j=1}^{g}[\alpha_j, \beta_j] = id$, where $[\alpha_j, \beta_j] = \alpha_j \circ \beta_j \circ \alpha_j^{-1} \circ \beta_j^{-1}$.

*Proof.* Take another closed Riemann surface $R^{'}$ of genus $g$ and a marking $\Sigma^{'}$ on it such that $[R, \Sigma] = [R^{'}, \Sigma^{'}]$ in $T_g$.

Then there exists a biholomorphic mapping $f : R \to R^{'}$ such that $f_*(\Sigma)$ is equivalent to $\Sigma^{'}$.

A lift $\tilde{f}$ of $f$ to H, which is an element of $Aut(H)$, is taken to satisfy:-

$\alpha_j^{'} = \tilde{f} \circ \alpha_j \circ \tilde{f}^{-1}$

and

$\beta_j^{'} = \tilde{f} \circ \beta_j \circ \tilde{f}^{-1}$,

where $\{\alpha_j^{'}, \beta_j^{'}\}_{j=1}^{g}$ is the canonical system of generators of a Fuchsian model of $R^{'}$ which satisfies the normalization conditions with respect $\Sigma^{'}$.

From condition (i), we have $\tilde{f}(z) = \lambda z$ for some $\lambda > 0$.

Further, by condition (ii), $\alpha_g$ and $\alpha_g^{'}$ have the common fixed point at 1,and hence $\lambda = 1$, i.e, $\tilde{f} = id$.

Thus we get $\alpha_j = \alpha_j^{'}$ and $\beta_j = \beta_j^{'}$.

□

## 1.9 Fricke Coordinates

**Proposition 1.13.** . *Let $\{\alpha_j, \beta_j\}_{j=1}^{g}$ be the canonical system of generators of the normalized Fuchsian model $\Gamma$ for a point $[R, \Sigma]$ in $T_g$.*

*If an element $\gamma(z) = \frac{az+b}{cz+d}$ of $\{\alpha_j, \beta_j\}_{j=1}^{g}$ does not coincide with $\beta_g$, then*

$\boldsymbol{bc \neq 0}$.

*Proof.* **Case:** $b = c = 0$, we have Fix($\gamma$)=Fix($\beta_g$)=$\{0, \infty\}$, and hence $\gamma$ and $\beta_g$ are commutative, a contradiction..

**Case:** $b = 0$ **and** $c \neq 0$: we get Fix($\gamma$)=Fix($\beta_g$)=$\{0\}$. Thus, $\gamma$ and $\beta_g$ being non-commutative.

Now by the result (ii) in the previous remark, we have that $\Gamma$ is not Fuchsian. Hence we have a contradiction. By the same argument, in the case where $b \neq 0$ and $c = 0$, we obtain a contradiction.

□

By this proposition, the canonical system $\{\alpha_j, \beta_j\}_{j=1}^{g}$ of generators of the normalizd Fuchsian model $\Gamma$ for a point $[R, \Sigma]$ in $T_g$ is written uniquely in the form,

$\alpha_j = \frac{a_j z + b_j}{c_j z + d_j}, a_j, b_j, c_j \in R, c_j > 0, a_j d_j - b_j c_j = 1,$

$\beta_j = \frac{a_j' z + b_j'}{c_j' z + d_j'}, a_j', b_j', c_j' \in R, c_j' > 0, a_j' d_j' - b_j' c_j' = 1$ for each $j = 1, 2, ..., g-1$

**Definition 1.14** (Fricke Coordinates)**.** $\mathfrak{F}_\mathfrak{g} : T_g \to R^{6g-6}$ *by*

$\mathfrak{F}_\mathfrak{g}([R, \Sigma]) = (a_1, c_1, d_1, a_1', c_1', d_1', ..., a_{g-1}, c_{g-1}, d_{g-1}, a_{g-1}', c_{g-1}', d_{g-1}')$

Fricke Space The image $F_g = \mathfrak{F}_g(T_g)$ is called the **Fricke space** of closed Riemann surface of genus $g$.

The topology of $F_g$ is introduced by the relative topology of $F_g$ in $R^{6g-6}$

### 1.9.1 Injectivity of Fricke Coordinates

**Theorem 1.15.** *The Fricke coordinates* $\mathfrak{F}_g : T_g \to R^{6g-6}$ *is injective.*

*Proof.* We need to show that every point $\mathfrak{F}_g([R, \Sigma]) = (a_1, c_1, d_1, a_1', c_1', d_1', ..., a_{g-1}, c_{g-1}, d_{g-1}, a_{g-1}', c_{g-1}', d$ in $F_g$ determines uniquely the canonical system $\{\alpha_j, \beta_j\}$ of generators of the normalized Fuchsian model $\Gamma$ for the point $[R, \Sigma] \in T_g$.

For each $j(j = I, 2, ..., 9 - l)$, $b_j$ is obtained from the relation $a_j d_j - b_j c_j = 1$ with $c_j > 0$, and hence $\alpha_j$ is determined uniquely by $\mathfrak{F}_g([R, \Sigma])$. By the same argument, $\beta_j(j = 1, 2, ..., g-1)$ is also determined.

■ What remains to show is that both $\alpha_g$ and $\beta_g$ are determined by $\mathfrak{F}_g([R, \Sigma])$.

By the normalization condition (i) for $\Gamma$, we have $\beta_g = \lambda z$ with $\lambda > 1$.

By the normalization condition (ii) for $\Gamma$, $\alpha_g$ has its attractive fixed point at 1, and hence $a_g + b_g = c_g + d_g$ ———— (0)

■ The fundamental relation : $\prod_{j=1}^{g} [\alpha_j, \beta_j] = id.$

Now putting $\gamma = \prod_{j=1}^{g-1} [\alpha_j, \beta_j]$, we have $\gamma \circ \alpha_g = \beta_g \circ \alpha_g \circ \beta_g^{-1}$.

Set $\gamma(z) = \frac{az+b}{cz+d}, a, b, c, d \in R, ad - bc = 1$

Replacing $a, b, c, andd$ by $-a, -b, -c, and - d$, respectively, if necessary, we may assume that the following equations hold:

$(a-1)a_g + bc_g = 0, - - - - - - - - - - - - (1)$

$ca_g + (d - \lambda^{-1})c_g = 0,$ – – – – – – – – – – – (2)

$cb_g + (d-1)d_g = 0.$ ———— (3)

Since at least one of $a_g$ or $c_g$ does not vanish, from (1) and (2) we have $a - 1 = \lambda(1-d)$.

If $a = 1$, then $d = 1$, and hence $tr^2(\lambda) = 4$, which implies that $\lambda$ is parabolic which is a contradiction.

Thus it follows that $a \neq 1, d \neq 1$, and $\lambda = \frac{a-1}{1-d}$.

Hence we determined $\beta_g$.

From (1) and (3) , we get

$a_g = \frac{bc_g}{1-a}$ ——— (4)

$d_g = \frac{cb_g}{1-d}$ ———— (5)

Substitution of (4) and (5) into (0) gives

$\frac{a+b-1}{1-a}c_g = \frac{c+d-1}{1-d}b_g$ ——— (6)

Here, if $c + d = 1$, then we have $a + b = 1$, because $c_g \neq 0$ by Proposition 2. Thus, from the relation $ad - bc = 1$, we find that $a + d = 2$, and hence $\gamma$ is parabolic which is again a contradiction. Therefore, we have determined $\alpha_g$ by $\mathfrak{F}_g([R, \Sigma])$.

□

We represented the Teichmüller space $T_g$ of genus $g(\geq 2)$ as a subset $F_g$ (named the Fricke space) of real $(6g-6)$-dimensional Euclidean space, by using Fuchsian models of surfaces.

Now we introduce another type of coordinates to $T_g$ by using hyperbolic geometry.

### 1.9.2 Pants Decomposition

Fix a point $[R, \Sigma]$ of $T_g$.

A set $\mathfrak{L}$ of **mutually disjoint simple closed geodesics** on $R$ is termed **maximal** if there is no set $\mathfrak{L}'$ which includes $\mathfrak{L}$ properly. We call a maximal set $\mathfrak{L} = \{L_j\}_{j=1}^N$ of mutually disjoint simple closed geodesics on $R$ a **syslem of decomposing curves**, and the family $\mathfrak{P} = \{P_k\}_{k=1}^M$ consisting of all connected components of $R - \cup_{j=1}^N L_j$ the **pants decomposition** of $R$ corresponding to $\mathfrak{L}$.

### 1.9.3 Geodesic Length Functions

Fix a point $[R, \Sigma]$ of $T_g$, and a system $\mathfrak{L} = \{L_j\}_{j=1}^N$ of decomposing curves on $R$.

For every $t$ in the Fricke space $F_g$, we denote by $[R_t, \Sigma_t]$ the point in $T_g$ corresponding to $t$.

■ Take a marking-preserving homeomorphism $f_t : R \to R_t$.

■ For every $L_j$ in $\mathfrak{L}$, let $L_j(t)$ be the unique closed geodesic in the free homotopy class of the closed curve $f_t(L_j)$ on $R_t$.

■ $\mathfrak{L}_t = \{L_j(t)\}_{j=1}^N$ is a system of decomposing curves on $R_t$.

For every $t$ in $F_g$ and every $j$, we denote the hyperbolic length $l(L_j(t))$ of $L_j(t)$ simply by $l_j(t)$. We consider $l_j(t)$ as a function on $F_g$ (or equivalently, on $T_g$) and call it the *geodesic length funcion* for $L_j$.

### 1.9.4 Real-analyticity of the length function on $F_g$

$\pi : \Delta \to R$ ( $R = \Delta/\Gamma$, $\Gamma$ is the Fuchsian model of $R$ on $\Delta$ )

■ Every $\gamma \in \Gamma$ corresponds to an element $[C_\gamma]$ of the fundamental group $\pi_1(R, p_0)$ of $R$. In particular, $\gamma$ determines the free homotopy class of $C_\gamma$, where $C_\gamma$ is a representative of the class $[C_\gamma]$.

■ We say that $\gamma$ *covers the closed curve* $C_\gamma$.

When $\gamma \in \Gamma$ is hyperbolic, the closed curve $L_\gamma = A_\gamma / < \gamma >$, the image on $R$ of the axis $A_\gamma$ by $\pi$, is the unique geodesic (with respect to the hyperbolic metric on $R$) belonging to the free homotopy class $C_\gamma$.

■ We call $L_\gamma$ the *closed geodesic corresponding to* $\gamma$, *or to* $C_\gamma$.

Let $R$ be a Riemann surface with universal covering surface $H$, and $\Gamma_1$ be a Fuchsian model of $R$ acting on H.

Let

$\gamma(z) = \frac{az+b}{cz+d}, a, b, c, d \in R, ad - bc = 1,$

be a hyperbolic element of $\Gamma_1$, and $L_\gamma$ be the closed geodesic on $R$ corresponding to $\gamma$.

Then,

$l(L_\gamma) = 2\log$ a.

*Proof.* Since $l(L_\gamma)$ is invariant under the conjugation of $\gamma$ by a element of $Aut(H)$, we may assume that $\gamma(z) = \lambda z$ $(\gamma > 1)$.

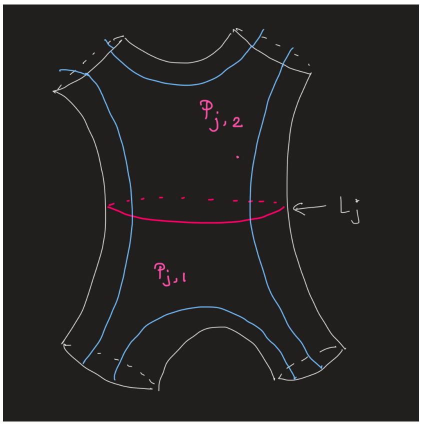
We may assume that $a = \sqrt{\lambda}, b = c = 0$, and $d = \frac{1}{\sqrt{\lambda}}$. In this case, we have

$l(L_\gamma) = \int_1^\lambda \frac{dy}{y} = \log \lambda = 2\log a$.

$\square$

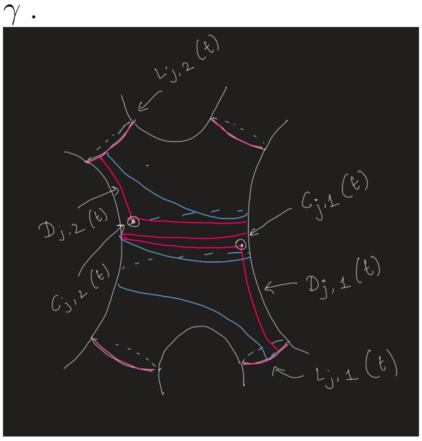
Every geodesic length function $l_j(t)$ is real-analytic on $F_g$.

## 1.10 Twisting Parameters

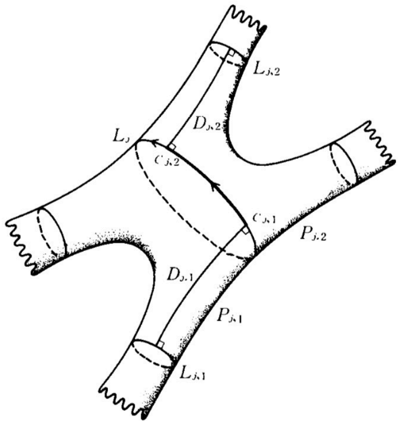


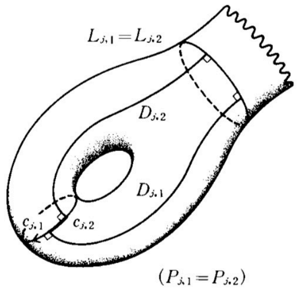

Choose simple arcs joining boundary components of each pair of pants.

Out of the two blue arcs intersecting $L_j$ choose the right one in the picture and name it $\gamma'$.




Here $f_t(\gamma')$ loops around the union of a pair of pants once.




For $j$, let $P_{j,1}$ and $P_{j,2}$ be two pairs of pants in $\mathfrak{P}$ having $L_j$ as a boundary component. Here we allow the case where $P_{j,1} = P_{j,2}$.

$P_{j,1}$ and $P_{j,2}$ admit the reflection $J_1$ and $J_2$, respectively.

■ Take a fixed point of $J_k$ on $L_j$ for each $P_{j,k}$ (k=1,2), and denote it by $c_{j,k}$.

■ Fix also an orientation on $L_j$.

As before, let $[R_t, \Sigma_t]$ be the point of $T_g$ corresponding to $t$ in $F_g$.
For every $t$ and $j$, let $P_{j,1}(t)$ and $P_{j,2}(t)$ be the connected components of $R_t - \cup_{j=1}^{N} L_j(t)$ (which are pair of pants of $R_t$) corresponding to $P_{j,1}$ and $P_{j,2}$, respectively.
Each $c_{j,k}$ (k=1,2) is the end point on $L_j$ of the geodesic $D_{j,k}$ joining $L_j$ and another boundary component, say $L_{j,k}$, in $P_{j,k}$.
Let $L_{j,k}(t)$ be the boundary component of $P_{j,k}(t)$ corresponding to $L_{j,k}$.
Denote by $D_{j,k}(t)$ the geodesic joining $L_j(t)$ and $L_{j,k}(t)$ in $P_{j,k}(t)$ with minimal length, and by $c_{j,k}(t)$ the point of $D_{j,k}(t)$ on $L_j(t)$.
Then each $c_{j,k}(t)$ (k=1,2) is a fixed point of the reflection of $P_{j,k}(t)$.

Let $T_j(t)$ be the oriented arc on $L_j(t)$ from $c_{j,1}(t)$ to $c_{j,2}(t)$. Since $L_j(t)$ has the natural orientation determined from that of $L_j$, we can define the signed hyperbolic length $\tau_j(t)$ of $T_j(t)$ (so that $\tau_j(t)$ is positive or negative according to whether the orientation of $T_j(t)$ is compatible with that of $L_j(t)$ or not). [10]

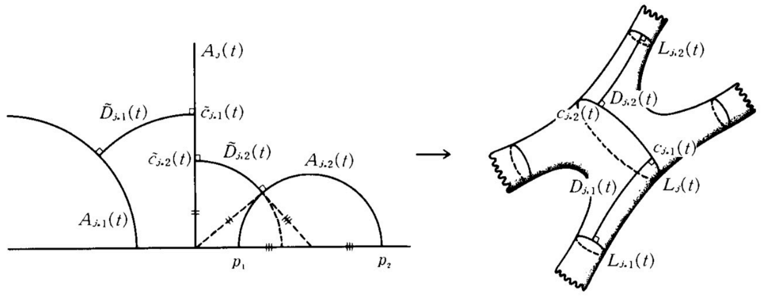
$$\text{Set } \boxed{\theta_j(t) = 2\pi \frac{\tau_j(t)}{l_j(t)}}$$

Then $\theta_j(t)$ is well-defined modulo $2\pi$.

We call $\theta_j(t)$ the *twisting parameter* with respect to $L_j$.

### 1.10.1 Real-analyticity of $\exp(i\theta_j(t))$

**Theorem 1.16.** *For every $j$, exp($i\theta_j(t)$) is well-defined and real-analytic on $F_g$.*



*Proof.* Fix $j$.

For every $t$ in $F_g$, let $\Gamma_t$ be the Fuchsian group represented $t$.

■ Take an element of $\Gamma_t$ which covers $L_j(t)$, and denote it by $\gamma_j(t)$.

Next, for each $k$ (=1,2), let $\gamma_{j,k}(t)$ be the element of $\Gamma_t$ which *covers* $L_{j,k}(t)$ and satisfies that the geodesic $\tilde{D}_{j,k}(t)$, connecting $A_j(t)$ and $A_{j,k}(t)$ with the minimal length, is projected onto $D_{j,k}(t)$, where $A_j(t)$ and $A_{j,k}(t)$ are the axes on $\gamma_j(t)$ and $\gamma_{j,k}(t)$, respectively. Here, we may assume that the fixed points of $\gamma_j(t)$, $\gamma_{j,1}(t)$, and $\gamma_{j,2}(t)$ move real-analytically on $F_g$.

Hence, when we take a conjugation of $\Gamma_t$ by an element of $Aut(H)$ so that $\gamma_j(t)$ goes to $\tilde{\gamma}_j(t)(z) = \lambda_{j(t)}z$ ($\lambda_j(t) > 1$), the fixed points of $\tilde{\gamma}_{j,k}(t)$ corresponding to $\gamma_{j,k}(t)$ move also real-analytically on $F_g$ for each $k$.

Now, $c_{j,k}(t)$ is the projection of the end point $\tilde{c}_{j,k}(t)$ of $\tilde{D}_{j,k}(t)$ to $A_j(t)$.

■ Hence, if we show that $\tilde{c}_{j,k}(t)$ moves real-analytically on $F_g$, the assertion follows by the definition of $\tau_j(t)$ and the analyticity of $l_j(t)$.

To show this, fix $k$, and let $p_1$ and $p_2$ be the fixed points of $\tilde{\gamma}_{j,k}(t)$. Set $\tilde{c}_{j,k}(t) = iy_k$ ($y_k > 0$).

Since

$$\boxed{y_k^2 + \left(\frac{p_1 - p_2}{2}\right)^2 = \left(\frac{p_1 + p_2}{2}\right)^2}$$

we see real-analyticity of $\tilde{c}_{j,k}(t)$.

□

# Chapter 2

# Complex Structure on Teichmüller Spaces

## 2.1 Steps for the construction of the complex structure

- First we will represent the Teichmüller Space $T(\Gamma)$ by quasiconformal mappings of the Riemann sphere $\hat{\mathbf{C}}$ which are conformal on the lower half-plane $H^*$
- Now, with the help of Schwarzian derivatives, we will construct an embedding (Ber's embedding), the image ( $T_B(\Gamma)$ ) of which will inherit the complex manifold structure of $A_2\left(H^*, \Gamma\right)$
- Then identifying $T(\Gamma)$ with $T_B(\Gamma)$, we get the intended complex structure.

## 2.2 Bers' Embedding

We will start with a concept called Simultaneous Uniformization.

For a given element $\mu \in B(H, \Gamma)_1$, i.e., a Beltrami coefficient $\mu$ on $H$ for $\Gamma$, we set

$$\tilde{\mu}(z) = \begin{cases} \mu(z), & z \in H \\ 0, & z \in \mathbf{C} - H. \end{cases}$$

By the below theorem, there exists uniquely a canonical $\tilde{\mu}$-qc mapping of $\hat{\mathbf{C}}$, i.e., a quasiconformal mapping of $\hat{\mathbf{C}}$ which has the complex dilatation $\tilde{\mu}$, and leaves 0 , 1 , and $\infty$ fixed, respectively.

We denote the quasiconformal mapping by $w_\mu$.

$L^\infty(D)$: the complex Banach space consisting of all bounded measurable functions on a domain $D$.

$||\mu||_\infty = ess.sup_{z \in D}|\mu(z)|,\ \mu \in L^\infty(D)$

$B(D)_1 = \{\mu \in L^\infty(D) |\ ||\mu||_\infty < 1\}$, the unit open ball of $L^\infty(D)$.

**Theorem 2.1.** *For every Beltrami coefficient* $\mu \in B(\mathrm{C})_1$*, there exists a homeomorphism* $f$ *of* $\hat{\mathbf{C}}$ *onto* $\hat{\mathbf{C}}$ *which is a quasiconformal mapping of* $\mathbf{C}$ *with complex dilatation* $\mu$*.*

*Moreover,* $f$ *is uniquely determined by the following normalization conditions:*

$$f(0) = 0, \quad f(1) = 1, \quad \textit{and} \quad f(\infty) = \infty$$

*We call this* $f$*, uniquely determined by the normalization conditions, the canonical* $\mu - q$ *c mapping of* $\hat{\mathbf{C}}$*, or the canonical quasiconformal mapping of* $\hat{\mathbf{C}}$ *with complex dilatation* $\mu$*, and denote it by* $f^\mu$*.*

**Theorem 2.2.** *Let* $\mu$ *be an arbitrary element of* $B(H)_1$*. Then there exists a quasiconformal mapping* $w$ *of* $H$ *onto* $H$ *with complex dilatation* $\mu$*.*

*Moreover, such a mapping* $w$ *(which can be extended to a homeomorphism of* $\bar{H} = H \cup \hat{\mathbf{R}}$ *onto itself by Corollary A.12 in Appendix) is uniquely determined by the following normalization conditions:*

$$w(0) = 0, \quad w(1) = 1, \quad \textit{and} \quad w(\infty) = \infty$$

*We call this unique* $w$ *satisfying the normalization conditions the canonical* $\mu - qc$ *mapping of* $H$*, and denote it by* $w^\mu$*.*

The quasiconformal mapping $w_\mu$ induces a quasiconformal mapping of $R = H/\Gamma$ to $R_\mu = H_\mu/\Gamma_\mu$ and a biholomorphic mapping of $R^* = H^*/\Gamma$ to $H^*_\mu/\Gamma_\mu$, where $R^*$ is the mirror image of $R = H/\Gamma$. Since two Riemann surfaces $R_\mu$ and $R^*$ are represented by $H_\mu/\Gamma_\mu$ and $H^*_\mu/\Gamma_\mu$, respectively, two Riemann surfaces $R_\mu$ and $R^*$ are uniformized simultaneously by a single quasi-Fuchsian group $\Gamma_\mu$. This is called Bers' simultaneous uniformization.

In particular, for any two closed Riemann surfaces $R$ and $S$ of genus $g$, we find a quasi-Fuchsian group $\Gamma_\mu$ which uniformizes simultaneously $R$ and $S$. In fact, pick a Fuchsian model $\Gamma$ of the mirror image $R^*$ of $R$, and take a quasiconformal mapping $f$ of $R^*$ to $S$.

Set $\mu = \mu_f$, the Beltrami coefficient of $f$. Then by Bers' simultaneous uniformization, we conclude that $S$ and the mirror image $R$ of $R^*$ are biholomorphic to $H_\mu/\Gamma_\mu$ and $H_\mu^*/\Gamma_\mu$, respectively.

Before we move on to the construction of the complex structure, let us review the concept of *Teichmüller distance*:

## 2.3 *Teichmüller Distance*

Let us take two points $p_1 = [S_1, f_1], p_2 = [S_2, f_2] \in T(R)$. Now define $d(p_1, p_2) = inf_{g \cong_{homotopic} f_2 \circ f_1^{-1}} log\ K(g)$, where $K(g)$ is the maximal dilation of $g$ (i.e., that of a lift of $g$).
We call this the *Teichmüller distance* on $T(R)$ between $p_1$ and $p_2$.
For verifications of the axioms of distance function one can refer page no. 125 of the book, "An Introduction to *Teichmüller* Spaces" by Y. Imayoshi and M. Taniguchi.

**Theorem 2.3** (Completeness of $T(R)$)**.** *The Teichmüller Space is complete w.r.t the metric defined above.*

**Theorem 2.4** (The *Teichmüller* space of $R$, $T(R)$ is independent of the base point chosen)**.** *Consider the base changing map, or the base point translating map* $[f_1]_* : T(R) \to T(R_1)$ *which takes* $[R, f]$ *to* $[R_1, f \circ f_1^{-1}]$.
*This map is then an isometrical homeomorphism w.r.t. the Teichmüller distances.*
*In particular,* $T(R) \cong_{homeomorphism} T(R_1)$.

**Lemma 2.5.** *For any two elements* $\mu, \nu \in B(H, \Gamma)_1$*, the following are equivalent:*

*(i)* $w^\mu = w^\nu$ *on* $\mathbf{R}$.

*(ii)* $w_\mu = w_\nu$ *on* $H^*$.

This theorem motivates us to define an equivalence relation:
For two elements $\mu, \nu \in B(H, \Gamma)_1, w_\mu$ and $w_\nu$ are said to be equivalent if $w_\mu = w_\nu$ on $H^*$.
Denote by $[w_\mu]$ the equivalence class of $w_\mu$ for every element $\mu \in B(H, \Gamma)_1$. Let $T_\beta(\Gamma)$ be

the set of these equivalence classes $[w_{\mu}]$.

Let's define a map from $T(\Gamma)$ to $T_\beta(\Gamma)$: $[w^\mu] \mapsto [w_\mu]$

By the above lemma this is a bijection.

The topology of $T_\beta(\Gamma)$ is induced from that of $T(\Gamma)$ under this correspondence. In other words, this correspondence gives a homeomorphism of $T(\Gamma)$ onto $T_\beta(\Gamma)$. In this way, we can identify $T_\beta(\Gamma)$ with $T(\Gamma)$ as topological spaces. We also call $T_\beta(\Gamma)$ the Teichmüller space of $\Gamma$.

Let $\beta$ be a mapping of $B(H,\Gamma)_1$ onto $T_\beta(\Gamma)$ given by $\beta(\mu) = [w_\mu]$. Then by the definition of topology of $T_\beta(\Gamma)$, we immediately obtain the following.

**Proposition 2.6.** *The mapping $\beta : B(H,\Gamma)_1 \to T_\beta(\Gamma)$ is a continuous surjection.*

## 2.4 Schwarzian Derivative

**Definition 2.7** (The Schwarzian Derivative)**.** *Let $f$ be analytic on a domain in $\mathbf{C}$. We define the Schwarzian derivative $\{f,z\}$ of $f$ by :*

$$\{f,z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

**Definition 2.8** (Norm of the Schwarzian Derivative)**.**

$$\|\{f,z\}\|_\infty = \sup_{z\in H^*} (\operatorname{Im} z)^2 |\{f,z\}|$$

**This norm is the ingredient for many classical results, in particular regarding the theory of Univalnent functions.*

### 2.4.1 Motivation

- We mainly study it because it's invariant under linear fractional transformations (i.e. Mobiüs transformations).

- If $f$ is a Mobiüs transformation then

$$\|\{f,z\}\|_2 = \sup_{z\in\mathbf{D}} (\operatorname{Im} z)^2 |\{f,z\}| = 1.$$

So, $\|\{f, z\}\|_2$ keeps the ability to measure how far the function $f$ is from being a Mobiüs transformation.

- The expression for the Schwarzian derivative itself is motivated by the fact that if we calculate the schwarzian derivative of a Mobiüs transformation then it comes out to be zero, and hence also directly keeps the potential to measure the difference of a conformal mapping on $H^*$ from a Möbius transformation.
- To be more context-specific, by using Schwarzian derivatives, we prove that the Teichmüller space $T(\Gamma)$ is realized as a bounded domain $T_B(\Gamma)$ in the space $A_2\left(H^*/\Gamma\right)$ of holomorphic quadratic differentials on the Riemann surface $H^*/\Gamma$, where $H^*$ is the lower half-plane.

**Lemma 2.9.** *If $f$ and $g$ are conformal mappings of $D$ and $f(D)$, respectively, then*

$$\{g \circ f, z\} = \{g, f(z)\} \cdot f'(z)^2 + \{f, z\}, \quad z \in D.$$

*Moreover, a conformal mapping of $D$ is a Möbius transformation if and only if $\{f, z\} = 0$ on $D$.*

**Lemma 2.10** (Nehari and Kraus)**.** *Every univalent function on $H^*$ satisfies the inequality*

$$\|\{f, z\}\|_\infty = \sup_{z \in H^*} (\operatorname{Im} z)^2 |\{f, z\}| \leqq \frac{3}{2}$$

## 2.5 Bers' Embedding

Set

$$\varphi_\mu(z) = \{w_\mu, z\}, \quad z \in H^*$$

for arbitrary $\mu \in B(H, \Gamma)_1$.

This $\varphi_\mu$ can be regarded as a holomorphic quadratic differential on a Riemann surface $H^*/\Gamma$, as it satisfies:

$$\varphi_\mu(\gamma(z))\gamma'(z)^2 = \varphi_\mu(z), \quad z \in H^*$$

for all $\gamma \in \Gamma$. (i.e. $\varphi_\mu$ is a holomorphic form w.r.t $\Gamma$)

**Lemma 2.11.** *For any two elements* $\mu, \nu \in B(H, \Gamma)_1, [w_\mu] = [w_\nu]$ *in* $T_\beta(\Gamma)$ *if and only if* $\varphi_\mu = \varphi_\nu$ *on* $H^*$.

**Definition 2.12** (Bers' Embedding). $\mathcal{B} : T_\beta(\Gamma) \to A_2(H^*, \Gamma) : \mathcal{B}([w_\mu]) = \varphi_\mu = \{w_\mu, z\}$, *the Schwarzian derivative of* $w_\mu$ *on* $H^*$.

The well-definedness and the injectivity of this embedding come from the above lemma.

## 2.6 Hyperbolic $L^\infty$-norm on $A_2(H^*, \Gamma)$

Before we introduce the norm let us mention a property of the already known Poincaré metric, $ds_{H^*}{}^2 = |dz|^2/(\operatorname{Im} z)^2$ on $H^*$ :
It is invariant under $\mathrm{PSL}(2, \mathbf{R})$, i.e.

$ds_{H^*}{}^2 = |dz|^2/(\operatorname{Im} z)^2 = |dz|^2/(\operatorname{Im}(\gamma(z)))^2$
We also have,

$$\varphi_\mu(\gamma(z))\gamma'(z)^2 = \varphi_\mu(z), \quad z \in H^*$$

for all $\gamma \in \Gamma$.

Both these equations give us that:

$$(\operatorname{Im}\gamma(z))^2|\varphi(\gamma(z))| = (\operatorname{Im} z)^2|\varphi(z)|, \quad z \in H^*, \gamma \in \Gamma$$

and for every element $\varphi \in A_2(H^*, \Gamma)$.
With this equality in mind, we can think of $(\operatorname{Im} z)^2|\varphi(z)|$ as a function on $R^* = H^*/\Gamma$.

**Definition 2.13** (Hyperbolic $L^\infty$-norm). *For* $\varphi \in A_2(H^*, \Gamma)$,

$$\|\varphi\|_\infty = \sup_{z\in H^*} (\operatorname{Im} z)^2|\varphi(z)|$$

Instead of taking the supremum on the whole of $H^*$, as $R^*$ is compact, we can pick a relatively compact subset in $H^*$ as such a domain.
So, $\|\varphi\|_\infty$ is finite for any $\varphi \in A_2(H^*, \Gamma)$. This makes $A_2(H^*, \Gamma)$ into a complex Banach space with this Hyperbolic $L^\infty$-norm.

**Proposition 2.14.** *Bers' embedding $\mathcal{B} : T_\beta(\Gamma) \to A_2(H^*, \Gamma)$ is continuous.*

**Proposition 2.15.** *Bers' embedding $\mathcal{B} : T_\beta(\Gamma) \to A_2(H^*, \Gamma)$ is a homeomorphism on to its image.*

*Proof.* We know that the Teichmüller space $T(\Gamma)$ is homeomorphic to $\mathbf{R}^{6g-6}$. Also, previously we showed that $T_\beta(\Gamma)$ is homeomorphic to $T(\Gamma)$, and hence it too is homeomorphic to $\mathbf{R}^{6g-6}$.
Now, Brouwer's theorem on the invariance of domains implies that the image $T_B(\Gamma)$ of the continuous injection $\mathcal{B} : T_\beta(\Gamma) \to A_2(H^*, \Gamma)$ is a domain in $A_2(H^*, \Gamma)$.
We also have that $\mathcal{B} : T_\beta(\Gamma) \to T_B(\Gamma)$ is a homeomorphism.
Here, $T_B(\Gamma)$ inherits the complex manifold structure of $(3g-3)$-dimensional complex vector space, $A_2(H^*, \Gamma)$.
As the spaces $T(\Gamma), T_\beta(\Gamma)$, and $T(R)$ are identified with $T_B(\Gamma)$, they too can be considered as $(3g-3)$-dimensional complex manifolds, where $R = H/\Gamma$

□

# Chapter 3

# Teichmüller Modular Groups

## 3.1 Definition of Teichmüller Modular Groups

Teichmüller Modular Group, $\mathrm{Mod}(R)$ = Quasiconformal self-mappings of $R/\sim$, where two Quasiconformal mappings are related iff they are homotopic.

Let us denote the equivalance classes (or the elements of the group) by $[g]$, where g is a quasiconformal self-mapping of $R$.

**Definition 3.1** (Action of the elements in $\mathrm{Mod}(R)$ )**.** *The action of the element* $[g] \in \mathrm{Mod}(R)$, $[g]_*$ *on* $T(R)$*:*

$$[g]_* ([S, f]) = \left[S, f \circ g^{-1}\right]$$

*for every* $[S, f] \in T(R)$

Let us call such $[g]_*$ a Teichmüller modular transformation of $T(R)$.

Now let us consider the below commutative diagrams:

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\omega_1"] \arrow[d, "\pi"'] & \mathbf{H} \arrow[d, "\pi"] \\ \mathbf{R} \arrow[r, "f_1"'] & \mathbf{R} \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\omega_2"] \arrow[d, "\pi"'] & \mathbf{H} \arrow[d, "\pi"] \\ \mathbf{R} \arrow[r, "f_2"'] & \mathbf{R} \end{tikzcd}$$

Here $\omega_i$ is a lift of $f_i$ for $i = 1, 2$.

Also, we have $\omega_i \Gamma (\omega_i)^{-1} = \Gamma$ for $i = 1, 2$.

Now let us note what it means for two elements in Mod($R$) to be equal:

Here $[f_1]$=$[f_2]$ in Mod($R$) $\implies$ $f_1 \circ (f_2)^{-1}$ is homotopic to the identity map $id$.

So $\omega_1 \circ (\omega_2)^{-1}$= $\gamma$ (for some $\gamma \in \Gamma$) holds on $\mathbf{R}$ and vice versa.

With this motivation in mind, we will define a similar group, namely Mod($\Gamma$) which will act on $T(\Gamma)$:

Let us first define an equivalence relation on the Quasiconformal self-mappings of $\mathbf{H}$ i.e. basically translating the above result as our definition for the relation.

For two Quasiconformal self-mappings of $\mathbf{H}$, $\omega_i$, ($i = 1, 2$), satisfying $\omega_i \Gamma (\omega_i)^{-1} = \Gamma$, we say $\omega_1 \sim \omega_2$ iff $\omega_1 = \omega_2 \circ \gamma$ holds on the real axis $\mathbf{R}$ for some $\gamma \in \Gamma$.

Let us denote the equivalence classes by $[\omega]$.

Now with this equivalence relation, we define our new group:

**Definition 3.2** (Teichmüller modular group Mod($\Gamma$) of $\Gamma$ )**.** Mod($\Gamma$)=$\omega$, *Quasiconformal self-mappings of* $\mathbf{H}$ — $\omega_i \Gamma (\omega_i)^{-1} = \Gamma$ / $\sim$, *where '$\sim$' is defined above.*

Having defined the group, now we're going to state how the elements of the group act:

**Definition 3.3** (Action of the elements of Mod($\Gamma$) )**.** *For* $[\omega_0] \in$ Mod($\Gamma$) *, the action* $[\omega_0]_*$ *is defined as follows:*

$[\omega_0]_*([\omega^\mu]) = [\alpha \circ \omega^\mu \circ \omega^{-1}]$ *for* $[\omega^\mu] \in T(\Gamma)$,

*where* $\alpha \in Aut(\mathbf{H})$ *such that* $\alpha \circ \omega^\mu \omega^{-1}$ *fixes each of 0, 1 and* $\infty$*. (Basically, we compose* *with an appropriate mobius transformation $\alpha$ so that the above term becomes canonical and hence becomes an element of $T(\Gamma)$).*

We know that $[\omega]_*$ is a biholomorphic automorphism of $T(\Gamma)$. Also, for all $[\omega] \in Mod(\Gamma)$, the action of it, $[\omega]_*$, is an isometry w.r.t the *Teichmüller* distance.

## 3.2 $Mod(\mathbf{R})$ is isomorphic to $Mod(\Gamma)$

The obvious function 'Φ' from $Mod(\mathbf{R})$ to $Mod(\Gamma)$ is as follows :
$\Phi([f]) = [\omega_f]$ where $\omega_f$ is the lift of $f : R \to R$.

- Φ is a homomorphism: 1) $\Phi([id]) = [\omega_{id}]$ where $\omega_{id}$ is a *Möbius* Transformation. Now, $id \circ \omega_{id} = \omega_{id}$ and hence $[\omega_{id}] = [id]$. Therefore, $\Phi([id]) = [id]$ 2) Claim: $\Phi([f_1] * [f_2]) = \Phi([f_1]) * \Phi([f_2])$ (where the group multiplication operators are denoted with the same symbol, '*' as an abuse of notation.) Proof: We need to prove that $[\omega_{f_1 \circ f_2}] = [\omega_{f_1}]*[\omega_{f_2}]=[\omega_{f_1} \circ \omega_{f_2}]$ Now consider the two commutative diagrams:

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\omega_{f_1 \circ f_2}"] \arrow[d, "\pi"'] & \mathbf{H} \arrow[d, "\pi"] \\ \mathbf{R} \arrow[r, "f_1 \circ f_2"'] & \mathbf{R} \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\omega_{f_2}"] \arrow[d, "\pi"'] & \mathbf{H} \arrow[r, "\omega_{f_1}"] & \mathbf{H} \arrow[d, "\pi"] \\ R \arrow[r, "f_2"'] & R \arrow[r, "f_1"'] & R \end{tikzcd}$$

  Here, $\omega_{f_1 \circ f_2} = \omega_{f_1} \circ \omega_{f_2}$

- Φ is a bijection.

*Proof.* ***Claim*** **:**[Φ is a Surjection]: Suppose $[\omega] \in Mod(\Gamma)$ and now we define a quasiconformal self-mapping $f_\omega$ from $R$ to $R$:

Suppose $x \in R$ and let $z \in \pi^{-1}(x)$ then define $f_\omega(x) = \pi(\omega(z))$. Here, the definition of the function might appear to depend on one of the representatives of the class $[\omega]$, namely $\omega$, but we know that if $\omega_1 = \omega \circ \gamma_0$ on $\mathbf{R}$ where $\gamma_0$ is some element of $\Gamma$, then $[f_{\omega_1}] = [f_{\omega_2}]$ where, $f_{\omega_1}$ *and* $f_{\omega_2}$ are defined in the above sense.
Hence we are done.

***Claim*** **:**[$\Phi$ is a Injection]: Suppose $[f_1], [f_2] \in Mod(\Gamma)$ s.t. $\Phi([f_1]) = \Phi([f_2]) \implies [\omega_{f_1}] = [\omega_{f_2}] \implies$ (by the definition of $Mod(\Gamma)$) $\exists\ \gamma \in \Gamma$ s.t. $\omega_{f_1} = \omega_{f_2} \circ \gamma$ on $\mathbf{R}$, the real axis. or, $\gamma^{-1} \circ \omega_{f_2}^{-1} \circ \omega_{f_2} = id_{\mathbf{H}}$ on $\mathbf{R}$
Here as $\gamma^{-1}$ is a deck transformation of $(\mathbf{H}, R, \pi)$, so $\gamma^{-1} \circ \omega_{f_2}^{-1} \circ \omega_{f_2}$ and $\omega_{f_2}^{-1} \circ \omega_{f_2}$ are both lifts of $f_2^{-1} \circ f_1$.
Now let us state a theorem which we will require to proceed further:

**Theorem 3.4.** *$f : R \to R$ is homotopic to $id_R$ iff $\tilde{f} : \mathbf{H} \to \mathbf{H}$ extends to $id_{\mathbf{H}}$ on $\bar{\mathbf{R}}$.*

So, with the help of this theorem, we conclude that $f_2^{-1} \circ f_1 \cong_{homotopy} id_R$, i.e. $f_1 \cong_{homotopy} f_2$.
Hence, $[f_1] = [f_2]$.
We are done. □

Now we prove the forward direction of the theorem that we used (the other direction can be found on Proposition 6.4.9 of *Teichmüller Theory* and Applications to Geometry, Topology, and Dynamics, Volume 1, by John H. Hubbard. This uses the Douady-Earle Extension, provided in the Appendix):

*Proof.* Here, $R = \mathbf{H}/\Gamma$. And let $\pi : \mathbf{H} \to R$ be the corresponding universal covering map.

$(\implies)$: Let $f_t, t \in [0, 1]$, be a homotopy with $f_0 = id$ and $f_1 = f$.
Define $\bar{f}_t : \mathbf{H} \to \mathbf{H}$ to be a lift of $f$ depending continuously on $t$, such that $\bar{f}_0 = id$,

and set $\bar{f}_1 = \bar{f}$.

Note that for $t$ and all $\gamma \in \Gamma$, the equation $\pi \circ \tilde{f}_t = f_t \circ \pi$ implies that there exists a unique $\gamma_t \in \Gamma$ s.t. $\gamma_t \circ \tilde{f}_t = f_t \circ \gamma$; Note: $\gamma_t$ depends continuously on $t$.

Since $\Gamma$ is discrete, the continuous map $t \to \gamma_t$ must be constant, i.e. $t \to \gamma$.

The equation $\gamma \circ \bar{f} = \bar{f} \circ \gamma$ extends by continuity to $\bar{R}$, and if $x \in \bar{R}$ is a fixed point of $\gamma$,

$\bar{f}(x) = \bar{f}(\gamma(x)) = \gamma(\bar{f}(x))$, so that $\bar{f}(x)$ is also a fixed point of $\gamma$.

We will show that $\tilde{f}(x) = x$.

Case 1: $\gamma$ is *parabolic*: There is only one fixed point of $\gamma$, and hence also fixed by $\tilde{f}$.

$\therefore\ \tilde{f}(x) = x$.

Case 2: $\gamma$ is *hyperbolic*: Suppose $x$ is an *attractive* fixed point of $\gamma$, i.e. $x = \lim_{n\to} \gamma^n(z)$.

Now consider the following equation:

$\tilde{f}(x) = \tilde{f}(\lim_{n\to} \gamma^n(z)) = \lim \tilde{f}(\gamma^n(z)) = \lim \gamma^n(\tilde{f}(z))$. This implies that $\tilde{f}(x)$ is also an *attractive* fixed point of $\gamma$. Now as there can be only one fixed point, we have: $\tilde{f}(x) = x$.

From Corollary A.4 of the Appendix, we have that the fixed point set of $\Gamma$ is dense in the limit set of $\Gamma$. Hence by continuity, $\tilde{f}$ extends to identity on $\bar{R}$. $\square$

- Compatibility of the action: Let $\phi$ be the identification map from $T(R)$ to $T(\Gamma)$, defined as follows: $\phi([S, f]) = [\omega^{\mu_f}]$ where $\mu_f$ is the Beltrami coefficient of $f$. (Here, $\omega^{\mu_f}$ is a lift of $f$.

  We need to prove $\phi([f_0]_*([s, f])) = [\omega_{f_0}]_*([\omega^{\mu_f}])$, or, $\phi([s, f \circ (f_0)^{-1}]) = [\omega_{f_0}]_*([\omega^{\mu_f}])$, or, $[\omega^{f \circ (f_0)^{-1}}]$=$[A \circ \omega^{\mu_f} \circ (\omega_{f_0})^{-1}]$, where $A \in Aut(\Gamma)$

  Now, the Beltrami coefficient of $\omega^{f \circ (f_0)^{-1}}$ is $\frac{f_{0_z}}{\overline{f_{0_z}}} \frac{\mu_f - \mu_{f_0}}{1 - \overline{\mu_{f_0}} \mu_f}$, which is equal to the Beltrami coefficient of $\omega^{\mu_f} \circ (\omega_{f_0})^{-1}$ and hence: $\omega^{f \circ (f_0)^{-1}} = A \circ \omega^{\mu_f} \circ (\omega_{f_0})^{-1}$ where $A \in Aut(\Gamma)$.

## 3.3 Moduli Sets

In this section, we develop some results which will amount to establishing the fact that the set $\{tr^2(\gamma)|\gamma \in \Gamma\}$ is discrete in $\mathbf{R}$, which then will prove to be one of the main ingredients for proving that the *Teichmüller* Modular group acts properly discontinuously on $T(\Gamma)$.

**Lemma 3.5.** *Suppose $K$ is a compact subset in the upper half-plane* $\mathbf{H}$. *Now, if $M$ is a positive number, then there can be at most finitely many elements $\gamma \in \Gamma$ satisfying the following inequality:*
$\min_{s\in K} \rho(z, \gamma(z)) \leqq M$*, where $\rho$ is the Poincaré distance on $H$.*

*of the lemma.* Suppose for the sake of contradiction, we have a sequence of infinite elements ($\in \Gamma$), $\{\gamma_n\}_{n=1}^{\infty}$ (where $\gamma_i \neq \gamma_j \forall i \neq j$) s.t. $\rho(z_n, \gamma_n(z_n)) \leq M$ for some $z_n \in K$.
Here $K$ is compact: Consider the sequence $\{z_n\}_{n=1}^{\infty}$. $\exists$ a subsequence $\{z_{n_i}\}_{i=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ such that $\{z_{n_i}\}_{i=1}^{\infty} \to z_0 \in K$.
Now, we also know that $\rho$ is complete: Hence the subsequence $\{\gamma_{n_i}(z_{n_i})\}_{i=1}^{\infty} \to w_0 \in \mathbf{H}$.
Here, $\{\gamma_{n_i}(z_{n_i})\}_{i=1}^{\infty}$ is a normal family on $\mathbf{H}$ and hence the sequence converges uniformly on the compact sets to a holomorphic function $\gamma_0$.
Then, $\gamma_0(z_0) = w_0$.
Now as $\gamma_0$ is not a constant, it's an element of $Aut(H)$. (Refer to lemma 2.18 of the book Imayushi, Taniguchi )
We know that for a subgroup $\Gamma$ of $Aut(H)$, $\Gamma$ is a Fuchsian group iff there exists no sequences of mutually distinct elements of $\Gamma$ which converge in $Aut(\Gamma)$. (Refer to lemma 2.16 of the book Imayushi, Taniguchi)
Hence as $\{\gamma_{n_i}(z_{n_i})\}_{i=1}^{\infty}$ converges to an element of $Aut(\Gamma)$, we have that $\Gamma$ is not a discrete subgroup of $Aut(\mathbf{H})$, a contradiction. □

**Proposition 3.6** (Discreteness of lengths of closed geodesics)**.** $\mathbf{R}$ *is a closed Riemann surface of genus $g \geq 2$.*

*The set $l(L)$, hyperbolic length of $L$ — $L$ is a closed geodesic in* $\mathbf{R}$ *is discerte in* $\mathbf{R}$.
*Also, suppose 't' is a hyperbolic length,*
*then the set $L$, a closed geodesic of* $\mathbf{R}$ — $l(L) = t$ *is finite.*

*Proof.* Suppose for the sake of contradiction there exists a sequence $\{L_n\}_{n=1}^{\infty}$ of mutually distinct closed geodesics on $\mathbf{R}$, which satisfies the following inequality:
$l(L_n) \leq M$ for some positive number $M$, where $l(L_n)$ is a hyperbolic length of $L_n$.
Choose $\gamma_n \in \Gamma$ such that it covers $L_n$ and $\tilde{F} \cap A_{\gamma_n} \neq \phi$, where $A_{\gamma_n}$ is the axis of $\gamma_n$.
Then $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence of mutually distinct elements of $\Gamma$ such that $\min_{\varepsilon \in F} \rho(z, \gamma_n(z)) = \ell(L_n) \leqq M$ for any $n$.
This contradicts the above lemma.

□

We know that for a closed geodesic $\gamma$ of $\mathbf{R}$, we have $tr^2(\gamma) = 4cosh^2(l(L_\gamma)/2)$.
And we have the following corollary by the lemma 3.4.:

The set $\{\mathrm{tr}^2(\gamma) \mid \gamma \in \Gamma\}$ is discrete in $\mathbf{R}$.
We end the section with a result that will prove to be useful in the next section where we prove the discontinuity of the $Teichmüller\ Modular\ Groups$.

**Theorem 3.7.** *Let $\Gamma$ be a Fuchsian model of a closed Riemann surface of genus $g(\geqq 2)$. Let $\{A_j\}_{j=1}^{m}$ be a system of generators for $\Gamma$ such that $A_1$ has the repelling fixed point 0 and the attractive fixed point $\infty$, and such that $A_2$ has the repelling fixed point $a$ with $a < 0$ and the attractive fixed point 1 . Then each $A_j$ is determined by the absolute values of traces of elements in the finite set*

$$\mathcal{G} = \left\{A_1 \circ A_2, A_j, A_1^{\pm 1} \circ A_k, A_2^{\pm 1} \circ A_k, (A_1 \circ A_2)^{\pm 1} \circ A_k\right\}$$

*where $j = 1, \ldots, m$, and $k = 3, \ldots, m$.*

## 3.4 Discontinuity of *Teichüller* Modular Groups

Let us state the main theorem for which we developed the previous results.

**Theorem 3.8.** *The Teichmüller modular group* $\mathrm{Mod}(\Gamma)$ *acts properly discontinuously on* $T(\Gamma)$ *as a subgroup of the biholomorphic automorphism group* $\mathrm{Aut}(T(\Gamma))$.

*Proof.* We know that we can choose generators of $\Gamma$ satisfying the hypothesis of the previous proposition.

Suppose for the sake of contradiction we assume that the action of $Mod(\Gamma)$ doesn't act properly discontinuously on $T(\Gamma)$.

Suppose $e \in T(\Gamma)$ and let $U_e$ be a neighborhood around $e$ in $T(\Gamma)$.

Now by our assumption, we have an infinite sequence of pairwise distinct elements of $Mod(\Gamma)$, namely $\{[g_n]\}_{n=1}^{\infty}$, such that:

$[g_n]_*(U_e) \cap U_e \neq \phi \forall n \in \mathbf{N}$

This implies that there exists a sequence $\{u_n\}_{n=1}^{\infty}$ in $U_e$ such that $[g_n]_*(u_n) \in U_e \forall n \in \mathbf{N}$.

Now, let $\bar{U_e}$ be the compact closure of $U_e$.

Then, there exists a point $u_0 \in \bar{U_e}$ such that $\{u_n\}_{n=1}^{\infty} \to u_0$.

And so $\{[g_n]_*(u_n)\} \to v_0 \in T(\Gamma)$. (for some $v_0$)

Let us denote $[g_n]$ by $g_n$ for convenience.

Set $G_n = g_n^{-1}, H_n = G_{n+1} \circ g_n$.

We know that the image $T_B(\Gamma)$ of the $Bers'$ embedding, with which $T(\Gamma)$ is identified, is contained in the open ball with center 0 and radius 3/2, in $A_2(\mathbf{H}^*, \Gamma)$.

Therefore, $\{g_n\}_{n=1}^{\infty}$ is a normal family.

This implies that there exists a subsequence $\{g_{n_i}\}_{i=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$, which converges uniformly on compact subsets in $T(\Gamma)$ to a holomorphic mapping $g_0$.

So, in particular, we have $g_0(u_0) = v_0$. Similarly, we have that: Some subsequence of $\{G_n\}_{n=1}^{\infty}$ and $\{H_n\}_{n=1}^{\infty}$ are converges uniformly on compact subsets in $T(\Gamma)$ to holomorphic mappings, $G_0$ and $H_0$ respectively.

Here $G_n \circ g_n = id$ and hence $G_n \circ g_n(u_n) = u_n$, which then implies that $G_n(v_0) = u_0$.

Also, $H_n(u_n) = G_{n+1} \circ g_n(u_n) \implies H_n(u_0) = G_0(v_0) = u_0$.

Thus, $\{H_n(u_0)\}_{n=1}^{\infty}$ converges to $u_0$.

Or, in other words, if we translate the base point of $T(\Gamma)$ into $u_0$, we may assume that $\{H_n[id]\}_{n=1}^{\infty}$ converges to $[id]$.

Now set $H_n = [\omega_n]_*$ where $\omega_n$ is some quasiconformal self-mapping of $\mathbf{H}$, s.t. $\omega_n \Gamma \omega_n^{-1} = \Gamma$.

Now by the definition of the action of the elements of $Mod(\Gamma)$, we have:

$[\omega_n]_*([id]) = [A_n \circ id \circ \omega_n^{-1}] = [A_n \circ \omega_n^{-1}]$, where $A_n \in Aut(\mathbf{H})$ s.t. $A_n \circ \omega_n^{-1}$ fixes each of 0, 1 and $\infty$.

$\{H_n[id]\}_{n=1}^{\infty}$ converges to $[id] \implies \{[A_n \circ \omega_n^{-1}]\}_{n=1}^{\infty}$ converges to $[id]$ in $T(\Gamma)$.

So, $\forall \gamma$, the sequence $\left\{(A_n \circ \omega_n^{-1}) \circ \gamma_\circ (A_n \circ \omega_n^{-1})^{-1}\right\}_{n=1}^{\infty}$ converges to $\gamma$.

$\therefore \lim_{n\to\infty} tr^2(A_n^{-1} \circ \gamma \circ A_n) = tr^2(\gamma), \gamma \in \Gamma$.

Now, we use a result from the previous section:

$\{\mathrm{tr}^2(\gamma) \mid \gamma \in \Gamma\}$ is discrete in $\mathbf{R}$.

From this, it follows that (as $A_n^{-1} \circ \gamma \circ A_n \in \Gamma \forall n \in \mathbf{N}$):

$tr^2(A_n^{-1} \circ \gamma \circ A_n) = tr^2(\gamma), \forall \gamma \in \Gamma$ for every sufficiently large n.

Or, in particular, we have:

$tr^2(A_n^{-1} \circ \gamma \circ A_n) = tr^2(\gamma), \forall \gamma \in \mathcal{G}$ ( where $\mathcal{G}$ is the set defined in $Theorem$ 3.6).

Again by $Theorem$ 3.6.(Or, one can refer to the corollary to Theorem 4 of the book, "*Teichmüller Theory* And Quadratic Differentials" by Frederick P. Gardiner ) $\exists$ an element $B_n \in Aut(\mathbf{H})$ such that:

$A_n^{-1} \circ \gamma \circ A_n = B_n^{-1} \circ \gamma \circ B_n, \gamma \in \Gamma$

This shows that $B_n$ belongs to the normalizer $N(\Gamma)$ of $\Gamma$ in $\mathrm{Aut}(H)$, and $[\omega_n]_* = [B_n]_*$. Thus every such $[\omega_n]_*$ fixes the base point $[id]$ of $T(\Gamma)$.

By the definition, it is easy to see that the isotropy subgroup of Mod($\Gamma$) at [id] is isomorphic to $N(\Gamma)/\Gamma$. On the other hand, it is well known that $N(\Gamma)/\Gamma$ is isomorphic to the biholomorphic automorphism group $\mathrm{Aut}(H/\Gamma)$ of the closed Riemann surface $H/\Gamma$, and that $Aut(H/\Gamma)$ is a finite group. Therefore, $\{[\omega_n]_*\}_{n=1}^{\infty}$ should be a finite set. This contradicts that $\{[g_n]\}_{n=1}^{\infty}$ consists of infinite elements.

$\square$

# Chapter 4

# Royden's Theorems

The main aim of this chapter is to establish a relation between the biholomorphic automorphism group $Aut(T(\Gamma))$ of the *Teichmüller* space and the *Teichmüller* modular group $Mod(\Gamma)$.

**Theorem 4.1** (Royden)**.**

$$\mathrm{Aut}(T(\Gamma)) \cong \begin{cases} \mathrm{Mod}(\Gamma)/\mathbf{Z}_2 & (g=2) \\ \mathrm{Mod}(\Gamma) & (g>2) \end{cases}$$

Define the homomorphism $i_*([\omega]) = [\omega]_* \ \ \forall [\omega] \in Mod(\Gamma)$.

Firstly let us quickly establish the injectivity of $i_*$:

$\boldsymbol{Case\ \ 1 : g > 2}$:

$\boldsymbol{Case\ \ 2 : g = 2 :}$

Any closed Riemann surface $R$ of genus two has a biholomorphic automorphism of order two, since $R$ is represented by a two-sheeted branched covering surface over $\hat{\mathbf{C}}$.

This implies that $ker(i_*)$ is isomorphic to $\mathbf{Z}_2$. (as $g = 2$)

Now, we will prove certain theorems (also due to Royden) to establish the surjectivity of $i_*$. But before that, we will introduce a new kind of distance, namely $Kobayashi$ distance, which is a generalization of the Poincare distance.

## 4.1 Kobayashi Distance

Let $M$ be a complex manifold.

Given two points $x, y \in M$, we set

$$\boldsymbol{d^1_M(x,y) = \inf \rho(a,b)}$$

where the infimum is taken over all points $a, b \in \Delta$ (unit disc) such that there exists a holomorphic mapping $f : \Delta \to M$ with $f(a) = x$ and $f(b) = y$.

For any positive integer $n$, we put

$$\boldsymbol{d_M^n(x,y) = \inf \sum_{i=1}^{n} d_M^1 \left(x_{i-1}, x_i\right)}$$

where the infimum is taken over all points $x_o, \ldots, x_n \in M$ with $x_o = x$ and $x_n = y$.

Here suppose $x_o, \ldots, x_n \in M$ with $x_o = x$ and $x_n = y$, then set $x_o^{'} = x_0, \ldots, x_{n-1}^{'} = x_{n-1}, x_n^{'} = x_{n-1}, x_{n+1}^{'} = x_n \in M$. Then :

$$d_M^{n+1}(x,y) = \inf \sum_{i=1}^{n+1} d_M^1 \left(x_{i-1}^{'}, x_i^{'}\right) = d_M^n(x,y) = \inf \sum_{i=1}^{n} d_M^1 \left(x_{i-1}, x_i\right)$$

(as $d_M^{n+1}(x_{n-1}^{'}, x_n^{'}) = 0$).

So, $d_M^{n+1}(x,y) \leq d_M^n(x,y)$ for $x, y \in M$ and $\forall n \in \mathbf{N}$.

**Definition 4.2** (Kobayashi pseudo-distance)**.**

$$\boldsymbol{d_M(x,y) = \lim_{n\to\infty} d_M^n(x,y)}$$

Here, $d_M : M \times M \to \mathbf{R}$ is continuous as $\inf \rho$ is continuous.

And, it also satisfies the axioms for pseudo-distance:

- $d_M(x,y) \geq 0$
- $d_M(x,y) = d_M(x,y)$
- $d_M(x,y) + d_M(y,z) \geqq d_M(x,z)$

$\forall x, y, z \in M$.

We call $d_M$ non-degenerate if $d_M(x,y) = 0 \implies x = y$.

If $d_M$ is non-degenerate then $d_M$ is *Kobayashi distance*.

And, $M$ is called *hyperbolic complex manifold*.

***Note :*** $d_M$ has a *distance decreasing property*:

let $M$ and $N$ be two complex manifolds and let $f : M \to N$ be a holomorphic mapping. Then it follows that

$$d_M(x, y) \geq d_N(f(x), f(y), \quad p, q \in M$$

So if $f$ is biholomorphic mapping of a hyperbolic complex manifold $M$ is an isometry w.r.t. $d_M$.

***Reason :***

$$\begin{tikzcd}
(a, b \in)\Delta \arrow[r, "\lambda"] \arrow[dr, "f\circ\lambda(f\circ\lambda(a)=f(p), f\circ\lambda(b)=f(q))"'] & (\lambda(a) = p, \lambda(b) = q \in)M \arrow[d, "f"] \\
 & N
\end{tikzcd}$$

$d^1_N(f(p), f(q)) =$

$inf\{\rho(a, b) | \exists \ holomorphic \ function \ g : \Delta \to N \ s.t. \ g(a) = f(p), g(b) = f(q)\}$

and,

$d^1_M(p, q) =$

$inf\{\rho(a, b) | \exists \ holomorphic \ function \ g : \Delta \to M \ s.t. \ g(a) = p, g(b) = q\}$

Now if there exists a function (holomorphic) $\lambda : \Delta \to M$ s.t. $\lambda(a) = p, \lambda(b) = q$, then the function, $f\circ\lambda : \Delta \to N$ as in the figure below. satisfies that $f\circ\lambda(a) = f(p), f\circ\lambda(b) = f(q)$. Hence the cardinality of the set $\{g : \Delta \to N(holomorphic) \ \ s.t. \ g(a) = f(p), g(b) = f(q)\}$ is atmost the cardinality of the set $\{g : \Delta \to M(holomorphic) \ s.t. \ g(a) = p, g(b) = q\}$.

$\therefore$

$$d_M(x, y)^1 \geq d^1_N(f(x), f(y), \quad p, q \in M$$

It follows from here that:

$$d_M^n(x, y) \geq d_N^n(f(x), f(y), \quad p, q \in M$$

And this inequality is true for the *Kobayashi metric* and we are done. □

In the next theorem, we prove the equality of the *Teichmüller* distance and the *Kobayashi distance*.

**Theorem 4.3.** *$T(\Gamma)$, Teichmüller space of a Fuchsian model $\Gamma$ of a closed Riemann surface of genus $g(\geqq 2)$.*
*Then the Teichmüller distance $d$ on $T(\Gamma)$ is equal to the Kobayashi distance $d_{T(\Gamma)}$.*

*Proof.* Suppose if $d^1_{T(\Gamma)} = d$, then $d^1_{T(\Gamma)}$ satisfies the triangle inequality. So, by definition we have $d^n_{T(\Gamma)} = d^1_{T(\Gamma)}$ for any positive integer $n$, and hence $d_{T(\Gamma)} = d$.
(Reason:
Consider, $d^n_{T(\Gamma)}(x, y) = \inf \sum_{i=1}^n d^1_{T(\Gamma)}\left(x_{i-1}, x_i\right)$
Here, $d^1_{T(\Gamma)}\left(x_{i-1}, x_i\right) + d^1_{T(\Gamma)}\left(x_i, x_{i+1}\right) \geq d^1_{T(\Gamma)}\left(x_{i-1}, x_{i+1}\right) = d(x_{i-1}, x_{i+1}) \forall i \in [n]$ (as $d^1_{T(\Gamma)} = d$) )

Therefore, it's sufficient to prove that $d^1_{T(\Gamma)} = d$.
For any $[w^\mu] \in T(\Gamma)$, we put $\Gamma^\mu = w^\mu \Gamma \left(w^\mu\right)^{-1}$, and denote by $d_\mu$ the Teichmüller distance on $T\left(\Gamma^\mu\right)$.
Now, $[w^\mu] \in T(\Gamma)$ induces a biholomorphic mapping $[w^\mu]_* : T(\Gamma) \to T\left(\Gamma^\mu\right)$.
Now, to prove $d^1_{T(\Gamma)} = d$, it suffices to prove the equality

$$d^1_{T(\Gamma^\mu)}\left([id], \left[w^\lambda\right]\right) = d_\mu\left([id], \left[w^\lambda\right]\right), \quad \left[w^\lambda\right] \in T\left(\Gamma^\mu\right)$$

for any $[w^\mu] \in T(\Gamma)$.
If this equality holds, then the relations

$$d\left([w^{\mu}],[w^{\nu}]\right)=d_{\mu}\left([id],\left[w^{\nu}\circ(w^{\mu})^{-1}\right]\right)$$

$$d^{1}_{T(\Gamma)}\left([w^{\mu}],[w^{\nu}]\right)=d^{1}_{T(\Gamma^{\mu})}\left([id],\left[w^{\nu}\circ(w^{\mu})^{-1}\right]\right)$$

imply that $d\left([w^{\mu}],[w^{\nu}]\right)=d^{1}_{T(\Gamma)}\left([w^{\mu}],[w^{\nu}]\right)$ for all $[w^{\mu}],[w^{\nu}]\in T(\Gamma)$.

We replace the notation, $\Gamma^{\mu}, d_{\mu}$, and $w^{\lambda}$ by $\Gamma, d$, and $w^{\mu}$, respectively for convenience.

$\boldsymbol{Claim:\ d^{1}_{T(\Gamma)}\left([id],[w^{\mu}]\right)\leqq d\left([id],[w^{\mu}]\right),\quad [w^{\mu}]\in T(\Gamma)}$

By Teichmüller's existence theorem, we have that:

$\exists\ \varphi\in A_2(H,\Gamma)$ such that $[w^{\mu_0}]=[w^{\mu}]$ with $\mu_0=k\bar{\varphi}/|\varphi|$ for some $k, 0\leqq k<1$.

By Teichmüller's uniqueness theorem, we have that:

$$d\left([id],[w^{\mu}]\right)=\log\frac{1+k}{1-k}$$

.

(Reason:

$$d\left([id],[w^{\mu}]\right)=\inf_{g\cong_{homotopic}w^{\mu}}\log K(g)$$

Here, $\mu_0$ is one with the lowest $k$ and as the logarithmic function and the function $k\to\frac{1+k}{1-k}$ are both increasing function, the above equality holds. )

Set $\mu_\tau=\tau\bar{\varphi}/|\varphi|$

Here, Now consider the mapping ,

$f_{\varphi}:\Delta\to T(\Gamma)$ given by $f_{\varphi}(\tau)=[w^{\mu_\tau}]$.

This is a holomorphic mapping which satisfies:

$f_{\varphi}(0)=[id], f_{\varphi}(k)=[w^{\mu}]$.

Therefore we get that:

$$d^{1}_{T(\Gamma)}\left([id],[w^{\mu}]\right)\leqq\rho(0,k)=\log\frac{1+k}{1-k}$$

$\Longrightarrow d^{1}_{T(\Gamma)}\left([id],[w^{\mu}]\right)\leqq d\left([id],[w^{\mu}]\right),\quad [w^{\mu}]\in T(\Gamma)$, which precisely is our claim.

Now, we got to prove the last part of our proof i.e. :

$$d^{1}_{T(\Gamma)}\left([id],[w^{\mu}]\right) \geqq d\left([id],[w^{\mu}]\right), \quad [w^{\mu}] \in T(\Gamma)$$

Before we move further into the proof let us introduce :

### 4.1.1 Infinitesimal metric on the Tangent Bundle to $T(\Gamma)$

Suppose $w$ is a quasiconformal mapping of $\mathbf{C}$ to $\mathbf{C}$ with Beltrami coefficient $\mu$ such that $w(\bar{z}) = w(\bar{z})$ and $w \circ A \circ w^{-1}$ is a $M\ddot{o}bius$ transformation for each $A \in \Gamma$.

Let $\Gamma_{\mu} = w \circ \Gamma \circ w^{-1}$

Now, $w$ induces an isometric mapping between $Teichm\ddot{u}ller$ spaces.

Let that isometry be $\beta : T(\Gamma) \to T(\Gamma_{\mu})$,

where $\beta([\nu_0]) = \ [w_{\nu_0} \circ w_{\mu}^{-1}]$. - - - - - - - - - - - - - - - - - - - - - - (1)

Notice: $\beta[\mu] = [0]$ and as $\beta$ is an isometry, we have that:

$d([\mu],\ [\nu]) = d([0],\ \beta[\nu])$ - - - - - - - - - - - - - - - - - - - - - - - - - - - (2)

Now using the above equality (2) let us calculate the infinitesimal length $F([\mu],\ \nu])$ of a tangent vector $\nu$ at an arbitrary point $[\mu]$.

By definition, $F$ is the derivative of the function $d\left([w^{\mu}], \left[w^{\mu+t\lambda}\right]\right)$, (or, $d([\mu],[\mu+t\lambda])$ w.r.t. $t$ at $t=0$.

We note a result about the $Teichm\ddot{u}ller$ distance which we will use:

$d(0,t\mu) = 2t\ sup\left| Re\ \iint_{F^{\mu}} \mu\ dxdy \right| + O(t^2)$ (Here, $F^{\mu}$ is a Fundamental domain for $\Gamma^{\mu}$ in $H$.)

So,

$d([\mu],\ [\mu+t\lambda]) =\ d([0],\ \beta([\mu+t\lambda])$

$= 2sup\left| Re\ \iint_{\Gamma^{\mu}} t\phi\dot{\alpha}(\lambda)dudv \right| + O(t^2)$ ,- - - - - - - - - - - - - - - - - (3)

where $\dot{\alpha}$ is the derivative at $\mu$ of $\alpha$ and $u+iv = w = w_{\mu}$. (The supremum is overall $\phi \in A(\Gamma_{\mu})$ with $||\phi|| = 1$.)

where the supremum is over $\phi \in A_2\left(H, \Gamma^{\mu}\right)$ for which $||\phi|| = 1$. Now, replace $[\nu]$ in (2) by $[\mu + t\lambda]$.

We know that $\alpha([\mu + t\lambda]) = \left[\frac{(\mu+t\lambda)-\mu}{1-\bar{\mu}(\mu+t\lambda)} \cdot \frac{1}{\theta}\right] \circ w_\mu^{-1}$

$= \left[\frac{t\lambda}{1-\bar{\mu}(\mu+t\lambda)} \cdot \frac{1}{\theta}\right] \circ w_\mu^{-1}$ where $\theta = \frac{\bar{p}}{p}$ and $p = (\partial/\partial z) w_\mu$. Here, as $\alpha([\mu]) = [0]$,

$\dot{\alpha}(\lambda) = \lim_{t\to 0} \frac{\left[\frac{t\lambda}{1-\bar{\mu}(\mu+t\lambda)} \cdot \frac{1}{\theta}\right] \circ w_\mu^{-1} - 0}{t}$

$= \frac{\lambda}{((1-|\mu|^2)\theta)}$ So,

$$F\left(\left[w^\mu\right], \lambda\right) = \lim_{t\to 0, t>0} \frac{d\left(\left[w^\mu\right], \left[w^{\mu+t\lambda}\right]\right)}{t}$$

$$= 2 \sup \left| \operatorname{Re} \iint_{F^\mu} \left[ \frac{\left(w^\mu\right)_z}{\overline{\left(w^\mu\right)_z}} \cdot \frac{\lambda}{1-|\mu|^2} \right] \circ \left(w^\mu\right)^{-1}(z) \varphi(z) dx dy \right|$$

where $F^\mu$ is a fundamental domain for $\Gamma^\mu$ in $H$, and the supremum is taken over all $\varphi \in A_2\left(H, \Gamma^\mu\right)$ with $\|\varphi\|_1 = 1$.

### 4.1.2 Integrated form

For an arbitrary piecewise smooth path $C : [0,1] \to T(\Gamma)$, we set

$$L(C) = \int_0^1 F\left(C(t), C'(t)\right) dt$$

For any two points $p, q \in T(\Gamma)$, we put

$$\bar{d}(p,q) = \inf_C L(\gamma)$$

where the infimum is taken over all piecewise smooth paths $C$ joining $p$ and $q$ in $T(\Gamma)$. Then it is shown that

•1) $\tilde{d} = d$.

•2)

It is verified that every holomorphic mapping $f : \Delta \to T(\Gamma)$ satisfies

$$F\left(f(\tau), f'(\tau)\right) \leqq \frac{2}{1-|\tau|^2}, \quad \tau \in \Delta$$

(We omit the proofs of 1, 2 )

•3) Take an arbitrary holomorphic mapping $f : \Delta \to T(\Gamma)$ with $f(a) = [\mathrm{id}]$ and $f(b) = [w^\mu]$ for some points $a, b \in \Delta$.

Then (ii) and (iii) imply that $d\left([id], [w^\mu]\right) = \bar{d}\left([id], [w^\mu]\right) \leqq \rho(a, b)$.

By the definition of $d^1_{T(\Gamma)}$, we get

$$d\left([id], [w^\mu]\right) \leqq d^1_{T(\Gamma)}\left([id], [w^\mu]\right), \quad [w^\mu] \in T(\Gamma)$$

And we are done.

□

## 4.2 Surjectivity of $i_*$

***Note* :**

Theorem 4.3. implies that $f$ is an isometry w.r.t. the *Teichmüller distance* on $T(\Gamma)$.

Take an element $f \in \mathrm{Aut}(T(\Gamma))$.

For every $p = [w^\mu] \in T(\Gamma)$, we set $q = f(p) = [w^\nu]$.

The derivative $\dot{f}$ of $f$ at $p$ is a complex linear isometry of $T_p(T(\Gamma))$ to $T_q(T(\Gamma))$ with respect to the infinitesimal metric $F$, where $T_p(T(\Gamma))$ and $T_q(T(\Gamma))$ denote the holomorphic tangent spaces of $T(\Gamma)$ at $p$ and $q$, respectively.

***Identification of*** *$T_p(T(\Gamma))$ with the dual space $A_2(\mathbf{H}, \Gamma)^*$ of $A_2(\mathbf{H}, \Gamma)$*:

Take $\mu \in B(\mathbf{H}, \Gamma)$ and send it to an element $\Lambda_\mu \in A_2(\mathbf{H}, \Gamma)^*$ which is a linear functional on $A_2(\mathbf{H}, \Gamma)$ defined by:

$\Lambda_\mu(\phi) = (\mu, \phi)_{\mathbf{R}} = \iint_F \mu(z)\phi(z)dxdy,\ \phi \in A_2(\mathbf{H}, \Gamma)$ (here, F is a fundamental domain)

Now, we have:

**Theorem 4.4.** *The mapping* $\Lambda : B(\mathbf{H}, \Gamma) \to A_2(\mathbf{H}, \Gamma)^*$ *($\mu \to \Lambda_\mu$), induces an isomorphism of* $B(\mathbf{H}, \Gamma)/N(\Gamma) \cong T_p(T(\Gamma))$ *onto* $A_2(\mathbf{H}, \Gamma)^*$.

Hence by this theorem we have that:

$T_p(T(\Gamma))^* \cong A_2(\mathbf{H}, \Gamma^\mu)$

And, similarly, $T_q(T(\Gamma))^* \cong A_2(\mathbf{H}, \Gamma^\nu)$.

$\therefore$ $\dot{f}$ induces a complex linear isometry $\alpha$ of $A_2(H, \Gamma^\mu)$ to $A_2(H, \Gamma^\nu)$ with respect to the infinitesimal cometric induced by the Teichmüller distance $d$.

We now write the last theorem of this section before we can start to show the surjectivity of the map $i_*$.

**Theorem 4.5** (Royden)**.** ⌈ *Let $\alpha$ be a complex linear isometry of $A_2(H, \Gamma^\mu)$ to $A_2(H, \Gamma^\nu)$ with respect to the infinitesimal cometric induced by the Teichmuller disiance $d$.*
*Then there exists a biholomorphic mapping $h : H/\Gamma^\nu \to H/\Gamma^\mu$ and a complex number $c$ with $|c| = 1$ such that $\alpha(\varphi) = c\varphi_0\bar{h} \cdot (\bar{h}')^2$ for all $\varphi \in A_2(H, \Gamma^\mu)$, where $\bar{h}$ is a lift of $h$ to $H$.* ⌋

*Proof of the surjectivity.* From the above discussion and the theorem 4.5., for every $f \in Aut(T(\Gamma))$ and every point $p \in T(\Gamma)$ there exists an element $[\omega_p] \in \mathrm{Mod}(\Gamma)$ with $[\omega_p]_*(p) = f(p)$

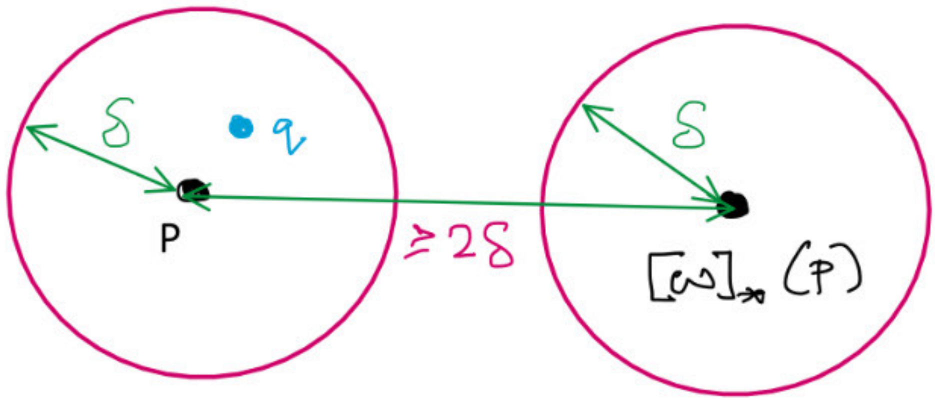
***Claim :*** $[\omega_p]$ can be chosen independently of $p$.

Fix a point $q \in T(\Gamma)$ arbitrarily.

Now we know that $T(\Gamma)$ is biholomorphic to the bounded domain and the *Teichmüller distance* $d$ is complete.

Also, from the previous chapter, we know that $\mathrm{Mod}(\Gamma)$ acts properly discontinuously on $T(\Gamma)$.

$\therefore \exists \delta > 0$ s.t. $\forall\ p \in T(\Gamma)$ $s.t.$ $d(q, p) < \delta$, we have, $d(p, [\omega]_*) > 2\delta$ ( for any $[\omega] \in Mod(\Gamma)$ with $[\omega]_*(p) \neq p$.




Thus we have:

$$\begin{aligned} d\left(p, [\omega_q]_*^{-1} \circ [\omega_p]_*(p)\right) &= d\left([\omega_q]_*(p), [\omega_p]_*(p)\right) \\ &\leqq d\left([\omega_q]_*(p), [\omega_q]_*(q)\right) + d\left([\omega_q]_*(q), [\omega_p]_*(p)\right) \\ &= d(p,q) + d(f(q), f(p)) \\ &= 2d(q,p) < 2\delta \end{aligned}$$

for all $p \in T(\Gamma)$ with $d(q,p) < \delta$.

So, $[\omega_q]_*^{-1}\,{}_0\,[\omega_p]_*(p) = p$, i.e., $[\omega_q]_*(p) = [\omega_p]_*(p) = f(p)$ for all $p \in T(\Gamma)$ with $d(q,p) < \delta$.

Now as $T(\Gamma)$ is connected, the uniqueness theorem for holomorphic functions says that $[\omega_q]_* = f$ on $T(\Gamma)$.

So we are done. $\square$

# Appendix A

## A.1 Fuchsian Group

**Definition A.1** (Fuchsian Group)**.** *A Fuchsian group is a discrete of* $\mathbf{Aut}(\mathbf{D})$.

Let $X$ be a hyperbolic Riemann surface and $\pi : \mathbf{D} \to X$ is a universal covering map. This covering map leads to the description of $X$ by $\mathbf{D}/\Gamma$, where $\Gamma$ is the covering transformation group. This group acts on both $\mathbf{D}$ and $S^1 = \partial D$.

Take a point $x \in \mathbf{D}$ and consider $\bar{\Gamma x}$, the closure of the orbit of $x$ in the closed disc $\bar{D}$.

We define $\boxed{\Lambda_\Gamma(x) = \bar{\Gamma x} \cap S^1}$

**Definition A.2** (Limit set of a Fuchsian Group)**.** *Suppose* $x_1, x_2 \in \mathbf{D}$*, then* $\Lambda_\Gamma(x_1) = \Lambda_\Gamma(x_2)$.

*Thus we can omit the dependency of* $x$ *from* $\Lambda_\Gamma(x)$ *and write* $\Lambda_\Gamma$.

*We call this set the limit set of* $\Gamma$.

**Proposition A.3** (Limit set is the smallest closed invariant set)**.** *Suppose* $Z \subset \partial\mathbf{D}$ *which is invariant under* $\Gamma$*, i.e.* $\Gamma Z = Z$*. Then we have that:* $\Lambda_Z \subset Z$

**Corollary A.4** (Fixed points are dense in the limit set)**.** *The Fixed point set of hyperbolic elements is dense in the limit set of a non-elementary Fuchsian group* $\Gamma$.

## A.2 *Teichmüller* Space of a Fuchsian Group

We assume that each of the points $0, 1$ and $\infty$ is a fixed point of some element of $\Gamma - \{id\}$.

If no additional information is given we will always assume that lift $\bar{f} : \mathbf{H} \to \mathbf{H}$ of a quasiconformal mapping $f : R \to S$, fixes each of $0, 1$ and $\infty$.

By Theorem 2.1. above we say that we determine $\tilde{f}$ uniquely.

We call such a lift a *canonical* lift of $f$ with respect to $\Gamma$.

Using this canonical lift $\bar{f}$, we consider the following injective homomorphism from $\Gamma$ to $PSL(2, \mathbf{R})$, sending $\gamma$ to $\tilde{f} \circ \gamma \circ \tilde{f}^{-}, \gamma \in \Gamma$. We name this map $\theta_{\tilde{f}}$.

Hence we have an isomorphism from one Fuchsian group $\Gamma$ to another Fuchsian group $\Gamma_1$ or $\tilde{f}\Gamma\tilde{f}^{-1}$.

### A.2.1 *Teichmüller* space of the Fuchsian Model $\Gamma$

Let us denote the set of all canonical quasiconformal mappings $\omega$ of $\hat{\mathbf{C}}$, s.t. $\omega\Gamma\omega^{-1}$ are also *Fuchsian* groups, by $QC(\Gamma)$.
Now we define the the *Teichmüller* space of the Fuchsian Model $\Gamma$,
$T(\Gamma) = \{\omega | \omega \in QC(\Gamma)/\sim$, where $\omega_1 \sim \omega_2$ iff $\omega_1 = \omega_2$ on $\mathbf{R}$.

If $\mathbf{R}$ is compact then $T(\mathbf{R})$ is identified with $T(\Gamma)$.

## A.3 Quasiconformal Maps

Let us give the Analytic definition of the Quasiconformal mapping: This definition depends on the notion of absolute continuity on lines, which we abbreviate by ACL.
We say the function $f(z) = u(x, y) + iv(x, y)$ is absolute continuity on lines if for every rectangle in the Jordan region with sides parallel to the $x - axis$ and $y - axis$, both $u(x, y)$ and $v(x, y)$ are absolutely continuous on almost every horizontal and almost every vertical line in $R$. The functions $u$ and $v$ will then have partial derivatives $u_x, u_y, v_x, v_y$ almost everywhere in the Jordan region.
The complex derivatives are, by definition:
$f_z = \frac{1}{2}(f_x - if_y)$ and $f_z = \frac{1}{2}(f_x + if_y)$.

**Definition A.5** (Analytic form)**.** *Let $f$ be a homeomorphism from a domain to a domain $'$. Then $f$ is $K-quasiconformal$ if*
*(i) $f$ is ACL in ; and (ii) $|f_{\bar{z}}| \leq k|f_z|$ almost everywhere, where $k = (K-1)/(K+1) < 1$*
*The minimal possible value of $K$ for which (ii) is satisfied is called the dilation of $f$.*

**Proposition A.6** (Quasiconformal Maps are closed under compositions and inverses)**.**
*1) If* $f : U \to V$ *is a* $K_1 - quasiconformal$ *and* $g : V \to W$ *is* $K_2 - quasiconformal$*,*
*then* $g \circ f : U \to W$ *is* $(K_1K_2) - quasiconformal$*.*
*2) If* $f : U \to V$ *is* $K - quasiconformal$*, then so is* $f^{-1} : V \to U$*.*

In the below figure, we see how the shrinkage of the circle depends on the Beltrami coefficient.

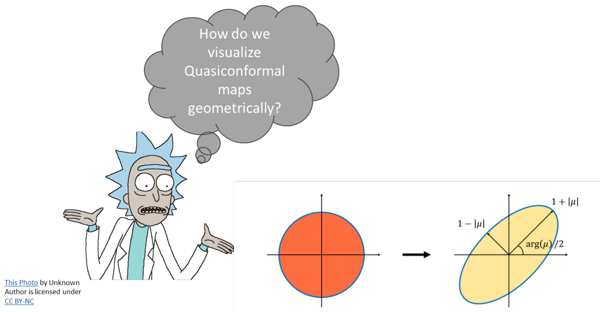



Figure A.1: Geometrical visualization of how a quasiconformal mapping transforms a circle to a ellipse

i.e. intuitively while conformal mappings send infinitesimal circles to infinitesimal circles, the Quasiconformal mappings send infinitesimal circles to infinitesimal ellipses whose eccentricity is bounded.

### A.3.1 Quasisymmetry

**Definition A.7** (Labeled Quasisymmetry)**.** *Let* $X, Y$ *be metric spaces, and let* $\beta : [0, \infty) \to [0, \infty)$ *be a homeomorphism.*
*A mapping* $g : X \to Y$ *is* $L - quasisymmetric$ *of modulus* $\beta$ *if for any three distinct points* $x, y, z \in W$ *we have:* $\boxed{\left|\dfrac{g(x) - g(y)}{g(x) - g(z)}\right| \leq \beta\left(\left|\dfrac{x - y}{x - z}\right|\right)}$

**Proposition A.8.** *Let* $U, V$ *be open sets of* $\mathbf{C}$*.*
*A homeomorphism* $f : U \to V$ *is* $K - quasiconformal$ *iff* $f$ *is* $L - quasisymmetric$ *with some modulus* $\beta$ *depending only on* $K$*.*

### A.3.2 The Mapping Theorem

**Theorem A.9** (The Mapping Theorem). *1) Let$U \subset \mathbf{C}$; open set. And, let $\mu \in L^{\infty}(U)$ s.t. $||\mu||_{\infty} < 1$. Then $\exists$ a quasiconformal mapping $f : U \to \mathbf{C}$ satisfying the Beltrami equation:*

$$\boxed{\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}}$$

*2) If $g$ is a map different from $f$ which also satisfies the same Beltrami equation as above then $\exists$ a function $A : f(U) \to \mathbf{C}$, which is injective and analytic, such that* $\boxed{g = A \circ f}$

The function $\mu$ in the above beltrami equation is called the *Beltrami coefficiant* of f. This coefficiant associated to $f$ serves as a measure of the non-conformality of $f$.

### A.3.3 Dependence of the existence of isothermal coordinates on the existance of the solution to the Beltrami equation

Suppose $R$ is a surface. And let $(U_j, (x_j, y_j))_j$, associated with a Riemannian metric which in local coordinates, $(x_j, y_j)$ takes the below form:

$ds^2 = \rho(x_j, y_j).(dx_j^2 + dy_j^2)$ where $\rho : R \to \mathbf{R}_+$ is a smooth function.

Consider the coordinate (complex valued):

$w_j = u_j + iv_j$.

We refer to this coordinate as *isothermal coordinates*.

We will now show that if our *Riemannian metric* is not given in the form $ds^2 = \rho(x_j, y_j).(dx_j^2 + dy_j^2)$, we can still find a set of coordinates such that the metric takes this form.

Now suppose $ds^2 = Fdx^2 + 2Hdxdy + Gdy^2$ in some local coordinates.

Write $z = x + iy$ then we get:

$ds^2 = \alpha|dz + \mu d\bar{z}|^2 = \alpha(dz + \mu d\bar{z})(d\bar{z} + \bar{\mu}dz)$

where $\alpha = 1/4(F + G + 2\sqrt{FG - H^2})$ and $\mu = \frac{F-G+2iH}{F+G+2\sqrt{FG-H^2}}$

We want to find a coordinate $w_p = q + ir$ so that

$ds^2 = \rho(dq^2 + dr^2) = \rho(|dw_p|^2) = \rho.\left|\frac{\partial w_p}{\partial z}\right| \cdot \left|dz + \frac{\partial w_p/\partial \bar{z}}{\partial w/\partial z} d\bar{z}\right|^2$

$\frac{\partial w_p/\partial \bar{z}}{\partial w/\partial z} d\bar{z} = \mu$

This concludes the fact that for an isothermal coordinate to exist we must have a solution to the Beltrami equation:

### A.3.4 Beltrami differential on $\mathbf{H}$ with respect to $\Gamma$

Suppose $[S, f] \in R$.

Let the complex dilation of the canonical lift $\tilde{f}$ of $f$ (w.r.t. $\Gamma$) be $\mu_f$.

Now $\theta_{\tilde{f}}(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma, \gamma \in \Gamma$.

$\therefore$ for almost every $z \in \mathbf{H}$ we have that:

$(\theta_{\tilde{f}}(\gamma)' \circ \tilde{f}) \cdot \tilde{f}_z = (\tilde{f}_Z \circ \gamma) \cdot \gamma'$

and, $(\theta_{\tilde{f}}(\gamma)' \circ \tilde{f}) \cdot \tilde{f}_{\bar{z}} = (\tilde{f}_Z \circ \gamma) \cdot \bar{\gamma}'$

$\implies$ $\boxed{\mu_{\tilde{f}} = (\mu_{\tilde{f}} \circ \gamma)\bar{\gamma}'/\gamma'}$ almost everywhere on $\mathbf{H}, \gamma \in \Gamma$.

All the bounded measurable function $\mu$ on $\mathbf{H}$ satisfying the above equality in the box with $\mu$ instead of $\mu_{\tilde{f}}$, are called *Beltrami Differentials* on $\mathbf{H}$ w.r.t $\Gamma$.

Define $B(\mathbf{H}, \Gamma)$: the set of all Beltrami differentials on $\mathbf{H}$ w.r.t. $\Gamma$.

And $B(\mathbf{H}, \Gamma)_1 = \{\mu \in B(\mathbf{H}, \Gamma) | ||\mu||_\infty < 1\}$.

We call the elements of this set the *Beltrami coefficient* on $\mathbf{H}$ w.r.t. $\Gamma$.

Suppose $f : R \to S$ from one Riemann Surface to another Riemann surface, is a quasi-conformal mapping.

Consider the complex dilation, $\mu_{\tilde{f}} \in B(\mathbf{H}, \Gamma)_1$ of the canonical lift of $f$.

Then it naturally determines an element $\mu \in B(R)_1$.

We define the Beltrami coefficiant of $f$ to be this $\mu$ and denote it by $\mu_f$.

**Definition A.10.** *(QC(U,V)) We denote by $QC(U, V)$, the set of all Quasi-conformal maps from $U$ to $V$, with the topology of uniform convergence on compact subsets.*

*The subset $QC_k(U, V)$ consists of those mappings $f \in QC(U, V)$ that are $K-quasiconformal$.*

### A.3.5 Analytic Dependence of the solution of the Beltrami equation on the Beltrami coefficient $\mu$

[$f^\mu$] We denote by $f^\mu$ the solution of :
$\boxed{\dfrac{\partial f^\mu}{\partial \bar{z}} = \mu \dfrac{\partial f^\mu}{\partial z}}$ s.t. $f^\mu(0) = 0, f^\mu(1) = 1, f^\mu(\infty) = \infty$

**Proposition A.11.** *The map from $L^\infty(\mathbf{C}) \to QC(\mathbf{C}, \mathbf{C})$ given by $\mu \hookrightarrow f^\mu$ is analytic.*

**Theorem A.12.** *Every $K - quasiconformal$ homeomorphism $f : \mathbf{H} \to \mathbf{H}$ extends continuously as a homeomorphism $\bar{\mathbf{R}} \to \bar{\mathbf{R}}$.*
*Also, the extension of $f$ to the Riemann sphere $\hat{\mathbf{C}}$ by $f(\bar{z}) = \bar{f(z)}$ is still a $K - quasiconformal$.*

**Corollary A.13.** *Let $U$ be a Jordan domain and suppose $f : \mathbf{D} \to U$ is quasiconformal, then there is an extension of $f$ from $\bar{D}$ to $\bar{U}$, which is homeomorphism.*

**Definition A.14** ($\mathbf{R} - quasisymmetric$)**.** *Let $f : \mathbf{R} \to \mathbf{R}$ be a homeomorphism.*
*Then $f$ is $\mathbf{R} - quasisymmetric$ with modulus $M$ if for all $x \in \mathbf{R}$ and all $t > 0$ it satisfies:*
$\boxed{\dfrac{1}{M} \leq \dfrac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq M}$

**Corollary A.15.** *If $K \leq 1$ then $\exists$ $M$ such that every $K - quasiconformal$ homeomorphism $f : \mathbf{H} \to \mathbf{H}$ with $f(\infty) = \infty$ extends to a homeomorphism $\mathbf{R} \to \mathbf{R}$ that is $\mathbf{R} - quasisymmetric$ with Modulus $M$.*

**Theorem A.16.** *If $f : \mathbf{R} \to \mathbf{R}$ is $\mathbf{R} - quasisymmetric$ with modulus $M$, then it extends to a homeomorphism $\tilde{f} : \mathbf{H} \to \mathbf{H}$ is $K - quasiconformal$ with $K$ depending only on $M$.*

## 4.4 The Douady-Earle Extension

Given that it's more convenient to work with quasisymmetric maps $f : S^1 \to S^1$, we provide two equivalent conditions:

**Lemma 4.17.** *If $f : S^1 \to S^1$, then TFAE: 1)$f$ is L-quasisymmetric with modulus $\eta$.*
*2) $\exists$ a constant $M$ such that for any $x \in S^1$, if the analytic isomorphism $A_1 : \mathbf{D} \to \mathbf{H}$ maps $\infty$ to $x$ and $A_2 : \mathbf{D} \to \mathbf{H}$ maps $f(a)$ to $\infty$, then the function $f_0 := A_2 \circ f \circ A_1$*

*satisfies:* $\boxed{\dfrac{1}{M} \leq \dfrac{f_0(x+t) - f_0(x)}{f_0(x) - f_0(x-t)} \leq M}$

**Notation 4.18.** $QS_M(S^1)$ *:the sapce of homeomorphism* $f : S^1 \to S^1$ *that are* $\mathbb{R}-$ *quasisymmetric with modulus* $M$

**Theorem 4.19** (The Douady-Earle extension theorem)**.** *For any* $M \geq 1$*, there exist* $K \geq 1$ *and a map* $\Phi : QS_M(S^1) \to QC_K(\mathbf{D})$ *such that the* $K-quasiconformal$ *map* $\Phi(f)$ *extends* $f$ *and for every* $A_1, A_2 \in Aut(\mathbf{D})$,

$\Phi(A_1 \circ f \circ A_2) = A_1 \circ \Phi(f) \circ A_2.$

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