# KNOTS AND KNOT INVARIANTS

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Abstract. This report is a summary of the topics studied during my project under **Professor Chitrabhanu.**
The report summarizes some basic definition of Knots , computes some Knot groups classifies Knots on a Torus (Solid Torus and $\mathbb{T}^2$) and uses those results to prove important results for tame and PL Knots such as Unknotting Theorem and Non-Cancellation Theorem.

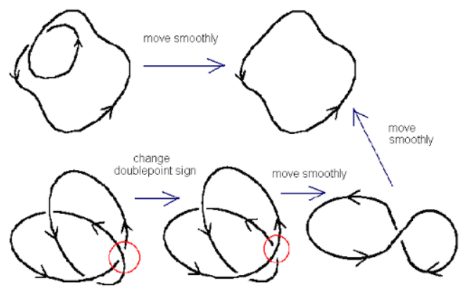
## Acknowledgement

This is to thank institute of **NISER** for giving me such a wonderful opportunity to showcase my talent and to acknowledge **Professor Chitrabhanu Chaudhari** for being patient, insightful and thank him for all his assistance for this project.

## Contents

## 1. Introduction

We will first introduce knots through a brief tour through history of the subject. This brings us to think about how do we define knots mathematically? What makes two knots the same or different? Can we find easy tests to distinguish knots and how do we classify them? We will then, proceed to the commonly accepted definition of a Knot and describe what makes two Knots Different.




### Brief History

Knots were used in tying sails, climbing and even in cloth. Knots were also considered having spiritual and non secular symbolism due to their aesthetic qualities in the past.Following the development of topology in the early 20th century, topologists such as Max Dehn, J. W. Alexander, and Kurt Reidemeister investigated knots. This gave rise to mathematical study now known as *Knot Theory*.

### Basic Definition of a Knot

The inspiration for definition of a knot comes from looking at the simplest of knots commonly referred to as the unknot, unknot is a simple closed loop lying on a 2-dimensional plane in $\mathbb{R}^3$. Since topologically, any closed loop is same as a circle, the unknot can be seen as circle in 3-D Space. In the same way, a general knot can be seen as a circle sitting in 3-D space which is twisted (in 3-D space).

**Definition 1.1.** K $\subset \mathbb{R}^3$ is a *knot* if $\exists$ a continuous map $\phi : \mathbb{S}^1 \to \mathbb{R}^3$ is continuous and $\phi$ is a homeomorphism onto K.
(In general for any topological space X, K $\subset$ X is a knot if $\phi : \mathbb{S}^1 \to$ X is continuous and $\phi$ is a homeomorphism onto K.)

Another equivalent definition is ,

If $\exists$ an embedding $K : \mathbb{S}^1 \to \mathbb{S}^3$ ( In General , any topological space X ) then, K is a *knot*

**Remark 1.2.** *These definitions are equivalent since the map $\phi$ in definition 1 is an embedding of $\mathbb{S}^3$ which is the map K*
*(By considering $\mathbb{S}^3$ as the one point compactification of $\mathbb{R}^3$.)*
*In Definition 2, K($\mathbb{S}^1$) is the Knot as per definition 1.*
*Commonly , definition 1 is used in* [1] *and definition 2 is used in* [2].

In general a link of n components is defined as follows,

**Definition 1.3.** If there is a embedding of the disjoint union of unit circle $\bigcup_{i=1}^{n} \mathbb{S}^1$ into the Euclidean space $\mathbb{R}^3$ whose image is L. Image of each of the circles is called a component of L.

## What makes two Knots different?

Knots are equivalent in 2 majors ways :
(i) Upto to Homeomorphism
(ii) Upto Ambient Isotopy

**Definition 1.4.** Two Knots K and K' in a topological space X are *equivalent upto homeomorphism* if there exists a homeomorphism h: X $\rightarrow$ X such that h(K) = K'

**Definition 1.5.** Two Knots K and $K'$ in a topological space $X$ are *equivalent upto ambient isotopy* if $\exists$ h: $X \times [0,1] \rightarrow X$ i $\in [0,1]$ continuous such that, $h_i$: = h( , i) is a homeomorphism $\forall$ i $\in$ [0,1] and, $h_0$ is the identity and $h_1$(K) = $K'$

**Remark 1.6.** Observe that the above two definitions each define an equivalence relation. And, the equivalence classes of these Knots are called *Knot types*. Commonly, *Knot types* refers to knots equivalent by homeomorphism (unless specified)

**Definition 1.7.** A knot is *polygonal* if it is the union of finitely many (edges) line segments with endpoints (vertices). A knot that has same knot type as a polygonal knot is called a *tame knot* else, it is a *wild knot*.

Equivalently, If $\exists$ a triangulation of $\mathbb{S}^1$ for which the embedding is piecewise linear, the knot K is called *tame*

Let us see some examples of this.

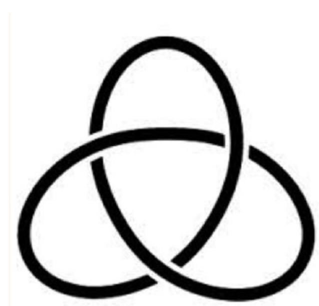


FIGURE 1.
This is the projection of a trefoil knot into the 2D plane. These points of overlap correspond to crossings in 3-D space.Trefoil is a common example of a tame Knot.

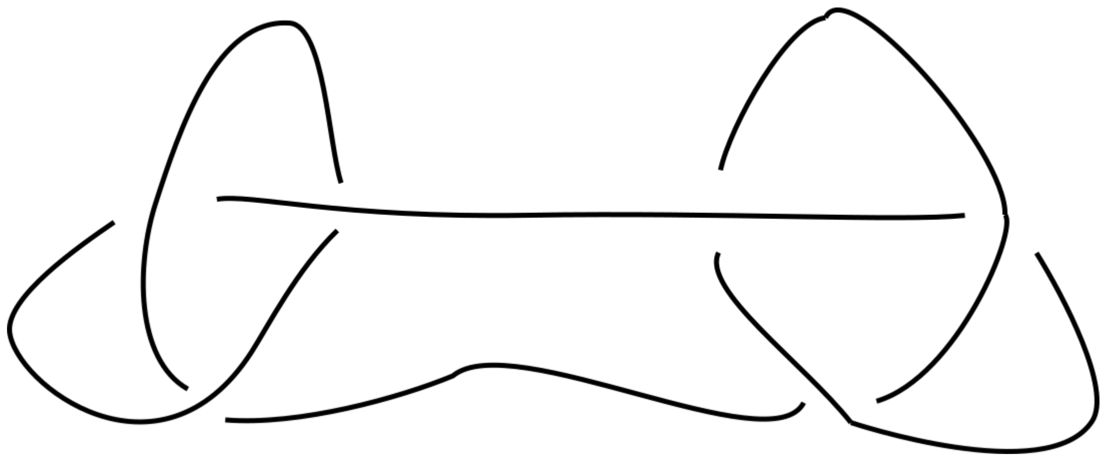


FIGURE 2. This is the projection the square Knot which is a connected sum of two trefoils

Some examples of Links , :

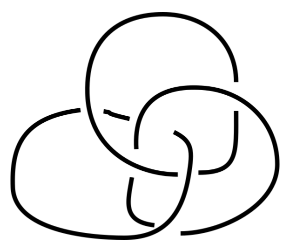


FIGURE 3. This is the Borromean Ring which a link where removing any one component results in the trivial Link

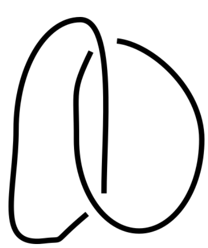


FIGURE 4. This is simplest Link called the Hopf Link.

## Connected Sum of Knots

We will now continue to look into connected sums of knots which is a common way of producing new knots from existing knots.

**Definition 1.8.** If M and M' are two n - manifolds, then their *connected sum of M and M* 'denoted by M # M' is defined as ,

$$M\#M' := M \setminus B^o \cup_h M' \setminus B'^0$$

where B and B' are n - balls contained in M and M' respectively and ,
h:$\partial B \to \partial B'$ is a homeomorphism

**Remark 1.9.** *If A and B are two sets with disjoint interiors $C = A \cup_h B$ is the quotient of the union $A \cup B$ with the association*

$x \sim y$ *in C if* $\begin{cases} y = h(x), x \in \partial A \quad \text{and} \quad y \in \partial B \\ x = y, \text{Otherwise.} \end{cases}$

*where, $h : \partial A \to \partial B$ is a continuous.*
*In general, the connected sum of two Manifolds, M and M' is $M \cup_h$ M' where, $h : \partial M \to \partial M'$ is continuous.*

Extending this idea we define the connected sum for the Pair (M,N) and (M',N') where N and N' are locally flat n -sub-manifolds of m - Manifolds M and M'.

Here by locally flat we just mean that for every point in N there is a neighbourhood, U of M such that $U \cap N \cong B^n$ and $U \cong B^m$.
Then their connected sum is denoted by

$$(M, N)\#(M', N') := (M \cup_{h1} M', N \cup_{h2} N')$$

Where h1:$\partial B^m \subset M \to \partial B^m \subset M'$ is a homeomorphism and,
h2:$\partial B^n \subset N \to \partial B^n \subset N'$ is the corresponding homeomorphism obtained from the locally flat neighbourhoods isomorphic $B^n$ in $B^m$ chosen for M.

In the case of Knots N and N' are the knot K and K' and M = M' = $\mathbb{S}^3$ and the connected sum is denoted by K # K' .

For example,

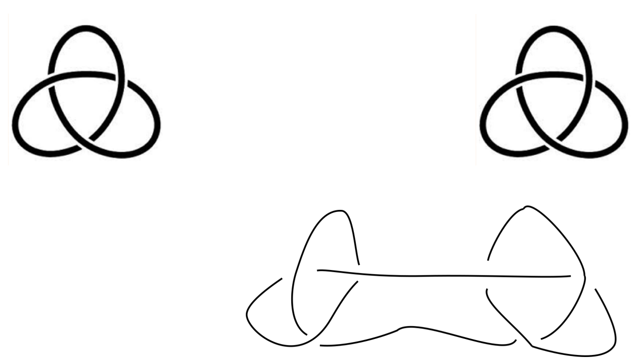


Figure 5. Observe that the Square Knot is connected sum of two Trefoils of opposite orientation obatined.

## Fundamental Groups , Knot Groups and Wirtinger Presentation

**Knot Invariants.**

A knot invariant is a function , f that assigns to Knot K an object f(K) such that equivalent knots are assigned to similair objects.
Thus, if two knots K and K' have different objects for the same invariant f ; (f(K) and f(K') are not similair) then, K and K' are of different type (ie: not equivalent). Hence choosing a good invariant is crucial to distinguishing knots. Some examples are as follows ,

Numerical And Algebraic Invariants :

- **Crossing Number** : Minimal number of simple "self-intersections" in amongst any of the projection of a knot or link.
- **Genus of Link** : Number of "handles" on a minimal surface S spanning the link L.
- **Minimax number** : This is the minimum number of local maxima of the knot K:$\mathbb{S}^3 \to \mathbb{R}^3$ in a given direction. (Relates to the Total Curvature of the Knot , [Milnor - 1950] )
- **Alexander Polynomial** :Polynomial associated with each knot type.
- **Alexander Matrix** : The presentation matrix whose determinant is the Alexander Polynomial
- **Torsion Numbers** :Invariant generated by the finite cyclic covering spaces of a knot complement.
- **Unknotting Number** : This is the minimum number of crossing that have to be changed to turn a projection of the Knot into the Unknot (Holds only for Polygonal Knots).

Miscellaneous Invariants:

- **Tricolorability** : This is simple invariants for polygonal knots which refers to whether or not a Knot can be coloured by using atleast 2 of the 3 distinct colours so that each crossing has all the same colours or all different colours. The Unknot is to not be tricolorable. However the trefoil is tricolorable.

For the rest of this report we will focus only on Topological Invariants.

Topological Invariants :

- **Knot Group** : Fundamental Group of the Knot Complement.Moreover, since knots of the same knot type have homeomorphic complements, the Knot group is the same for each element of the knot type.
- **Link Group** : Fundamental Group of the Link Complement
- **Knot Signature** : These are knot invariants obatined from Seifert Surfaces of the Knot.

**Remark 1.10.** The Link group unlike, the knot group is not a standard link Invariant.Since, there are two different links that have the same Link Group.
For Example,

FIGURE 6. Both of these links have homeomorphic compliments however they're not the same since one is made from two unknots and the other from an unknot and a trefoil.

**Proposition 1.11.** If B is a bounded subset of $\mathbb{R}^3$ and if $\mathbb{R}^3\setminus$ B is path connected then, the natural inclusion induces a map i : $\pi_1(\mathbb{R}^3 \setminus B) \to \pi_1(\mathbb{S}^3 \setminus B)$ which is an isomorphism

*Proof.* (Here, $\mathbb{S}^3$ is seen as , one point compactification of $\mathbb{R}^3$ with the point $\infty$ )
Let us choose U to be an open ball around $\infty$ contained inside $\mathbb{S}^3 \setminus B$.
(This is possible since B is bounded)
Then, U $\cong \mathbb{R}^3$ (As it is a open ball) and U $\cap\ \mathbb{R}^3 \cong \mathbb{R}^3 \setminus 0 \cong S^2$
So, U and U $\cap\ \mathbb{R}^3$ are simply connected.
So, by Van Kampen's Theorem taking U and $\mathbb{R}^3 \setminus B$
(This works since they are both path connected open subsets of $\mathbb{S}^3 \setminus B$).
The map i becomes an isomorphism as, U is simply connected.
Hence Proved. □

Computing the fundamental group in general is a tedious task.
However, computing the knot group for tame knots is rather simple.

**Wirtinger Presentation.** For this section the word overpass refers to part of the knot that crosses above the plane in its orthogonal projection (This will make more sense in the images).

**Definition 1.12.** A *projection* is a orthogonal projection in where the pullback of each point of projection has atmost 2 points and there are only finitely many point with pullback of 2 points which are called *crossings*.

**Theorem 1.13.** The Knot group of a knot K with a given projection , P has a the presentation

$$[x_1, x_2, ..., x_n \quad | \quad r_1, ..., r_n]$$

Where, each $x_i$ corresponds to an overpass in the projection of P and each $r_i$ is a conjugation relation in terms of $x_i$.
Moreover, the nth relation is combination of the other (n - 1) relations .

*Proof.* Let us consider an equivalent knot $K'$ with the n overpasses and the rest of the knot lies on the projected plane P.

(Since in equivalent knots , Knot complements are homeomorphic they have the same Knot group).
So we can divide the knot in to n cases similair to the case discussed below:

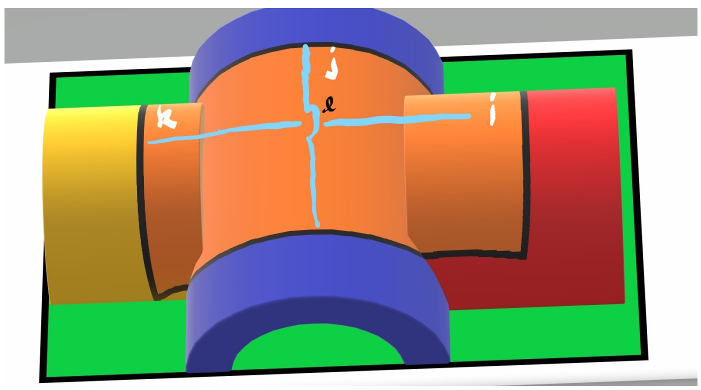



FIGURE 7. The blue line indicates the arcs of the knot

Here jth arc ($\alpha_j$) crosses over the ith arc ($\alpha_i$)and kth arc($\alpha_k$). Here the jth arc ($\alpha_j$) is an overpass.Let l ($\beta_l$) be the part of arc j ($\alpha_j$) that crosses over.

Now, corresponding to each crossing of the Knot we will have the above setup. Start with a rectangle R that covers the entire Knot K.Consider rectangular sheets $R_i$ chosen such that they cover the arcs $\alpha_i$ and are curved to run parallel to the arcs and meets R.In particular, we also choose part l ($\beta_l$) of arc j $\alpha_j$ to lie in the rectangular sheet $R_j$.
Over this arc l ( $\beta_l$ ) we place a rectangle $S_l$ such that two edges identify with arcs parallel to $\alpha_j$ crossing in the interior of $R_i$ and $R_k$ and other two sides with arcs parallel to $\alpha_k$ and $\alpha_i$ in the interior of $R_j$ respectively.
Repeating this process for each crossing we obtained a 2 - D complex X containing K' as a subspace. Finally, we lift K' slightly into the complement of X.

Now, $\mathbb{R}^3 \setminus$ K' deformation retracts onto this complement X (This is evident from a little bit of geometry).
Now , given any loop in X we give it a sequence $x_{i_1}^{\epsilon_1}...x_{i_k}^{\epsilon_k}$ where $\epsilon_i = 1$ if the loop crosses $x_i$ from right to left (or follows the right hand rule) or - 1 if the loop crosses from left to right (does not follow the right hand rule).

**Remark 1.14.** Here the right hand rule is the standard Right hand Thumb Rule (ie : you place thumb along arc $\alpha_i$ if the right hand curls in the direction the loop crosses over the right hand rule holds)

Every loop in X has a sequence associated with it.Since the rectangular sheets in X are simply connected,it follows that if two loops with common endpoints $\gamma$ and

$\gamma'$ have the same sequence then, they are homotopic.

Finally, consider the rectangle $S_l$,Since, $S_l$ is simply connected the constant loop from a corner point of $S_l$ is the same as the boundary loop of $S_l$.
This implies that the loop, This gives us the relation $x_j x_k x_j^{-1} x_i^{-1} = 1$.

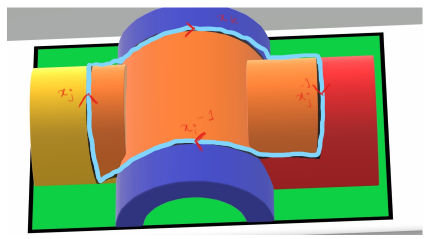


Figure 8. The boundary loop is $x_j x_k x_j^{-1} x_i^{-1}$

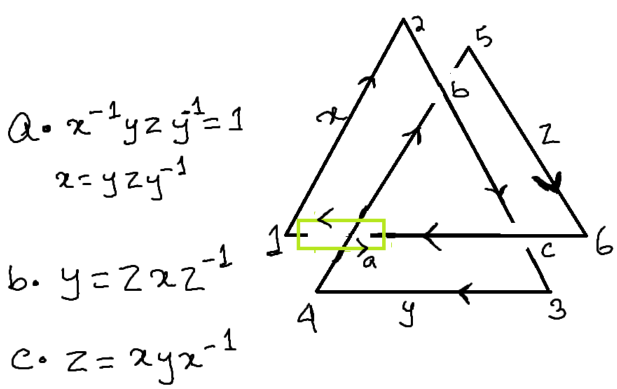
This is commonly written in the form

$$r_l : x_j x_k x_j^{-1} = x_i$$

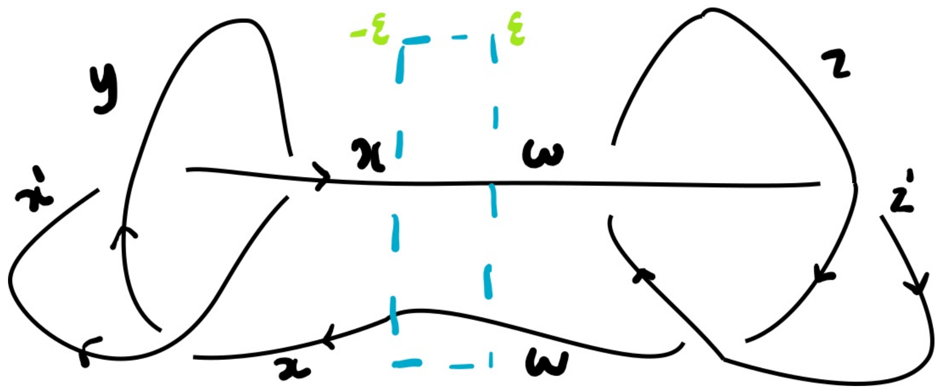
.
We repeat this for each of the n squares $S_l$ getting our n-relations.
Since, Deformation retraction induces isomorphism, we have,
$\pi_1(\mathbb{R}^3 - K) \cong \pi_1(\mathbb{R}^3 - K') \cong \pi_1(X) \cong [x_1, x_2, ..., x_n \quad | \quad r_1, ..., r_n]$
This gives the proof. One can also check that nth relation is a sequence of the other (n - 1) relations

□

## Application of the Wirtinger Presentation




**Knot Group of the Trefoil.** We can calculate for the knot group for trefoil using The Wirtinger presentation, we get the 3 relations as described above
Simplifying these relation we get that ,
The knot group of the trefoil is $[x, y : xyx = yxy]$




**Knot Group of the Square Knot and Granny Knot.** Using the trefoil we can calculate for the Square and Granny Knot.
Let the rectangle , R be $\{(x, 0, z)|a \leq x \leq b, -\epsilon \leq z \leq \epsilon\}$ ,
and K be the square Knot

A := $\{(x, y, z) \in \mathbb{R}^3|z < \epsilon\} \setminus K$ , B : = $\{(x, y, z) \in \mathbb{R}^3|z > -\epsilon\} \setminus K$

Let $X^*$ be one point-compactification of X.

Then, A $\cong \{(x, y, z) \in \mathbb{R}^3|z < \epsilon\}^* \setminus$ Trefoil

Similairly, B $\cong \{(x, y, z) \in \mathbb{R}^3|z > -\epsilon\}^* \setminus$ Trefoil

Since, $\{(x, y, z) \in \mathbb{R}^3|z < \epsilon\}$ is homeomorphic to $\mathbb{R}^3$

We have, $\pi_1(A) \cong$ [ x ,y : yxy = xyx ] , $\pi_1(B) \cong$ [ z ,w : zwz = wzw ]

Consider $\pi_1(A \cap B) \cong$ [ a , b : ] ($A \cap B \cong \mathbb{R}^3$ minus two parallel lines)

Where a and b loops corresponding to one of the parallel straight line segments in rectangle R both of which correspond to x in A and, we can show a and b loops both correspond to w in B
So, by Van Kampen's Theorem,
We have the knot group = $\pi_1(A \cup B)$
= [ x , y ,z , w : xyx = yxy , zwz =wzw , w = x]
= [x , y , z : xyx = yxy , zxz = xzx]

## Knots on a Torus and Solid Torus

**Theorem 1.15.** ***Chord Theorem***
*If X is a path connected subset of the plane and, C be a chord (straight line segment) with endpoints in X and length l(C).*
*Then for each $n \in \mathbb{N}, \exists$ a chord $C_n$ parallel to C with endpoints in X such that*

$$l(C_n) = \frac{l(C)}{n}$$

Using this we can show that:

**Theorem 1.16.** *A loop of class $(a, b)$ of $\pi_1(T^2)$ is a Knot iff $a = b = 0$ (or) GCD$(a, b)$ = 1*

*Proof.* Suppose (a,b) loop class has a non- trivial knot.Let d : = GCD(a,b).
Suppose, d > 1. P : $\mathbb{C} \to \mathbb{T}^2$ , P(x + iy ) = ($e^{ix}$,$e^{iy}$) is a covering map.
Using Homotopy Lifting Property,
Let w : [0,1] $\to \mathbb{T}^2$ be the knot and $\tilde{w} : [0, 1] \to \mathbb{C}$ be lift.
So, $\tilde{w}(1) - \tilde{w}(0) = 2\pi(a + ib)$
From the chord theorem, $\exists$ s , t $\in$ [0,1] such that, $\tilde{w}(s) - \tilde{w}(t) = 2\pi(\frac{a}{d} + i\frac{b}{d})$.
Since, d is a divisor of a and b, w(s) = w(t)
Contradiction

□

**Remark 1.17.** *There is only a single knot of type (a,b) GCD(a,b) = 1, which corresponds to the loop whose lift under the covering map P is a straight line with slope $\frac{b}{a}$ when $a \neq 0$ otherwise it has imaginary axis as its lift*

**Definition 1.18.** $h_L(e^{i\theta}, e^{i\phi}) := (e^{i(\theta-\phi)}, e^{i(\phi)})$
$h_M(e^{i\theta}, e^{i\phi}) := (e^{i\theta}, e^{i(\theta-\phi)})$
$h_I(e^{i\theta}, e^{i\phi}) := (e^{i\phi}, e^{i\theta})$
$h_S(e^{i\theta}, e^{i\phi}) := (e^{i(-\theta)}, e^{i(\phi)})$
These are the 4 *Twist Homeomorphisms* on $\mathbb{T}^2$
Given a loop of type (a,b),
$h_L^*$((a,b)) = (a - b,b)
$h_I^*$((a,b)) = (b,a)
$h_M^*$((a,b)) = (a, a - b)
$h_S^*(a, b)$ := (- a ,b)

**Lemma 1.19.** *Given any loop of type (a,b) and d : = GCD(a,b) then, $\exists$ a self-homeomorphism h on $\mathbb{T}^2$, such that $h^*$((a,b)) = (0,d)*

*Proof.* Proof is by using Euclid's Algorithm for finding GCD,
Using $h_I$ and $h_S$ we assume that,
WLOG Let 0 $\leq$ a $\leq$ b , $\exists$ $b_1$ and $q_1 \in \mathbb{Z}$ such that, b = $q_1$a + $b_1$,
$h_M^{*q_1}$((a,b)) = (a,$b_1$)

Repeating this process and composing these twist homeomorphisms,
We can produce homeomorphism h , $h^*((a,b)) = (0,d)$
Hence Proved. □

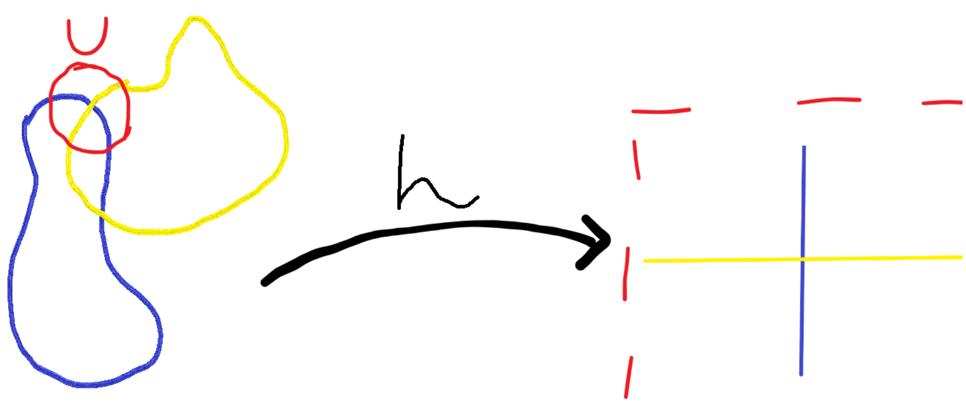
**Definition 1.20.** Consider the covering map q : $\mathbb{C}\setminus\{0\} \to \mathbb{T}^2$ $q(re^{i\theta}) = (e^{i\ ln(r)}, e^{i\theta})$
X $\subset \mathbb{C}\setminus\{0\}$ *lies in a fundamental region*,
if ∃ neighbourhood $U \subset \mathbb{C}\setminus\{0\}$ such that $q|_U$ is a homeomorphism.
Furthermore if, h is any self-homeomorphism on $\mathbb{C}\setminus\{0\}$ such that its support, supp(h) := $\{z \in \mathbb{C}\setminus\{0\} | h(z) \neq z\} \subset U$ , then, h induces a self-homeomorphism on $\mathbb{T}^2$ given by,

$$h' := \begin{cases} qhq^{-1}, x \in q(supp(h)) \\ Id, Otherwise \end{cases}$$

The loops (1,0) = { $(e^{i\theta}, 1)$ } and (0,1) are called *Longitude* and *Meridian* of $\mathbb{T}^2$ respectively.

**Definition 1.21.** If $J$ and $K$ are two knots *are transversal at a point* $p \in J \cap K$, if ∃ a small neighbourhood U and h : $U \to \mathbb{R}^2$ homeomorphism such that $h(U \cap J)$ and $h(U \cap K)$ are perpendicular lines meeting at h(p).

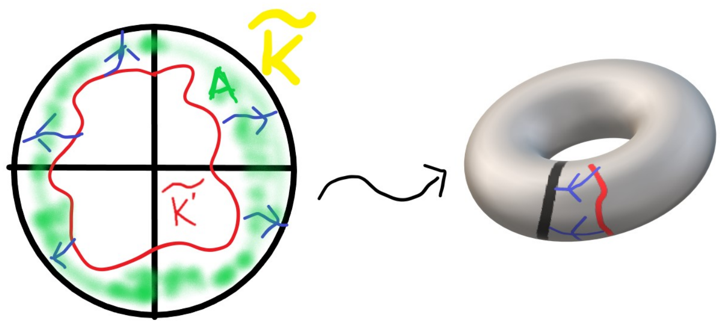
Example :



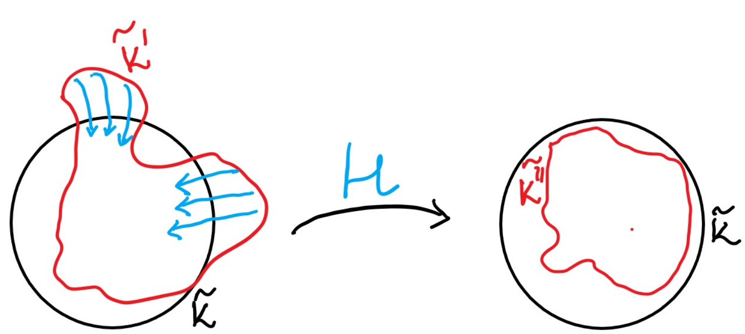
**Theorem 1.22.** *If K and $K'$ are two knots of class (0,± 1) then, K and K' are ambient isotopic.*

*Proof.* We will sketch the idea of the proof.
Case - 1 : If K and $K'$ are disjoint ,

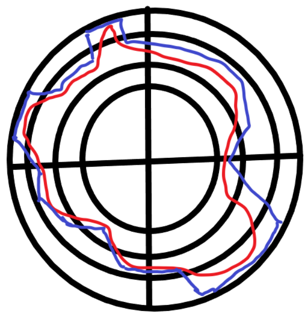
- Let A be the annular region between lifts of K and K' for covering map q (ie : $\tilde{K}$ and $\tilde{K'}$)
- Show A lies in a fundamental region and ,Int(A) has no other liftings .
- Use this to produce the ambient isotopy H with support of $H_t$ lies in a small enough neighbourhood of A taking $\tilde{K}$ to $\tilde{K}'$.
- Show that this produces ambient isotopy between K and $K'$.




Case - 2: K and K' tranversally intersect finitely many times
WLOG K is a meridian (ie : the loop (0,1) ), show that if $K'$ intersect K transversally finitely many times, it is ambient isotopic to $K''$ which is disjoint from K.



Finally, show that for any simple closed curve G in $\mathbb{C} \setminus \{0\}$ and $\epsilon > 0$. $\exists G'$ in the $\epsilon$ neighbourhood of G ( blue arc as shown below) that is homotopic to G and intersects $q^{-1}$(M) transversally atmost finitely many times , for any meridian M Using the Knot corresponding to G', we return to the previous 2 cases. □



Combining these results,

**Theorem 1.23.** *Any knot is of the knot type (0,1) upto homeomorphism. And,upto Ambient Isotopy, any knot types are of the form (a,b) , (-a,-b) GCD(a,b) = 1*

## Knots on a Solid Torus and Higher Dimensional Knots

**Definition 1.24.** A topological space V is called a *Solid Torus*; If $\exists$ a homeomorphism , h from $S^1 \times D^2$ to V . Such a homeomorphism h is called a *framing of V.*

Given simple closed curve J on the boundary of V ( J $\subset \partial$ V) we are interested in 2 kinds classified in the following way :

- J is an essential curve in $\partial$ V but, homologically trivial in V.
- J is an essential curve in $\partial$ V but, homotopically trivial in V.
- J is an essential curve in $\partial$ V and is the boundary of a disk in V.
- J = h( 1 $\times \partial D^2$ ), for some framing h of V.

If any one of the above conditions holds J is called a meridian. Otherwise, if J satisfies any of the following:

- J = h( $S^1 \times 1$) for some framing h of V
- J generators $H_1$(V)
- J intersects some meridian in V transversally at a single point.

J is then called a longitude of V.
As a consequence of these properties, A longitude is equivalent upto homeomorphism and meridians are equivalent upto ambient isotopy
X = $\overline{S^3 \setminus V}$ where, V is a solid torus.
The homology groups of X ( $H_i(X)$ ) are $\begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & Otherwise \end{cases}$
Combining all this results we get,

**Theorem 1.25.** *Upto ambient isotopy there is a unique longitude which is homologically trivial in X.*
*Moreover , if h is a framing of V, h($S^1 \times 1$) is the homologically trivial closed loop in X.*

Finally, We will be using a higher dimensional result.

**Definition 1.26.** f:$B^k \to M^n$ be a embedding is *flat* if $\exists$ neighbourhood $B^k \subset$ U open in $\mathbb{R}^n$ such that $\overline{f} : U \to M$ is an embedding and $\overline{f}(B^k)$ is a *flat ball.*

**Theorem 1.27.** *A knot K in $S^n$ is trivial iff K is the boundary of a flat 2 - ball in $S^n$*

## Properties of PL Knots

From this point on we will consider, each space to be a Simplicial Complex. (In particular, Knots).Each map after a suitable subdivision of the domain and the range sends simplexes linearly to simplexes. (such maps are called PL maps)
We will also assume that every 3-Manifold is homeomorphic with a simplicial complex.

**Theorem 1.28** (Dehn's Lemma)**.** *Suppose that* $f : \mathbb{D}^2 \to M^3$ *is a map into a 3 - Manifold such that,*
*If* $x \in \partial\mathbb{D}^2$ *and* $y \neq x \in \mathbb{D}^2 \implies f(x) \neq f(y)$.
*Then,* $\exists$ *an embedding* $g : \mathbb{D}^2 \to M^3$ *such that* $f(\partial\mathbb{D}^2) = g(\partial\mathbb{D}^2)$

**Corollary 1.29.** $J \subset \partial M$ *is a simple closed curve in a 3 - Manifold, M*
*If J is homotopically trivial in M, then, J bounds a properly bounded disk D ie:* $\partial D \subset \partial M$ *,* $Int(D) \subset Int(M)$

**Definition 1.30.** An embedding f: B $\to$ M from a k - Ball , B to a n - Manifold M is called *flat (Topologically)* extends to an embedding $f' :$ U $\to$ M where, U is a neighbourhood of B.
Then, $f$(B) is called a **flat ball** in M. In our case , k = 1 and n = 3

**Theorem 1.31.** *A knot , K is equivalent to the trivial (Unknot) knot in* $\mathbb{S}^3$ *iff* $\exists$ *a flat 2 - ball in* $\mathbb{S}^3$ *bounded by K.Moreover, any PL 2 - Ball (the disk) is flat.*

**Theorem 1.32.** *If K is a tame Knot then, K is trivial (ie : K is equivalent to the unknot) iff The knot group* $\pi_1(\mathbb{S}^3 \setminus K) \cong \mathbb{Z}$

*Proof.* The proof is as follows,

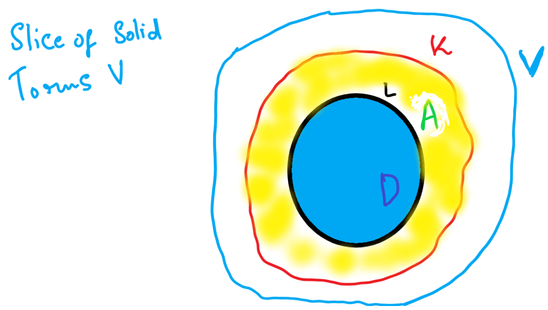
If K is trivial then, $\pi_1(\mathbb{S}^3 \setminus K) \cong \mathbb{Z}$

- Conversely, if $\pi_1(\mathbb{S}^3 \setminus K) \cong \mathbb{Z}$ for a tame knot K.

- Then, Consider the closed tubular neighbourhood, V.
  V $\cong \mathbb{S}^1 \times \mathbb{D}^2$.

- Since, there is a longitude of V ,L is homotopically trivial in $\overline{\mathbb{S}^3 \setminus V}$.
- So, L is homotopically trivial in $(\mathbb{S}^3 \setminus Int(V))$

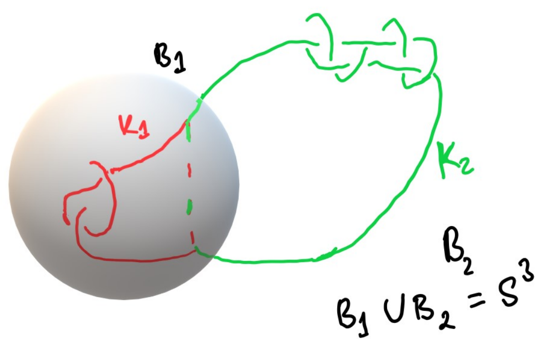
By Dehn's Lemma, L bounds a disk, D in $\mathbb{S}^3 \setminus Int(V)$
Since, L is a longitude, there is an annular region A, between L and the Knot K
Then, A $\bigcup$ D is now a PL disk whose boundary is K.
Therefore by the previous theorem, K is a trivial Knot.
Hence Proved. ∎




**Corollary 1.33.** *$K$ is not the trivial knot iff the natural inclusion of $\pi_1(\partial V) \to \pi_1(\mathbb{S}^3 \setminus Int(V)$ ) is injective*

**Theorem 1.34.** *Connected sum $K = K_1 \# K_2$ of two tame knots is trivial iff $K_1$ and $K_2$ Trivial*

*Proof.* From the definition of connected sum, we can produce two balls $B_1$ and $B_2$ that cover K.
$B_1$ is ball containing $K_1$ and $B_2$ is the ball containing $K_2$ such that their intersection $K_1 \cap K_2$ is on the boundary of B$_1$.
Now, $\pi_1(\mathbb{S}^3 \setminus K_1) \cong \pi_1(B_1 \setminus K)$
Also,$\pi_1(\mathbb{S}^3 \setminus K_2) \cong \pi_1(B_2 \setminus K)$
This gives the following commutative diagram.




$$\begin{array}{ccc} \bullet\pi_1(\partial B_1 \setminus K) & \longrightarrow & \bullet\pi_1(B_1 \setminus K) \\ \downarrow & & \downarrow \\ \bullet\pi_1(B_2 \setminus K) & \longrightarrow & \bullet\pi_1(\mathbb{S}^3 \setminus K) \end{array}$$

Now, if K$_1$ or K$_2$ are non-trivial and K is trivial, then both of these inclusions (from $\pi_1(\partial B_1 \setminus K)$ ) will be injective.
From, Van Kampen's Theorem, $\pi_1(\mathbb{S}^3 \setminus K)$ contains a copy of both these knot groups.
Contradiction.
Therefore, K$_1$ and K$_2$ are both unknots.

□

## Conclusions

(1) Knots on torus are of Knot type ( 0 , 1) and upto Ambient Isotopy [(a,b)] = { (a,b) , (-a,-b) }

(2) Connected Sum ( K # K' ) contains a copy of each of their knot groups.

(3) Connected Sum of n - Non-trivial Tame Knots is a Non-trivial Knot.

(4) Any PL Torus bounds a solid Torus on atleast one side.

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