Indian Institute of Science Education and Research, Pune

Semester Project

Differential Geometry of Curves and Surfaces

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Chapter 1

Theory Of Curves

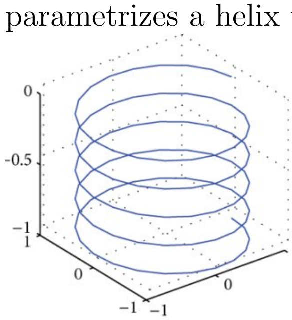
1.1 Parametrized Curves

Definition 1. *A parametrized differentiable curve is a differentiable map* $\alpha : I \to \mathbb{R}^3$ *of an open interval* $I = [a, b]$ *of real line into* $\mathbb{R}^3$.

Example

$$\alpha(t) = (\cos t, \sin t, bt)$$

parametrizes a helix with pitch b which is constant.



1.2 Regular curves, Arc Length Parametrization

Definition 2. $\alpha : I \to \mathbb{R}^3$ *is called regular if* $\alpha'(t) \neq 0 \ \forall \ t \in I$.

Definition 3. *Given a differentiable curve* $\alpha : I \to \mathbb{R}^3$ *the arclength of the curve* *from a point t_0 is defined as*

$$s(t) = \int_{t_0}^{t} \mid \alpha'(t) \mid dt.$$

Now, we can use arc length as parametrization then the velocity of the curve $\alpha'(t)$ will always be of length 1. Hence this parametrization is known as unit speed parametrization.

1.3 The Local theory of curves parametrized by arc length

Definition 4. *Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length $s \in I$. The number $\mid \alpha''(s) \mid = k(s)$ is called the curvature of α at s.*

At points where $k(s) \neq 0$, a unit vector in the direction of $\alpha''(s)$ is well defined by the equation $\alpha''(s) = k(s)n(s)$. Now

$$\alpha'(s) \cdot \alpha'(S) = 1$$

. By differentiating we obtain

$$\alpha''(s) \cdot \alpha'(s) = 0$$

which implies that $n(s)$ and $\alpha'(s)$ are perpendicular to each other.The vector $n(s)$ is known as normal vector to the curve at the point s.

Notation From here onwards the tangent vector $\alpha'(s)$ will be denoted as $t(s)$. Therefore we obtain

$$t'(s) = k(s)n(s).$$

Definition 5. *The plane determined by tangent and normal vector is known as osculating plane.*

Definition 6. *The unit vector $b(s) = t(s) \times n(s)$ is normal to the osculating plane and will be called as binormal vector. Then*

$$b'(s) = t'(s) \times n(s) + t(s) \times n'(s) = t(s) \times n'(s)$$

which implies $b'(s)$ is normal to $t(s)$. It follows that $b'(s)$ is parallel to $n(s)$. Therefore we can write

$$b'(s) = \tau(s)n(s)$$

for some function $\tau(s)$. The scalar function $\tau(s)$ is known as the torsion of the curve at the point s.

- Now consider the following equation

$$n'(s) = b'(s) \times t(s) + b(s) \times t'(s) = -\tau b - kt$$

The following set of equations that we derived is known as Frenet's equation

$$t'(s) = k(s)n(s), \tag{1.1}$$

$$n'(s) = -k(s)t(s) - \tau(s)b(s), \tag{1.2}$$

$$b'(s) = \tau(s)n(s). \tag{1.3}$$

Theorem 1. *Given differentiable functions $k(s) > 0$ and $\tau(s), s \in I$,$\exists$ a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ such that s is arc length k(s) is the curvature and $\tau(s)$ is the torsion of α. Moreover, any other curve $\bar{\alpha}$ satisfying the same conditions, differs from α by a rigid motion,i.e, there is an orthogonal linear map ρ of $\mathbb{R}^3$ with positive determinant and a vector C such that $\bar{\alpha} = \rho \circ \alpha + C$*

This theorem is known as **FUNDAMENTAL THEOREM OF LOCAL THEORY OF CURVES**.

1.4 The Local Canonical Form

Let $\alpha : I \to \mathbb{R}^3$ is unit speed parametrized differentiable curve without singular points of order 1[by which we mean that there is no point for which $\alpha''(s) = 0$. Then we can Taylor expand $\alpha(S)$ around any point s_0. Without loss of generality let us assume that $s_0 = 0$. By Taylor expanding

$$\alpha(s) = \alpha(0) + s\alpha'(0) + \frac{s^2}{2}\alpha''(0) + \frac{s^3}{6}\alpha'''(0) + R$$

where R is of $O(s^3)$. Applying Frenet's equation to this we obtain

$$\alpha(s) - \alpha(0) = (s - \frac{k^2 s^3}{6})t + (\frac{s^2 k}{2} + \frac{s^3 k'}{6})n - \frac{s^3}{6} k\tau b + R.$$

We can chose an orientation such that origin O agrees with $\alpha(0)$ and $t = (1, 0, 0), n = (0, 1, 0)$ and $b = (0, 0, 1)$. Under these condition $\alpha(s)$ is given by

$$\alpha(s) = (x(s), y(s), z(s))$$

where

$$x(s) = s - \frac{k^2 s^3}{6} + R_x, \tag{1.4}$$

$$y(s) = \frac{ks^2}{2} + \frac{s^3 k'}{6} + R_y, \tag{1.5}$$

$$z(s) = \frac{s^3}{6} k\tau + R_z. \tag{1.6}$$

This representation is called the local canonical form of α in a neighbourhood of $s = 0$. The physical interpretation of local canonical form is that each of the terms $x(s), y(s), z(s)$ represents the projection of the curve in $\tau b, bn, n\tau$ plane respectively.

1.5 Global Properties of plane curves

Definition 7. *A closed plane curve is a regular parametrized curve $\alpha : [a, b] \to \mathbb{R}^2$ such that α and all its derivatives agree on a and b, i.e,*

$$\alpha(a) = \alpha(b), \alpha''(a) = \alpha''(b), ...$$

The curve is simple if it does not have any other point of self-intersection.

Theorem 2. *Suppose, C is a simple closed plane curve in the plane $\mathbb{R}^2$. Then $\mathbb{R}^2 - C$ consists of exactly two connected components. One of these components is bounded(interior of C) and the other is unbounded(the exterior) and the curve lies in boundary of both these components.*

This theorem is known as **JORDAN CURVE THEOREM**

Theorem 3. ***The Isoperimetric Inequality*** *Let C be a simple closed plane curve of length l and let A be the area bounded by C. Then, $l^2 - 4\pi A \geq 0$. The equality holds if and only if C is a circle.*

1.6 Example

In this section we'll explicitly work out the example of helix. The helix is represented by the following map

$$\alpha : \mathbb{R} \to \mathbb{R}^3$$
$$\alpha(s) = (a\cos(\frac{s}{c}), a\sin(\frac{s}{c}), b(\frac{s}{c}))$$

where $c^2 = a^2 + b^2$. We'll first prove that the parametrization is arc length parametrization.

$$\int_0^s \mid \alpha'(t) \mid dt \tag{1.7}$$

Now $\mid \alpha'(t) \mid = 1$. Therefore, the value of the above integral is s, which essentially proves that the parametrization is indeed arc length parametrization. Let us calculate the torsion and the curvature of the curve.

$$\alpha'(s) = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}),$$

$$\alpha''(s) = (-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0),$$

Therefore,

$$k(s) = \mid \alpha''(s) \mid = \frac{a}{c^2}.$$

The normal vector will be

$$n(s) = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0).$$

Now let us calculate the binormal vector

$$b(s) = t(s) \times n(s) = (\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}),$$

$$b'(s) = (\frac{b}{c^2}\sin\frac{s}{c}, \frac{b}{c^2}\sin\frac{s}{c}, 0),$$

which implies that

$$b'(s) = \frac{b}{c^2}n(s).$$

Therefore, the torsion of the curve at the point s is given by $\tau(s) = \frac{b}{c^2}$. The torsion of the curve is constant at every point. The equation of the osculating plane is given by

$$(\frac{b}{c}\sin\frac{s}{c})x - (\frac{b}{c}\cos\frac{s}{c})y + (\frac{a}{c})z - \frac{ab}{c^2}s = 0.$$

This curve has an interesting property that every tangent makes an an constant angle of $\frac{\pi}{2}$ with the $Z - axis$, since $(0,0,1)\cdot t(s) = 0$ for all s and both of'em are unit vectors $\cos\theta = 0$ where θ is the angle between them.

We'll explain an interesting property of osculating plane. The osculating plane at s is the limit position of the plane determined by the tangent line at the point s and $\alpha(s+h)$ when $h \to 0$. To prove this let us take $s = 0$. We can chose an orientation such that origin O agrees with $\alpha(0)$ and $t = (1,0,0), n = (0,1,0)$ and $b = (0,0,1)$. Thus every plane containing tangent at $s = 0$ is of the form $z = cy$ or $y = 0$. The plane $y = 0$ is the rectifying plane which does not contain any point near $\alpha(0)$ except $\alpha(0)$ itself. The condition for the plane $z = cy$ to pass through $s + h$ is

$$c = \frac{z(h)}{y(h)} = \frac{-\frac{k}{6}\tau h^3 + ...}{\frac{k}{2}h^2 + \frac{k^2}{6}h^3 + ...}$$

Letting $h \to 0$ we see that $c \to 0$, i.e, the limit position of the plane $z = cy$ is the plane $z = 0$, that is, osculating plane as we wished.

Chapter 2

Regular Surfaces

2.1 Regular Surfaces

First we need to define the concept of a differential of a map.

Definition 8. *Let, $F : U \to \mathbb{R}^m$ where U is an open subset of $\mathbb{R}^n$, be a differentiable map. To each $p \in U$ we associate a linear map $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ and let $\alpha : (-\epsilon, \epsilon) \to U$ be a differentiable curve such that $\alpha(0) = 0, \alpha'(0) = w$. By the chain rule the curve defined as $\beta = F \circ \alpha : (-\epsilon, \epsilon) \to \mathbb{R}^m$ is also differentiable. Then, $dF_p(w) = \beta'(0)$.*

Proposition 1. *The differential of a map is well-defined ,i.e, it is independent of the choice of the curve α.*

Definition 9. *A subset $S \subset \mathbb{R}^3$ is a regular surface if there exists a neighbourhood V in $\mathbb{R}^3$ and a map $\mathbb{X} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that*

1. *$\mathbb{X}$ is differentiable.*
2. *$\mathbb{X}$ is a homeomorphism.*
3. *(The Regularity condition) For all $q \in U$,the differential $d\mathbb{X}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.*

Let us try to find out what is actually meant by the regularity condition. Suppose , $\mathbb{X} : U \to \mathbb{R}^3$ where U is an open set of $\mathbb{R}^2$. Suppose

$$\mathbb{X}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

Consider, the following Jacobian matrix of $\mathbb{X}$

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}.$$

If the map is regular at a point p, the linear transformation defined by the corresponding Jacobian matrix is one-to-one. Therefore, the two columns of the Jacobian matrix is linearly independent and it has rank 2. In other words, any of the 2×2 cofactor matrix has non-zero determinant.

Proposition 2. *If $f : U \to \mathbb{R}$ is a differentiable function on an open set U of $\mathbb{R}^2$. Then the graph of f ,i.e, the subset of $\mathbb{R}^3$ given by $(x, y, f(x, y)$ for $(x, y) \in U$, is a regular surface.*

Proof. We have to prove that $\mathbb{X}(u, v) = (u, v, f(u, v))$ satisfy all the conditions of Definition 9.
clearly, the map is smooth and the first two rows of the Jacobian matrix is linearly independent implying that it has rank 2. All that is left to prove is that $\mathbb{X}$ is a homeomorphism. $\mathbb{X}$ is injective and $\mathbb{X}^{-1}$ is the restriction to the graph of f of the continuous projection map of $\mathbb{R}^3$ onto xy plane , $\mathbb{X}^{-1}$ is continuous.

□

Definition 10. *Given a differentiable map $F : U \to \mathbb{R}^m$ where U is an open set in $\mathbb{R}^m$, We say that a point $p \in U$ is critical if $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective mapping. The image $F(p) \subset \mathbb{R}^m$ of a critical point is called a critical value of F. A point of $\mathbb{R}^m$ is called a regular value if it is not a critical value of F.*

Consider, a map $f : U \to \mathbb{R}$ where U is open in $\mathbb{R}^3$. The matrix of the linear map $df_p : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$df_p = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}.$$

To say that df_p is surjective is equivalent to saying all the three partial derivatives are not simultaneously zero. Hence $a \in f(U)$ is a regular value of f if and only if all the partial derivatives do not simultaneously vanish at any point of the pre-image of a

Proposition 3. *If $f : U \to \mathbb{R}$ where U is an open subset of $\mathbb{R}^3$ is a differentiable function and $a \in f(U)$ is a regular value of f then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$*

Proof. we'll outline the proof. Define a map $F : U \to \mathbb{R}^3$ by $F(x, y, z) = (x, y, f(x, y, z))$. Clearly, $\det(dF_p) \neq 0$ for a pre-image p of a regular value. Apply inverse function theorem on F to construct a coordinate neighbourhood. □

Proposition 4. *Let, $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then $\exists$ a neighbourhood v of p in S such that V is the graph of a differentiable function, which has one of the following three forms: $z = f(x, y), y = g(x, z), x = h(y, z)$.*

This proposition essentially asserts that every regular surface is locally expressible as a graph of a function.

Proposition 5. *Let $p \in S$ be a point of regular surface S and let $\mathbb{X} : U \to \mathbb{R}^3$ with $p \in \mathbb{X}(U) \subset S$ such that conditions 1 and 3 of Definition 9 holds. Assume that $\mathbb{X}$ is continuous, then $\mathbb{X}^{-1}$ is continuous.*

2.2 Change of Parameters, Differentiable functions on Surface

Now we are going to introduce a very important result about the coordinate charts.

Proposition 6. ***Change of Parameters*** *Let, p be a point of regular surface S, and let $\mathbb{X} : U \to S, \mathbb{Y} : V \to S$ be two parametrizations of S such that $p \in \mathbb{X}(U) \cap \mathbb{Y}(V) = W$. Then the "Change of coordinates"*

$$h = \mathbb{X}^{-1} \circ \mathbb{Y} : \mathbb{Y}^{-1}(W) \to \mathbb{X}^{-1}(W)$$

is a diffeomorphism.

Proof. $\mathbb{X}^{-1} \circ \mathbb{Y}$ is a homepormphism since it is composition of two homeomorphisms. Now, define a map

$$F(u, v, t) = (x(u, v), y(u, v), z(u, v) + t)$$

where $\mathbb{X}(u, v) = (x(u, v), y(u, v), z(u, v))$. By exploiting the fact that $\mathbb{X}$ is a parametrization we can orient our axis in such a way that $\det(dF_q) \neq 0$. Therefore, we can apply inverse function theorem to find our required neighbourhood. □

Now, we are going to define the differentiability of a function on a regular surface.

Definition 11. *Let,* $\mathbb{V} \to \mathbb{R}$ *be a function defined on an open subset of a regular surface* S*. Then* f *is said to be differentiable at* $p \in V$ *if for some parametrization* $\mathbb{X} : U \to S$ *with* $p \in \mathbb{X}(U) \subset V$ *the composition* $f \circ \mathbb{X} : \mathbb{R}^2 \to \mathbb{R}$ *is differentiable at* $\mathbb{X}^{-1}(p)$.

In a similar manner we can extend this definition for the functions from regular surface to regular surface.

Definition 12. ***Parametrized Surfaces*** *A parametrized surface* $\mathbb{X} : U \to \mathbb{R}^3$ *is a differentiable map* $\mathbb{X}$ *from an open set* $U \subset \mathbb{R}^2$ *into* $\mathbb{R}^3$*. The set* $\mathbb{X}(U)$ *is called the trace of* $\mathbb{X}$*.* $\mathbb{X}$ *is regular if the differential* $d\mathbb{X}_q : \mathbb{R}^2 \to \mathbb{R}^3$ *is one-to-one for all* $q \in U$*.*

Proposition 7. *Suppose,* $\mathbb{X} : U \to \mathbb{R}^3$ *is a regular parametrized surface and let* $q \in U$*. Then there exists a neighbourhood* V *of* q *in* $\mathbb{R}^2$ *such that* $\mathbb{X}(V) \subset \mathbb{R}^3$ *is a regular surface.*

2.3 The Tangent Plane; The Differential of a Map

Proposition 8. *Let* $\mathbb{X} : U \to S$ *be a parametrization of a regular surface* S *and* $q \in U$*. The vector subspace of dimension 2*

$$d\mathbb{X}_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

coincides with the set of tangent vectors to S *at* $\mathbb{X}(q)$*.*

Proof. Let, w be a tangent vector at $\mathbb{X}(q)$,i.e, $w = \alpha'(0)$ where $\alpha : (-\epsilon, \epsilon) :\to \mathbb{X}(U) \subset S$ is differentiable and $\alpha(0) = \mathbb{X}(q)$. Then $\beta = \mathbb{X}^{-1} \circ \alpha$ is the curve we are looking for.

For the converse part define the curve $\gamma : (-\epsilon, \epsilon) \to U$ given by

$$\gamma(t) = tv + q.$$

Take $\alpha = \mathbb{X} \circ \gamma$. □

This plane $d\mathbb{X}_q(\mathbb{R}^2)$ is known as the tangent plane of the surface S at p and hereafter will be denoted as $T_p(S)$.

The choice of parametrization determines a basis $(\frac{\partial \mathbb{X}}{\partial u}(q), \frac{\partial \mathbb{X}}{\partial v}(q))$. The coordinates of a vector w in this basis is given by $(u'(0), v'(0))$ where $(u(t), v(t))$ is the expression in the parametrization $\mathbb{X}$, of a curve whose velocity vector at $t = 0$ is w.

2.4 First Fundamental Form

Definition 13. *The quadratic form I_p on $T_p(S)$ is defined by*

$$I_p(S) : T_p(S) \to \mathbb{R}$$

$$w \mapsto \langle w, w \rangle_p$$

is called the first fundamental form of the regular surface $S \subset \mathbb{R}^3$ at $p \in S$.

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We'll express the fundamental form in the basis $\{\mathbb{X}_u, \mathbb{X}_v\}$

$$I_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0) \rangle_p = \langle \mathbb{X}_u u' + \mathbb{X}_v v', \mathbb{X}_u u' + \mathbb{X}_v v' \rangle_p = E(u')^2 + 2Fu'v' + G(v')^2$$

where

$$E = \langle \mathbb{X}_u, \mathbb{X}_u \rangle,$$
$$F = \langle \mathbb{X}_u, \mathbb{X}_v \rangle,$$
$$G = \langle \mathbb{X}_v, \mathbb{X}_v \rangle.$$

Now we are going to define area formally-

Definition 14. *Let, $R \subset S$ be a bounded region of regular surface contained in the coordinate neighbourhood of the parametrization $\mathbb{X} : U \to S$. The positive number*

$$\iint_Q |\ \mathbb{X}_u \times \mathbb{X}_v\ |\ du\, dv = A(R)$$

where $Q = \mathbb{X}^{-1}(R)$, is called the area of R.

2.5 Example

In this section we'll explicitly show the details for two different regular surfaces, namely $2-sphere$ and surface of revolution.

2.5.1 $2-sphere$

The $2-sphere$ is defined as the set $\{(x,y,z) \mid x^2+y^2+z^2=1\}$. First, let us prove that 2-sphere is a regular surface. We'll prove it using regular value theorem. Define the map

$$F : \mathbb{R}^3 \to \mathbb{R}$$

$$(x,y,z) \mapsto x^2+y^2+z^2.$$

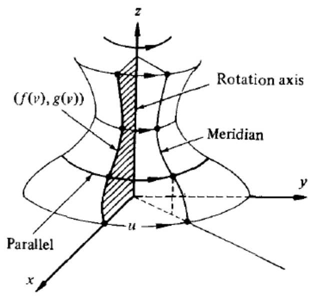
Therefore, $2-sphere$ is the inverse image of the value 1. we have to prove that 1 is a regular value of the map F. The Jacobian matrix of the map F is

$$\begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$

Now, x, y and z can't be simultaneously zero. Therefore, 1 is a regular value of the map F. Hence, it is a regular surface using proposition 3.

2.5.2 Surface of Revolution

Let, $S \subset \mathbb{R}^3$ be the set obtained by rotating a plane curve C about an axis in the plane which does not meet the curve. We shall take xz plane as the plane of curve and $z-axis$ as the axis of rotation.



Let,

$$x = f(v), y = g(v), a < v < b, f(v) > 0$$

be a parametrization for C and u is the rotation angle with respect to $z - axis$. Thus, we obtain a map

$$\mathbb{X}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

from the open set $U = \{(u, v) \in \mathbb{R}^2, 0 < u < 2\pi, a < v < b\}$ into S. $\mathbb{X}$ is clearly differentiable and the Jacobian matrix will be

$$\begin{bmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{bmatrix}$$

Since, the columns cannot be linearly dependent the differential is one-to-one. Therefore, $\mathbb{X}$ satisfies condition 3 of the definition 9. To prove that $\mathbb{X}^{-1}$ is continuous we have to prove that u is a continuous function of (x, y, z).

$$\tan\frac{u}{2} = \frac{\sin u}{1 + \cos u}$$

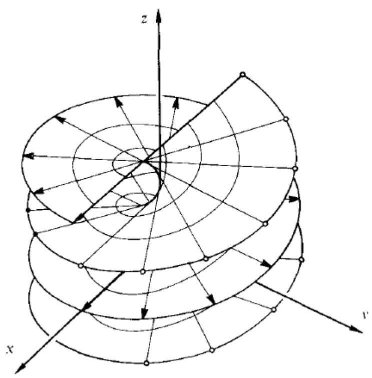
$$= \frac{\frac{y}{f(v)}}{1 + \frac{x}{f(v)}} = \frac{y}{1 + \sqrt{x^2 + y^2}} \Rightarrow u = 2\arctan\frac{y}{1 + \sqrt{x^2 + y^2}}$$

Thus u is a continuous function of (x, y, z). Therefore, $\mathbb{X}^{-1}$ is continuous and $\mathbb{X}$ is homeomorphism. The coefficients of the first fundamental form is given by

$$E = (f(v))^2, F = 0, G = (f'(v))^2 + (g'(v))^2$$

Note If $C \subset \mathbb{R}^2$ is simple closed regular curve which is symmetric about an axis r then the surface obtained by rotating C around this axis can also be proved to be a regular surface. We mention this surfaces as *extended surface of revolution.*

2.5.3 Helicoid



Consider, a helix given by $(\cos u, \sin u, au)$. Through each point of the helix draw a line parallel to the xy plane and intersecting $z - axis$. The surface generated by these is called *helicoid* and admits the following parametrization

$$\mathbb{X}(u, v) = (v \cos u, v \sin u, au), 0 < u < 2\pi, -\infty < v < +\infty.$$

The coefficients of the first fundamental form is given by

$$E(u, v) = v^2 + a^2, F(u, v) = 0, G(u, v) = 1.$$

The arc length of a parametrized curve is given by

$$s(t) = \int_0^t \sqrt{I\alpha'(t)}\, dt.$$

If $\alpha(t) = \mathbb{X}(u(t), v(t))$ is contained in a coordinate neighbourhood corresponding the parametrization $\mathbb{X}(u, v)$, the arclength will be

$$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, dt \tag{2.1}$$

Because, of this equation we are allowed to talk about the 'arclength element'

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

Chapter 3

The geometry of the Gauss map

3.1 The definition of the Gauss map and its fundamental properties

Given a parametrization $\mathbb{X} : U \to S$ of a regular surface S at a point $p \in S$, where U is an open subset of $\mathbb{R}^2$, we can choose a unit normal vector at each point of $\mathbb{X}(U)$ by the rule

$$N(q) = \frac{\mathbb{X}_u \times \mathbb{X}_v}{\mid \mathbb{X}_u \times \mathbb{X}_v \mid}(q), q \in \mathbb{X}(U)$$

More generally if $V \subset S$ is an open set and $N : \mathbb{X}(U) \to \mathbb{R}^3$ is a differentiable map which associates to each $q \in V$ a unit normal vector at q , we say that N is a differentiable field of unit normal vectors on V.

A regular surface is orientable if it admits a differentiable field of unit normal vectors defined on the whole surface; the choice of such a field is called an orientation of S.

Definition 15. *Let, $S \subset \mathbb{R}^3$ be a surface with an orientation. The map $N : S \to \mathbb{R}^3$ takes its values in the unit sphere*

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The map $N : S \to S^2$, thus defined , is called the Gauss map of .S

The differential dN_p of N at $p \in S$ is a linear map from $T_p(S)$ to $T_{N(p)}(S^2)$. Since, $T_p(S)$ and $T_{N(p)}(S^2)$ are the same vector spaces , dN_p can be looked upon as a linear map on $T_p(S)$.

The linear map $dN_p : T_p(S) \to T_p(S)$ operates as follows. For each parametrized curve $\alpha(t)$ in S with $\alpha(0) = p$, we consider the parametrized curve $N \circ \alpha(t) = N(t)$ in the sphere S^2. The tangent vector $N'(0) = dN_p(\alpha'(0))$ is a vector in $T_p(S)$.

Proposition 9. *The differential $dN_p : T_p(S) \to T_p(S)$ of the Gauss map is a self-adjoint linear map.*

Definition 16. *The quadratic form Π_p, defined in $T_p(S)$ by $\Pi_p(v) = -\langle dN_p(v), V\rangle$ is called the second fundamental form S at p.*

Definition 17. *Let C be a regular curve in S passing through $p \in S$, k the curvature C at p , and $\cos\theta = \langle n, N\rangle$ where n is the normal vector to C and N is the normal vector to S at p. The number $k_n = k\cos\theta$ is then called the normal curvature of $C \subset S$ at p.*

Proposition 10. *All curves lying on a surface and having at a given point $p \in S$ the same tangent line have at this point the same normal curvatures.*

This proposition allows us to speak of the normal curvature along a given direction at p. Given a unit vector $v \in T_p(S)$, the intersection of S with the plane containing v and $N(p)$ is called the normal section of S at p along p.

It can be shown that for each $p \in S$ there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$ such that $dN_p(e_1) = -k_1 e_1$ and $dN_p(e_2) = -k_2 e_2$. Moreover, k_1 and k_2 are the maximum and minimum of the second fundamental form Π_p restricted to the unit circle of $T_p(S)$.

Definition 18. *The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the principle curvatures at p; the corresponding directions , i.e, the directions given by eigenvectors $\{e_1, e_2\}$ are called the principle directions at p.*

Definition 19. *If a regular connected curve C on S is such that for all $p \in C$ is a principle direction at p, then C is said to be a line curvature of S.*

Proposition 11. *A necessary and sufficient condition for a connected regular curve C on S to be a line of a curvature of S is that*

$$N'(t) = \lambda(t)\alpha'(t),$$

for any parametrization $\alpha(t)$ of C where $N(t) = N \circ \alpha(t)$ and $\lambda(t)$ is a differentiable function of t. In this case, $-\lambda(t)$ is the principle curvature along $\alpha'(t)$.

Definition 20. *Let, $p \in S$ and let $dN_p : T_p(S) \to T_p(S)$ be the differential of the Gauss map. The determinant of dN_p is the Gaussian curvature K of S at p. The negative of half of the trace of dN_p is called the mean curvature H of S at p.*

Definition 21. *A point of a surface S is called*

1. *Elliptic if $det(dN_p) > 0$.*
2. *Hyperbolic if $det(dN_p) < 0$.*
3. *Parabolic if $det(dN_p) = 0$ with $dN_p \neq 0$.*
4. *Planar if $dN_p = 0$*

Definition 22. *I at $p \in S$, $k_1 = k_2$, then p is called an umbilical point S; in particular, the planar points ($k_1 = k_2 = 0$) are umbilical points.*

Proposition 12. *If all points of a connected surface S are umbilical points, then S is either contained in a sphere or in a plane.*

Definition 23. *Let p be a point in S. An asymptotic direction of S at p is a direction of $T_p(S)$ for which the normal curvature is zero. An asymptotic curve of S is a regular connected curve $C \subset S$ such that for each $p \in C$ the tangent line of C at p is an asymptotic direction.*

Definition 24. *Let, p be a point on a surface S. Two non-zero vectors $w_1, w_2 \in T_p(S)$ are conjugate if $\langle dN_p(w_1), w_2\rangle = \langle w_1, dN_p(w_2)\rangle = 0$. Two directions r_1, r_2 are conjugate if a pair of two non-zero vectors w_1, w_2 parallel to r_1, r_2 , respectively, are conjugate.*

3.2 The Gauss map in local coordinates

Let, $\mathbb{X}(u,v)$ be a parametrization at a point $p \in S$ of a surface S and let, $\alpha(t) = \mathbb{X}(u(t),v(t))$ be a parametrized curve on the surface S with$\alpha(0) = p$. The tangent vector to $\alpha(t)$ at p is

$$\alpha' = \mathbb{X}_u u' + \mathbb{X}_v v'$$
$$dN(\alpha') = N'(u(t),v(t)) = N_u u' + N_v v'.$$

Since, N_u and N_v belongs to $T_p(S)$ we can write

$$N_u = a_{11}\mathbb{X}_u + a_{21}\mathbb{X}_v, N_v = a_{12}\mathbb{X}_u + a_{22}\mathbb{X}_v, \tag{3.1}$$

and therefore

$$dN \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} \tag{3.2}$$

The expression of the second fundamental form in the basis $\{\mathbb{X}_u, \mathbb{X}_v\}$ is given by

$$\Pi_p(\alpha') = -\langle dN(\alpha'), \alpha' \rangle = -\langle N_u u' + N_v v', \mathbb{X}_u u' + \mathbb{X}_v v' \rangle = e(u')^2 + 2fu'v' + g(v')^2$$

where since $\langle N, \mathbb{X}_u \rangle = 0 = \langle N, \mathbb{X}_v \rangle$ we obtain

$$e = -\langle N_u, \mathbb{X}_u \rangle = \langle N, \mathbb{X}_{uu} \rangle,$$
$$f = -\langle N_u, \mathbb{X}_v \rangle = -\langle N_v, \mathbb{X}_u \rangle = \langle N, \mathbb{X}_{uv} \rangle = \langle N, \mathbb{X}_{vu} \rangle,$$
$$g = \langle N_v, \mathbb{X}_v \rangle.$$

Therefore, we obtain the following matrix equation

$$-\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{3.3}$$

Hence, we have

$$a_{11} = \frac{fF - EG}{EG - F^2}, a_{12} = \frac{gF - fG}{EG - F^2}, a_{21} = \frac{eF - fE}{EG - F^2}, a_{22} = \frac{fF - gE}{EG - F^2}. \tag{3.4}$$

These equations are known as *equations of Weingarten*. From this we obtain the Gaussian curvature

$$K = \frac{eg - f^2}{EG - F^2} \tag{3.5}$$

and the mean curvature as

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} \tag{3.6}$$

3.3 Examples

3.3.1 The Tractrix

The *Tractrix* is the curve ~~of~~ which the segment of the tangent line between the point of tangency and some line r ~~in r~~ in the plane, which does not meet the curve, constantly equals 1. First let us find the equation of the curve. Suppose, the curve lies in the $x-z$ plane and $z-axis$ is the line r. Let, the equation of the curve ~~is~~ $z = f(x)$. Take any arbitrary point $(h, f(h))$ on the curve. The equation of the tangent at this point is given by

$$\frac{z - f(h)}{x - h} = f'(h)$$

Therefore the z coordinate of the point where ~~the curve meet the $x - axis$~~ is given by

$$z = f(h) - hf'h.$$

Therefore, we obtain

$$h^2(1 + (f'(h))^2) = 1$$

$$\Rightarrow f'(h) = \frac{\sqrt{1-h^2}}{h}$$

$$\Rightarrow f(h) = \int \frac{\sqrt{1-h^2}}{h}\, dh$$

Substitute $h = \sin\theta$

$$\int \frac{\cos^2\theta}{\sin\theta}\, d\theta$$

If we assume that $f(\frac{\pi}{2}) = 0$ then we obtain the following parametrized curve

$$\theta \mapsto (\sin\theta, \log|\tan\frac{\theta}{2}| + \cos\theta)$$

Now, let us consider the surface of revolution obtained by rotating this curve w.r.t $z - axis$. The surface admits the following parametrization

$$\mathbb{X}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \log|\tan\frac{\theta}{2}| + \cos\theta)\ for\ 0 < \theta \leq \frac{\pi}{2}, 0 < \phi < 2\pi$$

Then, coefficients of the first fundamental form is given by

$$E = \langle \mathbb{X}_\theta, \mathbb{X}_\theta \rangle = \cot^2 \theta$$

$$F = \langle \mathbb{X}_\theta, \mathbb{X}_\phi \rangle = 0$$

$$G = \langle \mathbb{X}_\phi, \mathbb{X}_\phi \rangle = \sin^2 \theta$$

The Gauss map will be

$$\frac{\mathbb{X}_\theta \times \mathbb{X}_\phi}{\mid \mathbb{X}_\theta \times \mathbb{X}_\phi \mid} = (-\cos\theta\cos\phi, -\cos\theta\sin\phi, \sin\theta).$$

Hence, we obtain

$$e = \langle N, \mathbb{X}_{uu} \rangle = -\cot\theta$$

$$f = \langle N, \mathbb{X}_{uv} \rangle = 0$$

$$g = \langle N, \mathbb{X}_{vv} \rangle = \sin\theta\cos\theta$$

From equation 3.5 the Gaussian curvature obtained is

$$K = \frac{-\cot\theta\cos\theta\sin\theta}{\cot^2\theta\sin^2\theta} = -1$$

This surface with Gaussian curvature -1 is called the pseudosphere.

3.3.2 Beltrami-Enneper theorem

Theorem 4. *The absolute value of the torsion τ at a point of an asymptotic curve whose curvature is nowhere zero is given by*

$$\mid \tau \mid = \sqrt{-K}$$

where K is the Gaussian curvature of the surface at a given point.

Proof. Suppose, p is a point on the given surface S and $\alpha(t)$ is the asymptotic curve where $\alpha(0) = p$. Define, $\{e_1, e_2, e_3\}$ to be the coordinates of Frenet trihedron at the point p where e_1 is the tangent, e_2 is the normal and e_3 is the binormal vector at a point p. Since, $\alpha(t)$ is an asymptotic curve the binormal vector is equal to the unit normal to the surface upto a sign. The Gaussian curvature is given by

please define

$$K = \det(dN_p).$$

Since dN_P is self-adjoint and the normal to the curve $\alpha(t)$ lies on the tangent plane $T_p(S)$ then we obtain

$$K = \Pi_p(e_1)\Pi_P(e_2) - \langle dN_p(e_1), e_2\rangle^2$$

Since, α is asymptotic $\Pi_p(e_1) = 0$. The other term is given by

$$\langle dN_p(e_1), e_2\rangle = \langle N'(0), \alpha''(0)\rangle = \langle e_3', n\rangle.$$

Since, binormal and unit normal to the surface are equal upto sign, using Frenet equations the above expression equals to the torsion τ of the curve at the point p. Therefore,

$$K = -\tau^2 \Rightarrow \mid \tau \mid = \sqrt{-K}$$

□