SEMESTER PROJECT REPORT

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Abstract. This is a brief report summarizing the semester long project undertaken under the supervision of Dr. Chitrabhanu Chowdhuri. This is based on the books Differential Geometry of Curves and Surfaces by Manfredo P. Do Carmo and Multivariate Calculus by Sean Dineen.

Contents

1. Directed curves in $\mathbb{R}^n$

Definition 1.1. A directed (or oriented) curve in $\mathbb{R}^n$ is a quadruple $\{\Gamma, A, B, v\}$ where Γ is a set of points in $\mathbb{R}^n$; A and B are points in Γ , called respectively the initial and final points of Γ ; v is a unit vector in $\mathbb{R}^n$ called the initial direction, for which there exists a mapping $P : [a, b] \to \mathbb{R}^n$, called a parametrization of Γ, such that the following conditions hold:
(a) There exists an open interval I , containing $[a, b]$ and a mapping from I into I into $\mathbb{R}^n$ which has derivatives of all orders and which coincides with P on $[a, b]$ (regularity conditions).
(b) $P([a, b]) = \Gamma, P(a) = A, P(b) = B$ and $P'(a) = \alpha v$ for some $\alpha > 0$ (direction).
(c)$P'(t) \neq 0$ for all $t \in [a, b]$ (for unit speed parametrization).
(d) P is injective (i.e. one to one) on $[a, b)$and $(a, b]$ (prevent self intersection).

Definition 1.2. (Parametrized Curve) A continuous mapping $P : [a, b] \to \mathbb{R}^n$ which satisfies (a), (c) and (d) is called a parametrized curve. A parametrized curve determines precisely one directed curve ,

$$\left\{ P([a,b]), P(a), P(b), \frac{P'(a)}{\|P'(a)\|} \right\}$$

Proposition 1.3. *Length of* Γ, $l(\Gamma)$ *is given by,*

$$l(\Gamma) = \int_a^b \|P'(t)\| \, dt$$

1.1. **Unit speed Parametrization.**

Definition 1.4. If $P : [a, b] \to \mathbb{R}^n$is a parametrization of the directed curve Γ we define the length function by the formula

$$s(t) = \int_a^t \|P'(x)\| \, dx$$

If $l = l(\Gamma)$ then $s : [a, b] \to [0, l]$ and, by the one-variable fundamental theorem of calculus, $s'(t) = \|P'(t)\| > 0$. Hence s is strictly increasing, $S^{-1} : [0, l] \to [a, b]$ has derivatives of all orders on $[0, l]$ and $P \circ s^{-1}$ maps $[0, l]$ onto Γ. For the inverse function s^{-1} we have,

$$\left(s^{-1}\right)'(t) = \frac{1}{s'\left(s^{-1}(t)\right)} = \frac{1}{\|P'\left(s^{-1}(t)\right)\|}$$

and

$$\left\| \left(P \circ s^{-1}\right)'(t) \right\| = \frac{\left\|P'\left(s^{-1}(t)\right)\right\|}{s'\left(s^{-1}(t)\right)} = \frac{\left\|P'\left(s^{-1}(t)\right)\right\|}{\|P'\left(s^{-1}(t)\right)\|} = 1$$

Thus $P \circ s^{-1}$ is unit speed and we can easily verify that it is a valid parametrization.

Proposition 1.5. *Directed curves admit unit speed parametrizations.*

1.2. **Frenet-Seret Equations.**

We discuss curvature and torsion of directed curves. Vector-valued differentiation and orthonormal bases are the main tools used. We define geometric concepts associated with a directed curve and derive a set of equationsthe FrenetSerret equationswhich capture the fundamental relationships between them. We look at curves in $\mathbb{R}^2$ which motivate our analysis of surfaces in $\mathbb{R}^3$ along similar lines later.

Let $P : [a, b] \to \mathbb{R}^2$ denote a unit speed parametrization of the directed curve Γ and let $P(t) = (x(t), y(t))$ for all t in $[a, b]$. At $P(t) \in \Gamma$ the unit tangent, $T(t)$, is Γ given by

$$T(t) = P'(t) = (x'(t), y'(t))$$

The normal is given by,

$$N(t) = (-y'(t), x'(t))$$

FIGURE 1. Caption

We have $\langle T(t), T(t)\rangle = 1$ and differentiating we get, by the product rule,

$$T'(t) = \kappa(t)N(t)$$

Definition 1.6. The constant κ uniquely determined by above equation is known as curvature.

In terms of coorninates,

$$\begin{aligned}\kappa(t) &= \langle \kappa(t)N(t), N(t)\rangle = \langle T'(t), N(t)\rangle \\ &= (x''(t), y''(t)) \cdot (-y'(t), x'(t)) \\ &= y''(t)x'(t) - x''(t)y'(t)\end{aligned}$$

Definition 1.7. We call $|\kappa(t)|$ the absolute curvature of Γ at $P(t)$

$$|\kappa(t)| = \|T'(t)\| = \|P''(t)\|$$

In case of an arbitrary curve,

$$\kappa(t) = \frac{y''(t)x'(t) - x''(t)y'(t)}{\left((x'(t))^2 + (y'(t))^2\right)^{3/2}}$$

For curves in $\mathbb{R}^3$,

- $N(t) = \frac{T'(t)}{\|T'(t)\|}$
- $B(t) = T(t) \times N(t)$
- $N(t) \times B(t) = \pm T(t)$
- $B'(t) = \langle B'(t), N(t)\rangle N(t)$
- We define the torsion of Γ at $P(t)$, $\tau(t) = -\langle B'(t), N(t)\rangle$
- $N'(t) = -\kappa(t)T(t) + \tau(t)B(t)$
- The Frenet-Serret equations can be expressed in matrix form
$\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$
- $\begin{matrix} \{T(t), & \kappa(t) & N(t) & , & B(t) & \tau(t)\} \\ \| & \| & \| & & \| & \| \\ P'(t) & \|T'(t)\| & T'(t)/\kappa(t) & & T(t) \times N(t) & -\langle B'(t), N(t)\rangle \end{matrix}$

2. GEOMETRY OF SURFACES IN $\mathbb{R}^n$

2.1. **Basic definitions.** We consider cross section of $\mathbb{R}^3$ through the point p which contains the unit normal. we can find a unit tangent vector at $p, \mathbf{v}$, such that our cross-section has the form

$$p + \left\{x\mathbf{v} + y\mathbf{n}(p) : x, y \in \mathbb{R}^2\right\}$$

Definition 2.1. The intersection of this cross section with the surface, is a curve on the surface called a normal section of the surface.

Definition 2.2. We call $k_p(\mathbf{v})$ the *Normal curvature* at point p in the direction $\mathbf{v}$

Definition 2.3. (Principal curvatures)

$$k_1(p) = \max_{\|\mathbf{v}\|=1} k_p(\mathbf{v})$$
$$k_2(p) = \min_{\|\mathbf{v}\|=1} k_p(\mathbf{v})$$

2.2. **Geomtric interpretation of Gaussian Curvature.**

Definition 2.4. The Gaussian curvature, $K(p)$, at a point p on a surface S, is the product of the principal curvatures, $k_1(p)k_2(p)$.

Now, we consider various possibilities for the principal curvatures :

- $k_1(p) = k_2(p) \to$ umbilical point.
- $k_1(p) = k_2(p) = 0 \to$ flat spot
- $k_1(p) > k_2(p) \to$ non-umbilical point.

Proposition 2.5. *At a non-umbilic point on a surface S in $\mathbb{R}^3$ we have:*

$$K(p) > 0 \iff \textit{near } p, S \textit{ is shaped like an ellipsoid}$$
$$K(p) < 0 \iff \textit{near } p, S \textit{ is shaped like a saddle point,}$$
$$K(p) = 0 \iff \textit{near } p, S \textit{ is shaped like a cylinder or cone.}$$

At an umbilic point $K(p) \geq 0$ and

$$K(p) > 0 \iff \textit{nearp, Sis shaped like a sphere.}$$
$$K(p) = 0 \iff \textit{near p, S is very flat.}$$

2.3. **Gaussian curvature from Parametrizations.** Let ϕ be a parametrization of S.
We define:

$$l = \langle \phi_{xx}, \mathbf{n} \rangle = 0, \quad m = \langle \phi_{xy}, \mathbf{n} \rangle, \quad n = \langle \phi_{yy}, \mathbf{n} \rangle = 0$$
$$E = \phi_x \cdot \phi_x, \quad F = \phi_x \cdot \phi_y, \quad G = \phi_y \cdot \phi_y$$

Matrix for Weingarten operator is given by :

$$\begin{bmatrix} \frac{l}{E} & \frac{Em-Fl}{E\sqrt{EG-F^2}} \\ \frac{Em-Fl}{E\sqrt{EG-F^2}} & \frac{E^2n-2EFm+F^2l}{E(EG-F^2)} \end{bmatrix}$$
$$K(p) = \frac{ln-m^2}{EG-F^2}$$

3. Gaussian Curvature

3.1. **Gauss Map.**

Definition 3.1. Let $S \subset \mathbb{R}^3$ be an oriented surface. The Gauss map is the map $N : S \to S^2$ which assigns to $p \in S$ the unit normal. There are two unit normals; the meaning of the word oriented is that we have chosen one. Thus,

$$\|N(p)\| = 1, \quad \langle N(p), \mathbf{v} \rangle = 0 \text{ for } \mathbf{v} \in T_pS$$

The first fundamental form assigns to each $p \in S$ the quadratic form $I_p : T_pS \to \mathbb{R}$ defined by

$$I_p(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2$$

It assigns to each tangent vector $\mathbf{v} \in T_pS \subset \mathbb{R}^3$ the square of its length.

The second fundamental form is defined by :

$$I_p(\mathbf{v}) = \langle N(p), \alpha''(0) \rangle, \quad \mathbf{v} = \alpha'(0)$$

where $\alpha : (\varepsilon, \varepsilon) \to S$ is a curve whose tangent vector at p is v.

Lemma 3.2. *The second fundamental form is independent of the choice of curve α used to define it.*

Lemma 3.3. *The derivative $dN_p : T_pS \to T_{N(p)}S^2$ of the Gauss map is a map from a vector space to itself, i.e.*

$$T_pS = T_{N(p)}S^2$$

for $p \in S^2$

Lemma 3.4. *The derivative* $dN_p : T_pS \to T_pS$ *is self adjoint, i.e.*

$$\langle dN_p(\mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{u}, dN_p(\mathbf{v})\rangle$$

for $\mathbf{u}, \mathbf{v} \in T_pS$

Lemma 3.5. *Let* $\alpha : (-\varepsilon, \varepsilon) \to S$ *be a curve in* S *parameterized by arclength. By the geometric definition of the cross product, the vectors* $N, \alpha', N \wedge \alpha'$ *are orthonormal at each point* $\alpha(s)$*. The vector* α' *is a unit vector tangent to* S *and* $N(\alpha)$ *is a unit vector normal to* S *so* $N \wedge \alpha'$ *is a unit vector tangent to* S *and is orthogonal to both* N *and* α'*. Since* $\|\alpha'\| = 1$ *we also have* $\langle \alpha', \alpha''\rangle = 0$*. Hence the curvature vector can be written as :*

$$\alpha'' = k_nN + k_g\left(N \wedge \alpha'\right), \quad k_n := \langle \alpha'', N\rangle, \quad k_g := \langle \alpha'', N \wedge \alpha'\rangle$$

The coefficient k_n *is called the normal curvature and coefficient* k_g *is called the geodesic curvature. By definition*

$$I_\alpha\left(\alpha'\right) = -\left\langle \alpha'', N(\alpha)\right\rangle = -k_n$$

By the Pythagorean Theorem :

$$k^2 = k_n^2 + k_g^2$$

Lemma 3.6. *The eigenvalues* k_1, k_2 *of* dN_p *are called the principal curvatures and the determinant*

$$K := \det\left(dK_p\right) = k_1k_2$$

is called the ***Gauss curvature****. The average value*

$$H := \tfrac{k_1+k_2}{2}$$

of the principal curvatures is called the ***Mean curvature****. Thus* $\lambda = k_1$ *and* $\lambda = k_2$ *are the two solutions of the characteristic equation.*

$$\lambda^2 + 2H\lambda + K = 0$$

Definition 3.7. Weingarten Equations.

$$N_u = a_{11}\mathbf{x}_u + a_{12}\mathbf{x}_v. \quad N_v = a_{21}\mathbf{x}_u + a_{22}\mathbf{x}_v$$

where

$$\begin{array}{ll} a_{11} = \frac{fF-eG}{EG-F^2}, & a_{12} = \frac{gF-fG}{EG-F^2} \\ a_{21} = \frac{eF-fE}{EG-F^2},, & a_{22} = \frac{fF-gE}{EG-F^2} \end{array}$$

Corollary 3.8. *The Gauss curvature is given by :*

$$K = \tfrac{eg-f^2}{EG-F^2}$$

and the Mean curvature is given by :

$$H = \tfrac{1}{2}\tfrac{eG-2fF+gE}{EG-F^2}$$

3.2. **Isometries.**

Definition 3.9. A diffeomorphism $\varphi : \mathrm{S} \to \overline{\mathrm{S}}$ is an isometry if for all $\mathrm{p} \in \mathrm{S}$ and all pairs $\mathrm{w}_1, \mathrm{w}_2 \in \mathrm{T_p(S)}$ we have

$$\left\langle \mathrm{w}_1, \mathrm{w}_2\right\rangle_{\mathrm{p}} = \left\langle \mathrm{d}\varphi_{\mathrm{p}}\left(\mathrm{w}_1\right), \mathrm{d}\varphi_{\mathrm{p}}\left(\mathrm{w}_2\right)\right\rangle_{\varphi(\mathrm{p})}$$

The surfaces S and $\overline{\mathrm{S}}$ are then said to be isometric.

Definition 3.10. A map $\varphi : \mathrm{V} \to \overline{\mathrm{S}}$ of a neighborhood V of $p \in S$ is a local isometry at p if there exists a neighborhood $\overline{\mathrm{V}}$ of $\varphi(p) \in \overline{S}$ such that $\varphi : \mathrm{V} \to \overline{\mathrm{V}}$ is an isometry. If there exists a local isometry into $\overline{S}$ at every $p \in S$, the surface S is said to be locally isometric to $\overline{S}$ and $\overline{S}$ is locally isometric to S

Proposition 3.11. *Assume the existence of parametriztions* $\mathbf{x} : \mathrm{U} \to \mathrm{S}$ *and* $\overline{\mathbf{x}} : \mathbf{U} \to \overline{\mathrm{S}}$ *such that* $\mathrm{E} = \overline{\mathrm{E}}, \mathrm{F} = \overline{\mathrm{F}}, \mathrm{G} = \overline{\mathrm{G}}$ *in* U. *Then the map* $\varphi = \overline{\mathbf{x}} \circ \mathbf{x}^{-1} : \mathbf{x}(\mathrm{U}) \to \overline{\mathrm{S}}$ *is a local isometry.*

Definition 3.12. A diffeomorphism $\varphi : S \to \overline{S}$ is called a conformal map if for all $p \in S$ and all $v_1, v_2 \in T_p(S)$ we have

$$\left\langle \mathrm{d}\varphi_{\mathrm{p}}\left(\mathrm{v}_1\right), \mathrm{d}\varphi_{\mathrm{p}}\left(\mathrm{v}_2\right)\right\rangle = \lambda^2(\mathrm{p}) \left\langle \mathrm{v}_1, \mathrm{v}_2 \right\rangle_{\mathrm{p}}$$

where λ^2 is a nowhere-zero differentiable function on S; the surfaces S and $\overline{S}$ are then said to be conformal. A map $\varphi : \mathbf{V} \to \overline{\mathbf{S}}$ of a neighborhood V of $p \in S$ into $\overline{S}$ is a conformal map at p if there exists a nighborhood $\overline{V}$ of $\varphi(\mathrm{p})$ such that $\varphi : \mathrm{V} \to \overline{\mathrm{V}}$ is a conformal map. If for each $p \in S$, there exists a conformal map at p, the surface S is said to be conformal to $\overline{S}$.

$$\cos\overline{\theta} = \frac{\left\langle d\varphi\left(\alpha'\right), d\varphi\left(\beta'\right)\right\rangle}{|d\varphi(\alpha')||d\varphi(\beta')|} = \frac{\lambda^2\left\langle \alpha', \beta'\right\rangle}{\lambda^2|\alpha'||\beta'|} = \cos\theta$$

Proposition 3.13. *Let* $\mathbf{x} : \mathrm{U} \to \mathrm{S}$ *and* $\overline{\mathbf{x}} : \mathrm{U} \to \overline{\mathrm{S}}$ *be parametrizations such that* $\mathrm{E} = \lambda^2\overline{\mathrm{E}}, \mathrm{F} = \lambda^2\overline{\mathrm{F}}, \mathrm{G} = \lambda^2\overline{\mathrm{G}}$ *in* U, *where* λ^2 *is a nowhere-zero differentiable function in* U. *Then the map* $\varphi = \overline{\mathbf{x}} \circ \mathbf{x}^{-1} : \mathbf{x}(\mathbf{U}) \to \overline{\mathbf{S}}$ *is a local conformal map*

Theorem 3.14. *Any two regular surfaces are locally conformal.*

Definition 3.15. Isothermal coordinates,

$$E = \lambda^2(u, v) > 0, \quad F = 0, \quad G = \lambda^2(u, v)$$

3.3. **Equations of Compatibility.** We can express x_u, X_v and N in the basis determined by them as,

$$\begin{aligned}
\mathbf{x}_{uu} &= \mathbf{\Gamma}_{11}^1\mathbf{x}_u + \mathbf{\Gamma}_{11}^2\mathbf{x}_v + L_1N \\
\mathbf{x}_{uv} &= \mathbf{\Gamma}_{12}^1\mathbf{x}_u + \mathbf{\Gamma}_{12}^2\mathbf{x}_v + L_2N \\
\mathbf{x}_{vu} &= \mathbf{\Gamma}_{21}^1\mathbf{x}_u + \mathbf{\Gamma}_{21}^2\mathbf{x}_v + \overline{L}_2N \\
\mathbf{x}_{vv} &= \mathbf{\Gamma}_{22}^1\mathbf{x}_u + \mathbf{\Gamma}_{22}^2\mathbf{x}_v + L_3N \\
N_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v \\
N_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v
\end{aligned}$$

We determine the Cristoffel symbols by taking suitable inner products as follows,

$$\begin{cases} \Gamma_{11}^1E + \Gamma_{11}^2F = \langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle = \frac{1}{2}E_u \\ \Gamma_{11}^1F + \Gamma_{11}^2G = \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle = F_u - \frac{1}{2}E_v \end{cases}$$

$$\begin{cases} \Gamma_{12}^1E + \Gamma_{12}^2F = \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle = \frac{1}{2}E_v \\ \Gamma_{12}^1F + \Gamma_{12}^2G = \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle = \frac{1}{2}G_u \end{cases}$$

$$\begin{cases} \Gamma_{22}^1E + \Gamma_{22}^2F = \langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle = F_v - \frac{1}{2}G_u \\ \Gamma_{22}^1F + \Gamma_{22}^2G = \langle \mathbf{x}_{vv}, \mathbf{x}_v\rangle = \frac{1}{2}G_v \end{cases}$$

3.4. **Theorema Egregium.** We derive relations between the Cristoffel symbols as below and hence express Gaussian Curvature completely in terms of Christoffel symbols showing its invarian under isometries.

$$
\begin{aligned}
\left(\mathbf{x}_{uu}\right)_v-\left(\mathbf{x}_{uv}\right)_u&=0\\
\left(\mathbf{x}_{vv}\right)_u-\left(\mathbf{x}_{uu}\right)_v&=0\\
N_{uv}-N_{vu}&=0\\
A_1\mathbf{x}_u+B_1\mathbf{x}_v+C_1N&=0\\
A_2\mathbf{x}_u+B_2\mathbf{x}_v+C_2N&=0\\
A_3\mathbf{x}_u+B_3\mathbf{x}_v+C_3N&=0\\
A_i=0,\quad B_i=0,\quad C_i=0,\quad i&=1,2,3\\
\boldsymbol{\Gamma}_{11}^1\mathbf{x}_{uv}+\boldsymbol{\Gamma}_{11}^2\mathbf{x}_{vv}+eN_v+\left(\boldsymbol{\Gamma}_{11}^1\right)_v\mathbf{x}_u+\left(\boldsymbol{\Gamma}_{11}^2\right)_v\mathbf{x}_v+e_vN&\\
=\boldsymbol{\Gamma}_{12}^1\mathbf{x}_{uu}+\boldsymbol{\Gamma}_{12}^2\mathbf{x}_{vu}+fN_u+\left(\boldsymbol{\Gamma}_{12}^1\mathbf{x}_u+\left(\boldsymbol{\Gamma}_{12}^2\right)_u\mathbf{x}_v+f_uN\right)&\\
\Gamma_{11}^1\quad \Gamma_{12}^2+\Gamma_{11}^2\Gamma_{22}^2+ea_{22}+\left(\Gamma_{11}^2\right)_v&\\
=\Gamma_{12}^1\Gamma_{11}^2+\Gamma_{12}^2\Gamma_{12}^2+fa_{21}+\left(\Gamma_{12}^2\right)_u&\\
\left(\boldsymbol{\Gamma}_{12}^2\right)_u-\left(\boldsymbol{\Gamma}_{11}^2\right)_v+\boldsymbol{\Gamma}_{12}^1\boldsymbol{\Gamma}_{11}^2\quad&+\boldsymbol{\Gamma}_{12}^2\boldsymbol{\Gamma}_{12}^2-\boldsymbol{\Gamma}_{11}^2\boldsymbol{\Gamma}_{22}^2-\boldsymbol{\Gamma}_{11}^1\boldsymbol{\Gamma}_{12}^2\\
&=-E\tfrac{eg-f^2}{EG-F^2}\\
&=-EK
\end{aligned}
$$

Theorem 3.16. *The Gaussian curvature K of a surface is invariant by local isometries.*

Theorem 3.17. *(Bonnet) Let E, F, G, e, f, g be differentiable functions. defined in an open set* $\mathrm{V}\subset\mathrm{R}^2$, *with* $\mathrm{E}>0$ *and* $\mathrm{G}>0$. *Assume that the given functions satisfy formally the Gauss and Mainardi-Codazzi equations and that* $\mathrm{EG}-\mathrm{F}^2>0$. *Then, for every* $q\in\mathrm{V}$ *there exists a neighborhood* $U\subset\mathrm{V}$ *of* q *and a diffeomorphism* x*:* $U\rightarrow\mathrm{U}\rightarrow\mathrm{x}(\mathrm{U})\subset\mathrm{R}^3$ *such that the regular surface* $\mathbf{x}(\mathrm{U})\subset\mathrm{R}^3$ *has* $\mathrm{E},\mathrm{F},\mathrm{G}$ *and* $\mathrm{e},\mathrm{f},\mathrm{g}$ *as coefficients of the first and second fundamental forms, respectively. Furthermore, if* U *is connected and if*

$$\overline{\mathrm{x}}:\mathrm{U}\rightarrow\overline{\mathrm{x}}(\mathrm{U})\subset\mathrm{R}^3$$

is another diffeomorphism satisfying the same conditions, then there exist a translation T and a proper linear orthogonal transformation ρ in R^3 *such that* $\overline{\mathrm{x}}=\mathrm{T}\circ\rho\circ\mathbf{x}$

4. Parallel transport

Definition 4.1. Let w be a differentiable vector field in an open set $\mathrm{U}\subset\mathrm{S}$ and $\mathrm{p}\in\mathrm{U}$. Let $\mathrm{y}\in\mathrm{T}_{\mathrm{p}}(\mathrm{S})$. Consider a parametrized curve

$$\alpha:(-\epsilon,\epsilon)\rightarrow U$$

with $\alpha(0)=\mathrm{p}$ and $\alpha'(0)=\mathrm{y}$, and let $\mathrm{w}(\mathrm{t}),\mathrm{t}\in(-\epsilon,\epsilon)$, be the restriction of the vector field w to the curve α. The vector obtained by the normal projection of $(\mathrm{dw}/\mathrm{dt})(0)$ onto the plane $\mathrm{T}_{\mathrm{p}}(\mathrm{S})$ is called the covariant derivative at p of the vector field wrelative to the vector y. This covariant derivative is denoted by $(\mathrm{Dw}/\mathrm{dt})(0)$ or $\left(\mathrm{D}_{\mathrm{y}}\mathrm{w}\right)(\mathrm{p})$

Definition 4.2. *A* parametrized curve $\alpha:[0,l]\rightarrow\mathrm{S}$ is the restriction to $[0,l]$ of a differentiable mapping of $(0-\epsilon,l+\epsilon),\epsilon>0$, into S. If $\alpha(0)=\mathrm{p}$ and $\alpha(l)=\mathrm{q}$, we say that α joins p to q.α is regular if $\alpha'(t)\neq0$ for $\mathrm{t}\in[0,l]$

Definition 4.3. Let $\alpha:\mathrm{I}\rightarrow\mathrm{S}$ be a parametrized curve in S. A vector field w along α is a correspondence that assigns to each $\mathrm{t}\in\mathrm{I}$ *a* vector

$$\mathbf{W}(\mathrm{t})\in\mathrm{T}_{\alpha(\mathrm{t})}(\mathrm{S})$$

The vector field w is differentiable at $t_0 \in I$ if for some parametrization $\mathbf{x}(\mathrm{u}, \mathrm{v})$ in $\alpha\left(\mathrm{t}_0\right)$ the components a(t), b(t) of $\mathrm{w(t)} = \mathbf{a}\mathbf{x}_u + \mathrm{b}\mathbf{x}_v$ are are differentiable functions of t at t_0. w is differentiable in I if it is differentiable for every $t \in I$.

Definition 4.4. Let w be a differentiable vector field along $\alpha : \mathrm{I} \rightarrow \mathrm{S}$. The expression of $(\mathrm{Dw}/\mathrm{dt})(\mathrm{t}), \mathrm{t} \in \mathrm{I}$, is well defined and is called the covariant derivative of w at t.

Definition 4.5. A vector field w along a parametrized curve $\alpha : \mathrm{I} \rightarrow S$ is said to be parallel if $\mathrm{Dw}/\mathrm{dt} = 0$ for every $\mathrm{t} \in \mathrm{I}$

Proposition 4.6. *Let w and v be parallel vector fields along* $\alpha : \mathrm{I} \rightarrow \mathrm{S}$. *Then* $\langle \mathrm{w(t)}, \mathrm{v(t)} \rangle$ *is constant. In particular,* $|\mathrm{w(t)}|$ *and* $|\mathrm{v(t)}|$ *are constant, and the angle between v(t) and w(t) is constant.*

Proposition 4.7. *Let* $\alpha : \mathrm{I} \rightarrow S$ *be a parametrized curve in* S *and let* $\mathrm{w}_0 \in \mathrm{T}_{\alpha(\mathrm{t}_0)}(\mathrm{S}), \mathrm{t}_0 \in \mathrm{I}$. *Then there exists a unique parallel vector field w(t) along* $\alpha(\mathrm{t})$, *with* $\mathrm{w}\left(\mathrm{t}_0\right) = \mathrm{w}_0$.

Definition 4.8. Let $\alpha : \mathrm{I} \rightarrow \mathrm{S}$ be a parametrized curve and $\mathrm{w}_0 \in \mathrm{T}_{\alpha(\mathrm{t}_0)}(\mathrm{S}), \mathrm{t}_0 \in \mathrm{I}$. Let w be the parallel vector field along α, with $\mathrm{w}\left(\mathrm{t}_0\right) = \mathrm{w}_0$. The vector $\mathrm{w}\left(\mathrm{t}_1\right), \mathrm{t}_1 \in \mathrm{I}$, is called the parallel transport of w_0 along α at the point t_1

5. Geodesic Curvature

Definition 5.1. A nonconstant, parametrized curve $\gamma : \mathrm{I} \rightarrow \mathrm{S}$ is said to be geodesic at $t \in I$ if the field at it's tangent vectors $\gamma'(\mathrm{t})$ is parallel along γ at t, this is :

$$\frac{\mathrm{D}\gamma'(\mathrm{t})}{\mathrm{dt}} = 0$$

γ is a parametrized geodesic if it is a geodesic for all $t \in I$.

Definition 5.2. A regular connected curve C is S is said to be a geodesic if, for every $p \in C$, the parametrisation $\alpha(s)$ of a coordinate neighborhood of p by the arc length s is a parametrized geodesic; that is, $\alpha'(\mathrm{s})$ is a parallel vector field along $\alpha(\mathrm{s})$

Definition 5.3. Let w be a differentiable field of unit vectors along a parametrized curve $\alpha : \mathrm{I} \rightarrow \mathrm{S}$ on an oriented surface S. Since w(t), $\mathrm{t} \in \mathrm{I}$, is a unit vecto field, $(\mathrm{dw}/\mathrm{dt})(\mathrm{t})$ is normal to w(t), and therefore

$$\frac{\mathrm{Dw}}{\mathrm{dt}} = \lambda(\mathrm{N} \wedge \mathrm{w(t)})$$

The real number $\lambda = \lambda(\mathrm{t})$, denoted *by*[Dw/dt] is called the algebraic value of the covariant derivative of w at t.

Definition 5.4. Let C be an oriented regular curve contained in an oriented surface S, and let $\alpha(s)$ be a parametrization of C, in a neighborhood of $p \in S$, by the arc length s. The algebraic value of the covariant derivative $[\mathrm{D}\alpha'(\mathrm{s})/\mathrm{ds}] = \mathrm{k_g}$ of $\alpha'(\mathrm{s})$ at p is called the geodesic curvature of C at p.

Lemma 5.5. *Let a and be differentiable functions in I with* $\mathrm{a}^2 + \mathrm{b}^2 = 1$ *and* φ_0 *be such that* $\mathrm{a}\left(\mathrm{t}_0\right) = \cos\varphi_0, \mathrm{b}\left(\mathrm{t}_0\right) = \sin\varphi_0$. *Then the differentiable function*

$$\varphi = \varphi_0 + \int_{t_0}^{t} \left(ab' - ba'\right) dt$$

is such that $\cos\varphi(\mathrm{t}) = \mathrm{a(t)}, \sin\varphi(\mathrm{t}) = \mathrm{b(t)}, \mathrm{t} \in \mathrm{I}$, *and* $\varphi\left(\mathrm{t}_0\right) = \varphi_0$.

Lemma 5.6. *Let v and w be two differentiable vector fields along the curve* $\alpha : \mathrm{I} \rightarrow \mathrm{S}$, *with* $|\mathrm{w(t)}| = |\mathrm{v(t)}| = 1, \mathrm{t} \in \mathrm{I}$. *Then*

$$\left[\frac{\mathrm{Dw}}{\mathrm{dt}}\right]-\left[\frac{\mathrm{Dv}}{\mathrm{dt}}\right]=\frac{\mathrm{d}\varphi}{\mathrm{dt}}$$

where φ is one of the differentiable determinations of the angle from v to w, as given by the previous Lemma.

Proposition 5.7. *Let $\mathbf{x}(\mathbf{u},\mathbf{v})$ be an orthogonal parametrization (that is, $\mathrm{F}=0$) of a neighborhood of an oriented surface S, and w(t) be a differentiable field of unit vectors along the curve $\mathbf{x}(\mathrm{u}(\mathrm{t}),\mathrm{v}(\mathrm{t}))$. Then,*

$$\left[\frac{\mathrm{Dw}}{\mathrm{dt}}\right]=\frac{1}{2\sqrt{\mathrm{EG}}}\left\{\mathrm{G}_{\mathrm{u}}\frac{\mathrm{dv}}{\mathrm{dt}}-\mathrm{E}_{\mathrm{v}}\frac{\mathrm{du}}{\mathrm{dt}}\right\}+\frac{\mathrm{d}\varphi}{\mathrm{dt}}$$

where $\varphi(\mathrm{t})$ is the angle from $\mathbf{x}_{\mathrm{u}}$ to $\mathrm{w}(\mathrm{t})$ in the given orientation.

Proposition 5.8. *(Liouville). Let $\alpha(\mathrm{s})$ be a parametrization by arc length of a neighborhood of a point $\mathrm{p}\in\mathrm{S}$ of a regular oriented curve C on an oriented surface S. Let $\mathbf{x}(\mathrm{u},\mathrm{v})$ be an orthogonal parametrization of S in p and $\varphi(\mathrm{s})$ be the angle that $\mathbf{x}_{\mathrm{u}}$ makes with $\alpha'(\mathrm{s})$ in the given orientation. Then*

$$\mathrm{k}_{\mathrm{g}}=(\mathrm{k}_{\mathrm{g}})_1\cos\varphi+(\mathrm{k}_{\mathrm{g}})_2\sin\varphi+\frac{\mathrm{d}\varphi}{\mathrm{ds}}$$

where $(\mathrm{k}_{\mathrm{g}})_1$ and $(\mathrm{k}_{\mathrm{g}})_2$ are the geodesic curvatures of the coordinate curves $\mathrm{v}=\mathrm{const.}$ and $\mathrm{u}=\mathrm{const.}$ respectively.

Proposition 5.9. *Given a point $\mathrm{p}\in\mathrm{S}$ and a vector $\mathrm{w}\in\mathrm{T}_{\mathrm{p}}(\mathrm{S}),\mathrm{w}\neq 0$ there exist an $\epsilon>0$ and a unique parametrized geodesic $\gamma:(-\epsilon,\epsilon)\to\mathrm{S}$ such that $\gamma(0)=p,\gamma'(0)=\mathrm{w}$.*

6. Gauss-Bonnet Theorem and its applications

Theorem 6.1. *(of Turning Tangents) We have for plane curves :*

$$\sum_{i=0}^{k}\left(\varphi_i\left(t_{i+1}\right)-\varphi_i\left(t_i\right)\right)+\sum_{i=0}^{k}\theta_i=\pm 2\pi$$

where the sign depends on the orientation of α.

6.1. **Local Gauss-Bonnet.**

Theorem 6.2. ***GAUSS-BONNET THEOREM (Local)*** *Let Let x: $\mathrm{U}\to S$ be an isothermal parametrization of an oriented surface S, where $\mathrm{U}\subset\mathrm{R}^2$ is homeomorphic to an open disk and x is compatible with the orientation of S.*

Let $\mathrm{R}\subset\mathbf{x}(\mathrm{U})$ be a simple region of S and let $\alpha:I\to S$ be such that $\partial\mathrm{R}=\alpha(\mathrm{I})$. Assume that α is positively oriented, parametrized by arc length s,and let $\alpha(\mathrm{s}_0),\ldots,\alpha(\mathrm{s}_{\mathrm{k}})$ and $\theta_0,\ldots,\theta_{\mathrm{k}}$ be, respectively, the vertices and the external edges of α. Then

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}}k_g(S)ds+\iint_{\mathbb{R}}Kd\sigma+\sum_{i=0}^{k}\theta_i=2\pi$$

where $\mathrm{k}_{\mathrm{g}}(\mathrm{s})$ is the geodesic curvature of the regular arcs of α and K is the Gaussian curvature of S

Proof. Let $u=u(s),v=v(s)$ be the expression of α in the parametrization of $\mathbf{x}$. We have :

$$k_g(s)=\frac{1}{2\sqrt{EG}}\left\{G_u\frac{dv}{ds}-E_v\frac{du}{ds}\right\}+\frac{d\varphi_i}{ds}$$

$$\begin{aligned}\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}}k_g(s)ds=&\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}}\left(\frac{G_u}{2\sqrt{EG}}\frac{dv}{ds}-\frac{E_v}{2\sqrt{EG}}\frac{du}{ds}\right)ds\\&+\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}}\frac{d\varphi_i}{ds}ds\end{aligned}$$

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \left(P\frac{du}{ds} + Q\frac{dv}{ds}\right) ds = \iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) dudv$$

It follows that:

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)ds = \iint_{\mathbf{x}^{-1}(R)} \left\{ \left(\frac{E_v}{2\sqrt{EG}}\right)_v + \left(\frac{G_u}{2\sqrt{EG}}\right)_u \right\} dudv$$
$$+ \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds$$

$$\begin{aligned} \left(\frac{E_v}{2\sqrt{EG}}\right)_v + \left(\frac{G_u}{2\sqrt{EG}}\right)_u &= \frac{1}{2} \left\{ \left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u \right\} \\ &= \frac{1}{2\lambda} \left\{ (\log\lambda)_{vv} + (\log\lambda)_{uut} \right\} \lambda \\ &= \frac{1}{2\lambda} (\Delta \log\lambda)\lambda = -K\lambda \end{aligned}$$

$$\sum_{i=0}^{k} \int_{s_i}^{s_i+1} k_g(s)ds = -\iint_R K\lambda dudv + \sum_i \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} ds$$

on the other hand, by the theorem of turning tangents,

$$\begin{aligned} \sum_{i=0}^{k} \int_{s_i}^{s_i+1} \frac{d\varphi_i}{ds} ds &= \sum_{i=0}^{k} \left(\varphi_i\left(s_{i+1}\right) - \varphi_i\left(s_i\right)\right) \\ &= \pm 2\pi - \sum_{i=0}^{k} \theta_i \end{aligned}$$

Putting these facts together, we obtain :

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)ds + \iint_R K d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi.$$

□

6.2. **Global Gauss-Bonnet.**

Proposition 6.3. *Every regular region of a surface admits a triangulation*

Proposition 6.4. *Let S be an oriented surface and* $\{\mathbf{x}_\alpha\}, \alpha \in \mathrm{A}$, *a family of parametrizations compatible with the orientation of S. Let* $R \subset \mathrm{S}$ *be a regular region of S. Then there is a triangulation of* $\mathfrak{J}$ *of R such that every triangle* $\mathrm{T} \in \mathfrak{J}$ *is contained in some coordinate neighborhood of the family* $\{\mathbf{x}_\alpha\}$. *Furthermore, if the boundary of every triangle of* $\mathfrak{J}$ *is positively oriented, adjacent triangles determine opposite orientations in the common edge*

Proposition 6.5. *If* $R \subset S$ *is a regular region of a surface S, the Euler-Poincar characteristic does not depend on the triangulation of R. It is convenient, therefore, to denote it by* $\chi(\mathrm{R})$

Proposition 6.6. *Let* $\mathrm{S} \subset \mathrm{R}^3$ *be a compact connected surface; then one of the values* $2, 0, -2, \ldots, -2\mathrm{n}, \ldots$ *is assumed by the Euler-Poincar characteristic* $\chi(\mathrm{S})$. *Furthermore, if* $\mathrm{S}' \subset \mathrm{R}^3$ *is another compact surface and* $\chi(\mathrm{S}) = \chi(\mathrm{S}')$, *then* S *is homeomorphic to S.*

Proposition 6.7. *With the above notation, the sum*

$$\sum_{j=1}^{k} \iint_{\mathbf{x}_\mathrm{j}^{-1}(\mathrm{T_j})} \mathrm{f}\left(\mathrm{u_i}, \mathrm{v_j}\right) \sqrt{\mathrm{E_j G_j - F_j^2}} \mathrm{du_j dv_j}$$

does not depend on the triangulation $\mathfrak{J}$ or on the family $\{\mathbf{x}_{\mathrm{j}}\}$ of parametrizations of S.

Theorem 6.8. ***GLOBAL GAUSS-BONNET THEOREM*** *Let $R \subset \mathrm{S}$ be a regular region of an oriented surface and let $\mathrm{C}_1, \ldots, \mathrm{C}_{\mathrm{n}}$ be the closed, simple, piecewise regular curves which form the boundary $\partial\mathrm{R}$ of R. Suppose that each C_{i} is positively oriented and let $\theta_1, \ldots, \theta_{\mathrm{p}}$ be the set of all external angles of the curves $\mathrm{C}_1, \ldots, \mathrm{C}_{\mathrm{n}}$. Then*

$$\sum_{i=1}^{n} \int_{c_i} k_g(s)ds + \iint_R K d\sigma + \sum_{l=1}^{p} \theta_l = 2\pi\chi(\mathrm{R})$$

where s denotes the arc length of C_i, and the integral over C_i means the sum of integrals in every regular arc of C_{i}.

Proof.

$$\sum_i \int_{c_i} k_g(s)ds + \iint_R K d\sigma + \sum_{j,k=1}^{F,3} \theta_{jk} = 2\pi F$$

$$\sum_{j,k} \theta_{jk} = \sum_{j,k} \pi - \sum_{j,k} \varphi_{jk} = 3\pi F - \sum_{j,k} \varphi_{jk}$$

$$\sum_{j,k} \theta_{jk} = 2\pi E_i + \pi E_e - \sum_{j,k} \varphi_{jk}$$

$$\sum_{j,k} \theta_{jk} = 2\pi E_i + \pi E_e - 2\pi V_i - \pi V_{et} - \sum_l (\pi - \theta_i)$$

$$\begin{aligned}\sum_{j,k} \theta_{jk} &= 2\pi E_i + 2\pi E_e - 2\pi V_i - \pi V_e - \pi V_{et} - \pi V_{ec} + \sum_l \theta_i \\ &= 2\pi E - 2\pi V + \sum_i \theta_i\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n} \int_{C_i} k_g(s)ds + \iint_R K d\sigma + \sum_{i=1}^{p} \theta_l &= 2\pi(F - E + V) \\ &= 2\pi\chi(R)\end{aligned}$$

□

Corollary 6.9. *If R is a simple region of S, then*

$$\sum_{i=0}^{k} \int_{s_i}^{s_i+1} k_g(s)ds + \iint_R K d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi$$

Corollary 6.10. *Let S be an orientable compact surface; then*

$$\iint_{\mathrm{s}} \mathrm{K}d\sigma = 2\pi\chi(\mathrm{S})$$

6.3. **Applications. Applications of the Gauss-Bonnet Theorem :**

1. A compact surface of positive curvature is homeomorphic to a sphere.
2. Let S be an orientable surface of negative or zero curvature. Then two geodesics γ_1 and γ_2 which start from a point $p \in S$ cannot meet again at a point $\mathrm{q} \in \mathrm{S}$ in such a way that the traces of γ_1 and γ_2 constitute the boundary of a simple region R of S.
3. Let S be a surface diffeomorphic to a cylinder with Gaussian curvature $\mathrm{K} < 0$. Then S has at most one simple closed geodesic.
4. If there exist two simple closed geodesics Γ_1 and Γ_2 on a compact connected surface S of positive curvature, then Γ_1 and Γ_2 intersect.
5. Let $\alpha : \mathrm{I} \to \mathrm{R}^3$ be a closed, regular, parametrized curve with nonzero curvature. Assuming that the curve described by the normal vector n(s) in the unit sphere S^2 is simple. Then n(I) divides S^2 in two regions with equal areas.

(6) Let T be a geodesic triangle in an oriented surface S. Assuming that the Gauss curvature K does not change sign in T Let $\theta_1, \theta_2, \theta_3$ be the external angles of T and let $\varphi_1 = \pi - \theta_1$ $\varphi_2 = \pi - \theta_2, \varphi_3 = \pi - \theta_3$ be its interior angles. By the Gauss-Bonnet theorem,

$$\iint_T K d\sigma + \sum_{i=1}^{3} \theta_i = 2\pi$$

Thus,

$$\iint_T K d\sigma = 2\pi - \sum_{i=1}^{3} (\pi - \varphi_i) = -\pi + \sum_{i=1}^{3} \varphi_i$$

It follows that the sum of the interior angles $\sum_{i=1}^{3} \varphi_i$ of a geodesic triangle is :

(a) Equal to π if $K = 0$
(b) Greater than π if $K > 0$
(c) Smaller than π if $K < 0$