The Morse Homology of $Gr_{n,n+k}(\mathbb{C})$

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MASTER OF SCIENCE

by

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DECLARATION

I hereby declare that I am the sole author of this thesis in partial fulfillment of the requirements for a postgraduate degree from National Institute of Science Education and Research (NISER). I authorize NISER to lend this thesis to other institutions or individuals for the purpose of scholarly research.

Prosondip Sadhukhan
Signature of the Student
Date: 30/04/2024

The thesis work reported in the thesis entitled The Morse Homology of $Gr_{n,n+k}$ was carried out under my supervision, in the school of Mathematical Science at NISER, Bhubaneswar, India.

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Signature of the thesis supervisor
School: SMS
Date: 30/04/2024

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ABSTRACT

In the realm of differential topology, Morse theory stands out as a powerful technique for analyzing the topology of manifolds through the study of differentiable functions defined on them. Marston Morse's foundational insights suggest that a differentiable function on a manifold typically reflects its topology in a direct manner. By leveraging Morse theory, researchers can uncover CW-complex structures and handle decompositions of manifolds, thereby gaining significant insights into their homology.

Morse theory offers a straightforward approach to understanding the topology of manifolds by examining the critical points of Morse functions. These critical points play a central role in constructing Morse complexes, discrete structures that capture essential topological information about the manifold. Through Morse theory, one can establish a deep connection between the geometry of a manifold and its homological properties, providing a powerful tool for topological classification and analysis.

Furthermore, Morse theory facilitates the computation of homology groups of manifolds, offering a systematic way to quantify their topological features. By imposing the Morse-Smale condition, which ensures the genericity of Morse functions, researchers can construct boundary operators and define Morse homology, which is isomorphic to singular homology. This equivalence enables the translation of geometric intuition into algebraic language, facilitating rigorous mathematical analysis.

In recent years, Morse theory has found diverse applications in mathematics and physics, ranging from symplectic geometry to algebraic topology. Particularly noteworthy is its utility in the study of Grassmannian manifolds, where Morse theory provides a direct approach to computing homology groups and unraveling the intricate topological structure of these spaces.

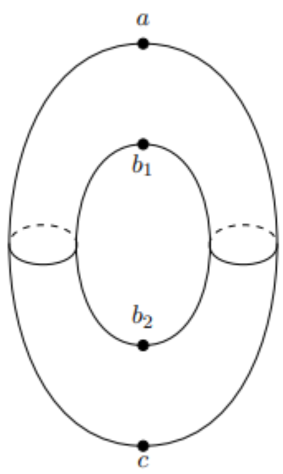
Here we will discuss all of these aspects with its application by computing the homology of Grassmannian.

Contents

Chapter 1

Introduction

One of the fundamental inquiries in smooth manifolds revolves around the quest for topological invariants—properties of a manifold contingent solely upon its underlying topology. Morse theory offers a mechanism to construct such invariants by leveraging the critical points of certain suitably smooth functions $f : M \longrightarrow \mathbb{R}$. To illustrate this concept, let's consider the torus, depicted below within $\mathbb{R}^3$, with critical points marked corresponding to a height function.



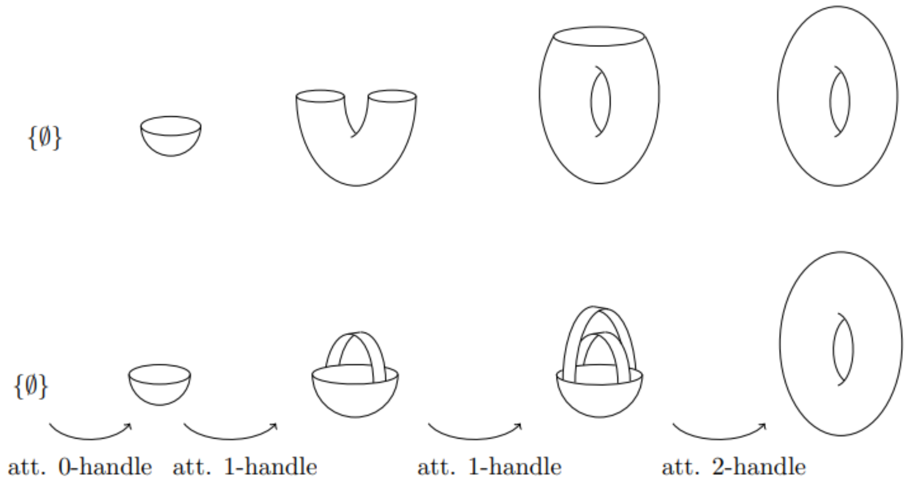
Imagine the torus immersed in space, gradually submerged under rising water. As the water level ascends, covering points c, b_2, b_1, and finally a, observe how the topology of the submerged portion evolves:

- Before point (c), the torus remains above the water level.
- After the water crosses point c, the submerged region becomes homeomorphic to a disk—a contractible space.
- Between points c and b_2, the topology remains unchanged.
- After the water passes through point b_2, the topology becomes more intricate. The

submerged region resembles a disk with an attached strip of the torus—homotopic to an open cylinder.

- Upon covering point b_1, another strip is added to the manifold, rendering it homotopic to a cylinder with a 1-dimensional cell attached.
- Upon reaching point a, the entire torus is submerged, completing the manifold. Specifically, the addition from the previous step forms a disk—a 2-dimensional cell.

In this example, we've intuitively constructed a cell skeleton of $\mathbb{T}^2$ using the analogy of rising water. However, this intuitive concept can be formalized by examining the critical points of the height function restricted to the torus.



Chapter 2

Homology

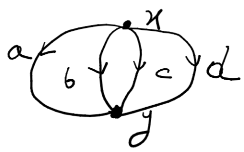
2.1 Idea of Homology

Let's first see some example

Example 2.1.

Seeing the figure, we can define:

- C_1 be the free abelian group with basis the edges a, b, c, d.
- C_0 be the free abelian group with basis the vertices x, y.
- Elements of C_1 are chains of edges or 1-d chains.
- Elements of C_0 are linear combinations of vertices or 0-d chains.



X_1

Now, define a homomorphism.

$$\partial : C_1 \to C_0 \quad \begin{cases} a \longmapsto y - x \\ b \longmapsto y - x \\ c \longmapsto y - x \\ d \longmapsto y - x \end{cases}$$

If $P \in C_1 \Rightarrow P = Ka + lb + mc + nd$ where $k, l, m, n \in \mathbb{Z}$

$$\partial(ka + lb + mc + nd) = (k + l + m + n)y - (k + l + m + n)x$$

so the kernels of ∂ is precisely the cycles, the chain which enters y as $k + l + m + n$ times & enters x as $-(k + l + m + n)$ times. Cycle enters & leaves a vertex the same number of times. So $a-b, b-c, c-d$ forms a basis for the kernel.

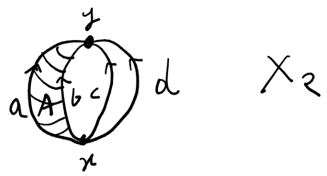
- Let $P \in \text{Ker}(\partial)$ & if $P = ka + lb + mc + nd \Rightarrow k + l + m + n = 0$ then $P = ka + lb + mc + nd = k(a-b) + (k+l)(b-c) + (k+l+m)(c-d)$. We can write this as $k + l + m + n = 0$.

- now as
$$\partial(a-b) = (y-x) - (y-x) = 0$$
$$\partial(b-c) = (y-x) - (y-x) = 0 \qquad \text{Homomorphism}$$
$$\partial(c-k) = (y-x) - (y-x) = 0$$
$$\forall P \in \langle (a-b), (b-c), (c-d) \rangle \quad \partial(P) = 0$$
So $\text{Ker}(\partial) = \langle (a-b), (b-c), (c-d) \rangle$

Example 2.2.

After previous example,

- Now we are attaching a cell A along the cycle $a-b$.
- Now, this cycle is homotopically trivial as it can be contracted to a point.
- We form a quotient of the group of cycles in Example 2.1 by factoring out the subgroup generated by $a-b$.
- So $a-b$ and $b-c$ become equivalent.



Let's define a pair of homomorphism $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ where C_2 is the infinite cyclic group generated by A.

$$\partial_2(A) = a - b$$

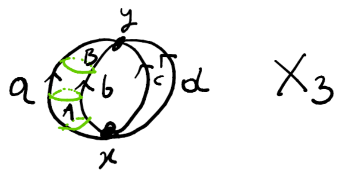
$\& \partial_1, C_1, C_0$ are same as Example 2.1.

$$H_1(X_2) = \frac{Ker\partial_1}{Im\partial_2} = \frac{\langle a-b, b-c, c-d \rangle}{\langle a-b \rangle} = \langle b-c, c-d \rangle$$

So $H_1(x_2)$ is a free abelian group on two generators. $(b-c),(c-d)$. So it denotes the fact that by filling in the 2 -cell A we have reduced the number of 'holes' from three to two.

Example 2.3.

Now we are allocting another 2-cell B along $a-b$ cycle.



$$C_2 = \langle A - B\rangle \quad \partial(A) = \partial(B) = a - b \quad C_3 = \langle 0\rangle$$

Lets define three of homomorphism $\underset{\langle 0\rangle}{C_3} \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$

$$H_1(X_2) = \frac{Ker\partial_1}{Im\partial_2} = \frac{\langle a-b, b-c, c-d\rangle}{\langle a-b\rangle} = \langle b-c, c-d\rangle$$

$$H_2(X_3) = \frac{Ker\partial_2}{Im\partial_3} = \frac{\langle A-B\rangle}{\langle 0\rangle} = \langle A-B\rangle$$

$A-B$ is a 2-dim cycle (hole).

Example 2.4.

Now we are attaching a 3-cell C along the 2-spheres formed by $A\&B$.

$$C_3 = \langle C\rangle$$

$$\partial_3(C) = A - B$$

Let's define three homomorphisms.

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

& $C_2, \partial_1, \partial_2, \partial_3, C_1, C_0$ is same as Example 2.3. Now

$$H_1(X_2) = \frac{Ker\partial_1}{Im\partial_2} = \frac{\langle a-b, b-c, c-d\rangle}{\langle a-b\rangle} = \langle b-c, c-d\rangle$$
$$H_2(X_3) = \frac{Ker\partial_2}{Im\partial_3} = \frac{\langle A-B\rangle}{\langle C\rangle} = \langle 0\rangle$$

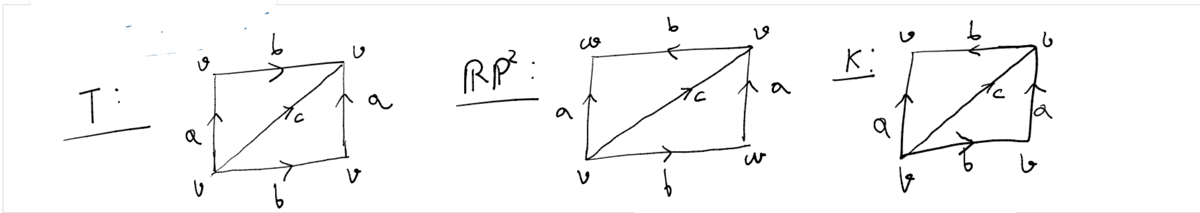
So, from the general pattern of examples, we get that.

- For a cell complex X one has chain groups $C_n(X)$ which are free abelian group with basis consisting n-cells of X,
- There is a boundary homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$.
- Now the homology group $H_n(x) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$

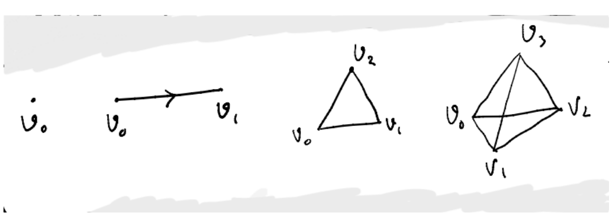
Now the hurdles are

- Orientation of higher dim
- Arbitrary polyhedra can be subdivided into special polyhedrals called simplex
- For simplices, there is no difficulty in handling orientation & defining boundary maps.
- Decompose into simplices - consider the collection of all possible continuous maps of simplices into a given space X. This map generates a large chain group $C_n(x)$. Singular Homology $\cong$ Simplicial Homology.

2.2 Δ - Complexes



Every surface can be constructed by identifying the edges of some triangles.



The concept of an n-simplex is foundational in geometry, particularly in the topics of convex sets within Euclidean spaces $\mathbb{R}^m$. Let's delve into a detailed explanation of the properties and characteristics outlined in the following points:

- **Definition and Basic Properties:-**An n-simplex is defined as the smallest convex set in an m-dimensional Euclidean space $\mathbb{R}^m$ that contains $n+1$ distinct points, denoted as $v_0, v_1, \ldots, v_n$. It's essential to note that an n-simplex cannot lie entirely within a hyperplane of dimension less than n. A hyperplane is defined as the solution set of a system of linear equations.

- **Linear Independence:-**A crucial criterion for identifying an n-simplex is the linear independence of the difference vectors between its vertices. Specifically, the vectors $v_1 - v_0, v_2 - v_0, \ldots, v_i - v_0$ must be linearly independent.

- **Notation and Representation:-**Mathematically, an n-simplex is denoted by $[v_0, v_1, \ldots, v_n]$, where each v_i represents a vertex of the simplex.

- **Standard n-Simplex:-**The standard n-simplex, denoted as Δ^n, is defined as the set of points $(t_0, t_1, \ldots, t_n)$ in $\mathbb{R}^{n+1}$ satisfying two conditions:
  1. $\sum_i t_i \leq 1$, ensuring that the points lie within the unit hyperplane.
  2. $t_i \geqslant 0$ for all i, guaranteeing non-negativity along each coordinate axis.

  The vertices of the standard simplex are precisely the unit vectors along the coordinate axes.

- **Ordering and Orientation:-**Ordering the vertices of an n-simplex determines the orientation of its edges. By convention, the vertices are ordered such that the subscript increases with t_0. This ordering establishes a canonical homeomorphism between the standard simplex Δ^n and any n-simplex $[v_0, v_1, \ldots, v_n]$, preserving the sequence of vertices. The mapping is defined as:

$$(t_0, \dots, t_n) \mapsto \sum_i t_i v_i$$

Here, t_i represents the barycentric coordinates of the point $\sum_i t_i v_i$ within the simplex $[v_0, \dots, v_n]$.

- **Faces of an n-Simplex:-**Removing one of the $(n+1)$ vertices from an n-simplex yields n remaining vertices that span an $(n-1)$-simplex. This $(n-1)$-simplex is termed a face of the original n-simplex and is denoted as $[v_0, \dots, \hat{v}_j, \dots, v_n]$, where $\hat{v}_j$ indicates the omission of the vertex v_j.

In a geometric structure like a simplex, the vertices of a face, or any subsimplex formed by a subset of these vertices, are consistently arranged according to their original order within the larger simplex.

The composite of all faces comprising the simplex Δ^n constitutes its boundary, denoted as $\partial\Delta^n$. On the other hand, the open simplex $\mathring{\Delta}^n$ represents the interior of Δ^n, defined as the set difference between Δ^n and its boundary $\partial\Delta^n$.

A Δ-complex configuration over a space X entails a collection of mappings $\sigma_\alpha : \Delta^n \to X$, where the dimension n depends on the index α. This structure adheres to the following conditions:

(i) The mapping σ_α restricted to the interior of the simplex $(\mathring{\Delta}^n)$ is injective, and each point in X lies in the image of precisely one such restriction σ_α.

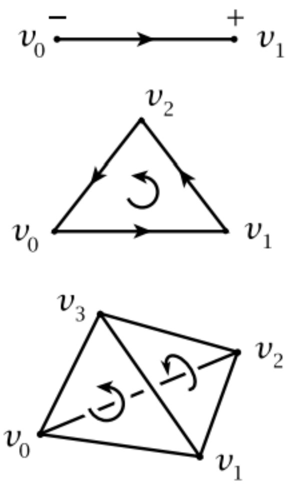
(ii) Each restriction of σ_α to a face of Δ^n corresponds to one of the mappings $\sigma_\beta : \Delta^{n-1} \to X$. This correlation stems from identifying the face of Δ^n with Δ^{n-1} through a linear homeomorphism that conserves the ordering of the vertices.

(iii) A subset $A \subset X$ is deemed open if and only if $\sigma_\alpha^{-1}(A)$ is an open set in Δ^n for every σ_α.

The objective now lies in establishing the simplicial homology groups of a Δ-complex X. Let $\Delta_n(X)$ denote the free abelian group generated by the open n-simplices e_α^n

of X. Elements within $\Delta_n(X)$, referred to as ***n*-chains**, are expressed as finite formal summations $\sum_\alpha n_\alpha e_\alpha^n$, where coefficients n_α belong to the integers. Alternatively, one could represent these chains as $\sum_\alpha n_\alpha \sigma_\alpha$, where $\sigma_\alpha : \Delta^n \to X$ represents the characteristic map of e_α^n, with its image being the closure of e_α^n as described earlier. This summation $\sum_\alpha n_\alpha \sigma_\alpha$ essentially represents a finite collection or "chain" of n-simplices within X, each with integer multiplicities denoted by the coefficients n_α.

To establish the boundary of an n-simplex $[v_0, \cdots, v_n]$, it consists of various $(n-1)$-dimensional simplices $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$, where the hat symbol indicates the exclusion of the corresponding vertex. While one might initially consider expressing this boundary as the sum of these $(n-1)$-dimensional faces, it's more effective to introduce certain signs. Thus, the boundary of $[v_0, \cdots, v_n]$ is represented as $\sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_n]$. These signs are introduced to maintain coherence in orientations, ensuring all faces of a simplex possess consistent orientations.



$$\partial[v_0, v_1] = [v_1] - [v_0]$$

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

To further clarify, consider the orientations depicted in the accompanying figure. The orientations of the concealed faces are also counterclockwise when viewed from outside the 3-simplex.

With this geometric perspective in mind, we define a boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ for a general Δ-complex X. This homomorphism is determined by its values on basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha \mid [v_0, \cdots, \hat{v}_i, \cdots, v_n]$$

It's worth noting that the right side of this equation indeed lies within $\Delta_{n-1}(X)$, as each restriction $\sigma_\alpha \mid [v_0, \cdots, \hat{v}_i, \cdots, v_n]$ represents the characteristic map of an $(n-1)$-simplex within X.

Lemma 2.5.

The composition $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ is zero.

Proof. We have $\partial_n(\sigma) = \sum_i (-1)^i \sigma \mid [v_0, \cdots, \hat{v}_i, \cdots, v_n]$, and hence

$$\begin{aligned}\partial_{n-1}\partial_n(\sigma) = &\sum_{j<i} (-1)^i(-1)^j \sigma \mid [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n] \\ &+ \sum_{j>i} (-1)^i(-1)^{j-1} \sigma \mid [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n]\end{aligned}$$

The latter two summations cancel since after switching i and j in the second sum, it becomes the negative of the first. □

Now we have a homomorphism of abelian groups

$$\ldots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

Now we can define n^{th} homology group $H_n = \frac{\operatorname{Ker} \partial n}{\operatorname{Im} \partial_{n+1}}$ as a quotient group.

- Elements of Kernel ∂_n are called cycles.
- Elements of Image ∂_{n+1} ore called boundaries.
- Elements of H_n are coset of $\operatorname{Im} \partial_{n+1}$, called homology classes.
- Two cycles are called homologous if for $C_1, C_2 \in \operatorname{Ker} \partial_n \quad C_1 - C_2 \in \operatorname{Im} \partial_{n+1}$
- For $C_n = \Delta_n(x)$ the homology group $\frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}$ will be denoted $\mathrm{H}_n^{\Delta}(X)$ and called n^{th} simplicial homology group of X.

Example 2.6.

Let $X = S^1$ Then $\Delta_0(S^1) \cong \mathbb{Z}$ as it has only 1-vertices.

$\Delta_1(S^1) \cong \mathbb{Z}$ as it has only 1-edge.

$$0 \xrightarrow{\partial_2} \Delta_1\left(S^1\right) \xrightarrow{\partial_1} \Delta_0\left(S^1\right) \xrightarrow{\partial_0} 0 \quad \partial(e) = v - v = 0$$

So $H_0^{\Delta}\left(S^1\right) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\langle v\rangle}{\langle 0\rangle} \equiv \mathbb{Z}$

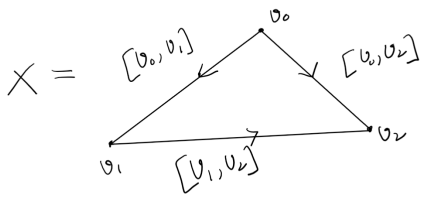
$H_1^{\Delta}\left(S^1\right) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\langle e\rangle}{\langle 0\rangle} \cong \mathbb{Z}$

$H_n^{\Delta}\left(S^1\right) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}} = \frac{\langle 0\rangle}{\langle 0\rangle} \cong \langle 0\rangle \quad n > 1$

Example 2.7.

$$K_0 = \{v_0, v_1, v_2\}$$

$$K_1 = \{[v_0, v_1], [v_1, v_2], [v_0, v_2]\}$$



$$d_0[v_i, v_j] = v_i, \quad d_1[v_i, v_j] = v_j,$$

$e_1^{\Delta}(x) \cong \mathbb{Z}^3$ generated by $[v_0, v_1], [v_1, v_2], [v_0, v_2]$.

$e_0^{\Delta}(x) \cong \mathbb{Z}^3$ generated by v_0, v_1, v_2.

$e_n^{\Delta}(x) = 0$ for $n \geqslant 2$. The chain complex $\Delta^{\circ}(x)$ is

$$0 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^3 \xrightarrow{\partial_0} 0$$

$$\partial_1([v_0, v_1]) = v_0 - v_1,$$

$$\partial_1([v_1, v_2]) = v_1 - v_2,$$

$$\partial_1([v_0, v_2]) = v_0 - v_2.$$

Hence if $\alpha = a\,[v_0, v_1] + b\,[v_1, v_2] + c\,[v_0, v_2] \in \operatorname{Ker}(\partial_1)$, then

$$\partial_1 \alpha = \underbrace{(a+c)}_{n_0} v_0 + \underbrace{(b-a)}_{n_1} v_1 - \underbrace{(b+c)}_{n_2} v_2 = 0$$

$$\Rightarrow \quad a+c=0, \quad b-a=0, \quad b+c=0 \Rightarrow a=b=-c$$

$$\Rightarrow \quad \alpha = a\left([v_0, v_1] + [v_1, v_2] - [v_0, v_2]\right).$$

Hence $H_1^{\Delta}(x) \cong \ker(\partial_1) \cong \mathbb{Z}$,

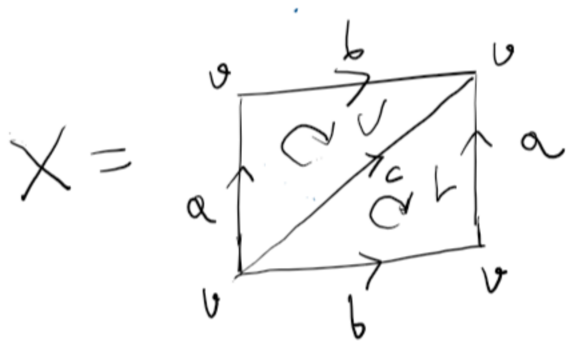
$$\text{generated by } [v_0, v_1] + [v_1, v_2] - [v_0, v_2].$$

Similarly $n_0 v_0 + n_1 v_1 + n_2 v_2 \in \operatorname{Im}(\partial_1) \Leftrightarrow n_0 + n_1 + n_2 = 0$ Hence, $H_0^{\Delta}(x) \cong e_0^{\Delta}(x) / \operatorname{Im}(\partial_1) \cong \mathbb{Z}$, generated by $v_1 + v_2 + v_3$ for example. Hence we see again that.

$$H_n^{\Delta}(x) \cong \begin{cases} \mathbb{Z} & n = 0, 1; \\ 0, & n \geqslant 2. \end{cases}$$

Example 2.8.

$$e_0^{\Delta}(x) = \mathbb{Z}\langle v \rangle,$$
$$e_1^{\Delta}(x) = \mathbb{Z}\langle a, b, c \rangle,$$
$$e_2^{\Delta}(x) = \mathbb{Z}\langle u, L \rangle,$$
$$e_n^{\Delta}(x) = 0 \text{ for } n \geqslant 3.$$



The simplicial chain complex is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \longrightarrow 0$$

Now $\partial_2(U) = a - c + b, \quad \partial_2(L) = b - c + a$, hence

$$nU + mL \in \operatorname{Ker}(\partial_2) \Rightarrow (n+m)(a+b-c) = 0 \Rightarrow m+n=0.$$

Thus $H_2^{\Delta}(X) \cong \text{Ker}\,(\partial_2) = \mathbb{Z}\langle U - L\rangle \cong \mathbb{Z}$.

And $\text{Im}\,(\partial_2) = \mathbb{Z}\langle a + b - c\rangle \subseteq \mathbb{Z}\langle a, b, c\rangle. = \Delta_1(X)$.

Now $\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0 \Rightarrow \partial_1 = 0$, hence

$$\text{Ker}\,(\partial_1) = e_1^{\Delta}(x)$$

Thus $H_1^{\Delta}(X) = \frac{\text{Ker}(\partial_1)}{\text{Im}(\partial_2)} = \frac{\mathbb{Z}\langle a,b,c\rangle}{\mathbb{Z}\langle a+b-c\rangle} \cong \mathbb{Z}\langle a, b\rangle$. Finally since $\partial_1 = 0, \text{Im}\,(\partial_1) = 0$, hence

$$H_0(X) = \frac{\Delta_0(X)}{\text{Im}\,(\partial_1)} \cong \mathbb{Z}\langle v\rangle \cong \mathbb{Z}.$$

We have,

$$H_n^{\Delta}(X) \cong \begin{cases} \mathbb{Z}, & n = 0; \\ \mathbb{Z} \oplus \mathbb{Z}, & n = 1; \\ \mathbb{Z}, & n = 2; \\ 0, & n \geqslant 3. \end{cases}$$

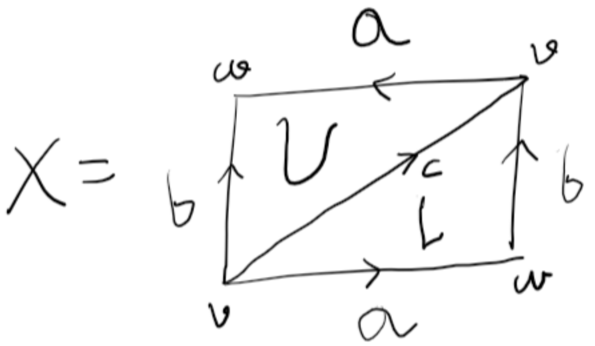
Example 2.9.

$$e_0^{\Delta}(x) = \mathbb{Z}\langle v, w\rangle,$$

$$e_1^{\Delta}(x) = \mathbb{Z}\langle a, b, c\rangle,$$

$$e_2^{\Delta}(x) = \mathbb{Z}\langle U, L\rangle,$$

$$e_n^{\Delta}(x) = 0 \text{ for } n \geqslant 3.$$



Now, $\partial_2(u) = a - b + c, \quad \partial_2(L) = b - a + c$, hence

$$\partial_2(mU + nL) = (m - n)(a - b) + (m + n)c.$$

$$\partial_2(mU + nL) = 0 \Rightarrow m - n = m + n \Rightarrow 0 \Rightarrow m = 0 \text{ and } n = 0.$$

Thus ∂_2 is injective and $\text{Ker}\,(\partial_2) = 0, H_2^{\Delta}(X) = \text{Ker}\,(\partial_2) = 0$.

Now $\partial_1(a) = \partial_1(b) = v - \omega$ and $\partial_1(c) = v - v = 0$.

Thus $\text{Ker}\,(\partial_1) = \mathbb{Z}\langle a - b, c\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ Note that if $\alpha \in \text{Im}\,(\partial_2)$ then

$$\alpha = \underbrace{(m-n)(a-b) + (m+n)c}_{=} = (m-n)(a-b+c) + 2nc.$$

$$l(a-b+c) + 2nc$$

for some integers l, n.

We can take $a - b + c$ and c as a basis of $\text{Ker}\,(\partial_1)$, then $\text{Im}\,(\partial_2)$ is a_n index 2 subgroup of $\text{Ker}\,(\partial_1)$. Thus $H_1^{\Delta}(X) = \frac{\text{Ker}(\partial_1)}{\text{Im}(\partial_2)} \cong \mathbb{Z}/2\mathbb{Z}$. Finally $\text{Im}\,(\partial_1) = \mathbb{Z}\langle v - \omega\rangle$, hence

$$H_0^{\Delta}(X) = \frac{e_o^{\Delta}(X)}{\text{Im}\,(\partial_1)} = \frac{\mathbb{Z}\langle v, \omega\rangle}{\mathbb{Z}\langle v - \omega\rangle} \cong \mathbb{Z}.$$

Hence

$$H_n^{\Delta}(X) \cong \begin{cases} \mathbb{Z}, & n = 0; \\ \mathbb{Z}/2\mathbb{Z}, & n = 1; \\ 0, & n \geq 2. \end{cases}$$

Now we will discuss about **Chain Complexes**.

A chain complex is a sequence of homomorphisms,

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where C_i are abelian groups such that

$$\partial_n \cdot \partial_{n+1} = 0 \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Hence $\text{Ker}\,(\partial_n) \supseteq \text{Im}\,(\partial_{n+1})$. This chain complex is denoted by $C.$. Terminology:

$$Z_n(C.) = \text{Ker}\,(\partial_n) - n \text{ cycles of } C.$$

$$B_n(C.) = \text{Im}\,(\partial_{n+1}) - n \text{ boundaries of } C.$$

Definition 2.10.

If $C.$ is a chain complex, then we define the n-th homology group of e to be

$$H_n\,(C.) = \frac{Z_n\,(C.)}{B_n\,(C.)} = \frac{\text{Ker}\,(\partial_n)}{\text{Im}\,(\partial_{n+1})}.$$

Definition 2.11.

A morphism of chain complexes

$$f : C. \longrightarrow D.$$

is a sequence of homomorphisms, $f_n : C_n \to D_n$ such that

$$f_{n-1} \circ \partial_n = \partial_n \circ f_n.$$

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ D_n & \xrightarrow{\partial_n} & D_n \end{array}$$

Such morphisms are called Chain maps.

Proposition 2.12.

A chain map $f : C. \longrightarrow D.$ includes homomorphism $f_* : H_n\left(C.\right) \longrightarrow H_n(D.)$

Proof. If $\alpha \in Z_n(C.)$, then $\partial_n \alpha = 0$ and $f_n(\alpha) \in Z_n(D.)$ because $\partial_n\left(f_n(\alpha)\right) = f_{n-1}\left(\partial_n \alpha\right) = 0$. So we have a map

$$f_n : Z_n\left(C.\right) \to Z_n\left(D.\right)$$

If $\alpha \in B_n\left(C.\right)$, then $\alpha = \partial_{n+1}\beta$ for some $\beta \in C^{n+1}$. Then

$$f_n(\alpha) = f_n\left(\partial_{n+1}\beta\right) = \partial_{n+1}\left(f_{n+1}(\beta)\right)$$
$$\Rightarrow f_n(\alpha) \in B_n(D.).$$

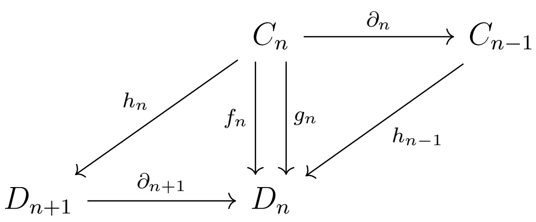
$$\begin{array}{ccc} Z_n(C.) & \xrightarrow{f_n} & Z_n(D.) \\ \downarrow & & \downarrow \\ H^n(C.) & \xrightarrow{f_*} & H(D.) \end{array}$$

Thus we get the induced maps f_*. □

Definition 2.13.

If $f, g : C. \to D.$ are two chain maps, a chain homotopy h from f to g is a sequence of homomorphisms $h_n : C_n \longrightarrow D_{n+1}$ such that

$$g_n - f_n = \partial_{n+1} h_n + h_{n-1}\partial_n.$$



Proposition 2.14.

If $f, g : C. \longrightarrow D.$ are chain maps and h is a chain homotopy from f to g, then

$$f_* = g_* : H_n(C.) \to H_n(D.).$$

Proof. If $\alpha \in Z_n(C.)$, then $\partial_n \alpha = 0$.

$$\begin{aligned} f_n(\alpha) - g_n(\alpha) &= \partial_{n+1}(h_n(\alpha)) + \underbrace{h_{n-1}(\partial_n \alpha)}_{0} \\ &= \partial_{n+1}(h_n(\alpha)) \in B_n(D.). \end{aligned}$$

Hence $f_* = g_*$ on $H_n(C.)$. □

Definition 2.15 (Exact Sequence).

An exact sequence of abelian groups is a sequence of homomorphisms

$$\ldots \to A_n \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \ldots$$

such that

$$\operatorname{Ker}(\alpha_n) = \operatorname{Im}(\alpha_{n+1}).$$

Definition 2.16 (SES).

A short exact sequence $0 \to C. \xrightarrow{i} D. \xrightarrow{j} E. \to 0$ is a sequence of chain maps such that $0 \to C_n \xrightarrow{i_n} D_n \xrightarrow{j_n} E_n \longrightarrow 0$ is exact for each n.

If $0 \to C. \xrightarrow{i} D. \xrightarrow{j} E. \to 0$ is an SES of chain complexes, then we have a connecting homomorphism $\delta : H_n(\text{ E. }) \to H_{n-1}(\text{ C. })$.

From this, we get an important result

Theorem 2.17.

If $0 \to A. \xrightarrow{i} B. \xrightarrow{j} C. \to 0$ is an SES of chain complexes, then there are connecting homomorphisms $\delta : H_n(C.) \to H_{n-1}(A.)$ such that the sequence

$$\cdots \to H_n(A.) \xrightarrow{i_*} H_n(B.) \xrightarrow{j_*} H_n(C.) \xrightarrow{\delta} H_{n-1}(A.) \xrightarrow{i_*} H_{n-1}(B.) \to \cdots$$

is exact.

Proof. Refer to Algebraic Topology by Allen Hatcher. □

Assuming the basic knowledge of **Singular Homology**, we are stating some important results here by referring to the proofs from Algebraic Topology by Allen Hatcher.

Proposition 2.18.

If X is path connected, then $H_0(X) \cong \mathbb{Z}$. If X is not path connected and X_α are the path Components of X, then

$$H_0(x) \cong \bigoplus_\alpha \mathbb{Z}.$$

Proposition 2.19.

If X is a point, then $H_n(X) = 0$ for $n \geqslant 1$ and $H_0(X) \cong \mathbb{Z}$.

2.3 Reduced Homology Groups

Now we will discuss about **Reduced Homology Groups**

Note that we have the augmentation homomorphism $\varepsilon : C_0(x) \to \mathbb{Z}$ gives by

$$\varepsilon\left(\sum n_i x_i\right) = \sum n_i \quad \text{for} \quad x_i \in X, n_i \in \mathbb{Z}.$$

Also $\varepsilon_0 \partial_1 = 0$, hence we have an augmented Chain complex $\tilde{C}_i(x)$

$$\ldots \longrightarrow C_n(x) \xrightarrow{\partial_n} C_{n-1}(x) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1(x) \xrightarrow{\partial_1} C_0(x) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

The reduced homology groups of X are defined as

$$\tilde{H}_n(x) = H_n(\tilde{C}.(x)).$$

Note that $\tilde{H}_n(X) = H_n(X)$ for $n > 0$, but

$$H_0(x) = \tilde{H}_0(x) \oplus \mathbb{Z}.$$

For a point $X, \tilde{H}_n(X) = 0$ for all n.

Now let's discuss some results on **Induced homomorphism**.

If $f : X \to Y$ is a continuous map, then we get homomorphisms $f_\# : C_n(X) \longrightarrow C_n(Y)$ given by $f_\#(\sigma) = f \circ \sigma$, where $\sigma : \Delta^n \longrightarrow X$ is a singular n-simplex, and $f_\# \left(\sum n_i \sigma_i\right) = \sum n_i f_\# (\sigma_i)$ for $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$ singular n-simplices.

Claim 2.20.

$f_\#$ is a chain map from $C_0(X)$ to $C_0(Y)$. We need to show

$$f_\# \cdot \partial = \partial \cdot f_\#$$

Then the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial} & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \cdots \\
 & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \\
\cdots & \xrightarrow{\partial} & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \xrightarrow{\partial} & \cdots
\end{array}$$

Proof. If σ is a singular n-simplex in X, then

$$\begin{aligned}
\partial (f_\# \sigma) = \partial(f \circ \sigma) &= \sum_{i=0}^{n} (-1)^i f_\circ \sigma \Big|_{d_i \Delta^n} \\
&= f_\# \left(\sum_{i=0}^{n} (-1)^i \sigma \Big|_{d_i \Delta^n} \right) \\
&= f_\#(\partial \sigma).
\end{aligned}$$

□

As a consequence $f_\#$ induces homomorphisms

$$f_* : H_n(X) \to H_n(Y).$$

Proposition 2.21.

(a) If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then

$$(f \circ g)_* = f_* \circ g_* : H_n(X) \to H_n(Z).$$

(b) If $Id_X : X \to X$ is the identity map, then

$$(Id_X)_* = Id : H_n(X) \to H_n(X).$$

Proof. Refer to Algebraic Topology by Allen Hatcher. □

Here is an important result of the Homotopy invariance discussed below.

Theorem 2.22.

If $f, g : X \to Y$ are homotopic maps, then $\quad f_x = g_* : H_n(X) \longrightarrow H_n(Y)$ are equal.

Proof. Refer to Algebraic Topology by Allen Hatcher. □

Corollary 2.23.

If $f : X \to Y$ is a homotopy equivalence, then $\quad f_* : H_n(x) \to H_n(Y)$ is an isomorphism.

Proof. Refer to Algebraic Topology by Allen Hatcher. □

2.4 Relative Homology

Let's Now discuss **Relative Homology** as it will be a useful tool.

Suppose $A \subseteq X$, then $C_n(A) \subseteq C_n(X)$ is a subgroup. Let $C_n(X, A) = \frac{C_n(X)}{C_n(A)}$, moreover, $\partial\left(C_n(A)\right) \subseteq C_{n-1}(A)$. Hence we have a chain complex $C^{\cdot}(X, A)$

$$\ldots \longrightarrow C_{n+1}(X, A) \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \xrightarrow{\partial} \cdots$$

because $\partial^2 = 0$.

$$\begin{array}{ccccc}
C_{n+1}(X) & \xrightarrow{\quad\partial\quad} & C_n(X) & \xrightarrow{\quad\partial\quad} & C_{n-1}(X) \\
\downarrow & & \downarrow & & \downarrow \\
C_{n+1}(X, A) & \xrightarrow{\quad\partial\quad} & C_n(X, A) & \xrightarrow{\quad\partial\quad} & C_{n-1}(X, A)
\end{array}$$

Definition 2.24.

Relative homology groups of the pair (X, A) are defined as the homology groups of the chain complex $C^{\cdot}(X, A)$,

$$H_n(X, A) = \frac{\text{Ker}\,(\partial : C_n(X, A) \to C_{n-1}(X, A))}{\text{Im}\,(\partial : C_{n+1}(X, A) \to C_n(X, A))}.$$

Note that there are exact sequences

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{q} C_n(X, A) \to 0$$

such that $i \circ \partial = \partial \circ i$ and $q \circ \partial = \partial \circ q$. Hence, we get an SES of chain complexes

$$0 \to C^{\cdot}(A) \to C^{\cdot}(X) \to C^{\cdot}(X, A) \to 0$$

$$\begin{array}{ccc} C_{n+1}(A) & \xrightarrow{\partial} & C_n(A) \\ {\scriptstyle i}\downarrow & \circlearrowleft & \downarrow{\scriptstyle i} \\ C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ C_{n+1}(X, A) & \xrightarrow{\partial} & C_n(X, A) \end{array}$$

Proposition 2.25.

We have a long, exact sequence of relative Homology

$$\to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \to \ldots \to H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{q_*} H_0(X, A) \to 0$$

Remark 2.26.

- An element of $H_n(X, A)$ is represented by a relative cycle $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$.
- A relative cycle α is trivial is $H_n(X, A)$ if it is a relative boundary of form $\alpha = \partial\beta + \gamma$ where $\beta \in C_{n+1}(x)$ and $\gamma \in C_n(A)$.
- Connecting homomorphism $S : H_n(X, A) \to H_{n-1}(A)$ is given by

$$\delta[\alpha] = [\partial\alpha].$$

Now we will discuss some aspects of **Reduced Homology**.

We have a short, exact sequence of chain complexes

$$0 \to \widetilde{C}^{\cdot}(A) \xrightarrow{i} \widetilde{C}^{\cdot}(X) \xrightarrow{q} \widetilde{C}^{\cdot}(X,A) \to 0$$

where $\widetilde{C}^{\cdot}(A)$ is the chain complex,

$$\dots \xrightarrow{\partial} C_n(A) \xrightarrow{\partial} C_{n-1}(A) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_0(A) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$\widetilde{C}^{\cdot}(X)$ is the chain complex,

$$\dots \xrightarrow{\partial} C_n(x) \xrightarrow{\partial} C_{n-1}(x) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$\widetilde{C} \cdot (X, A)$ is the chain complex

$$\dots \to C_n(X,A) \to C_{n-1}(X,A) \to \dots \to C_0(X,A) \to 0$$

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_n(A) & \longrightarrow & \dots & \longrightarrow & C_0(A) & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow i & & \downarrow i & & & & \downarrow & & \downarrow id & & \\
\dots & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_n(X) & \longrightarrow & \dots & \longrightarrow & C_0(X) & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & C_{n+1}(X,A) & \longrightarrow & C_n(X,A) & \longrightarrow & \dots & \longrightarrow & C_0(X,A) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

This SES gives us a LES in reduced homology groups

$$\to \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X,A) \to \tilde{H}_{n-1}(A) \to \dots \to \tilde{H}_0(A) \to \tilde{H}_0(X) \to H_0(X,A) \to 0$$

Corollary 2.27.

If x_0 is a point in X, then $H_n(X, x_0) \cong \widetilde{H}_n(X)$ for all n.

Proof. We have a LES

$$\dots \to \tilde{H}_n(x_0) \to \tilde{H}_n(X) \xrightarrow{\cong} H_n(X, x_0) \to \tilde{H}_{n-1}(x_0) \to \dots$$

which gives the isomorphisms as $\tilde{H}_n(x_0)$ is 0 and $\tilde{H}_{n-1}(x_0)$ is also 0 as x_0 is just a point. □

If $A \subseteq X$ and $B \subseteq Y, f : X \to Y$ continuous, such the $f(A) \subseteq B$, then we wite

$$f : (X, A) \to (Y, B).$$

Then, there is an induced homomorphism.

$$f_* : H_n(X, A) \to H_n(Y, B).$$

The idea of the proof is given in Algebraic Topology by Hatcher.

Proposition 2.28.

If $f, g : (X, A) \to (X, B)$ are homotopic by a homotopy $H_t : (X, A) \to (Y, B)$ then

$$f_* = g_* : H_n(X, A) \longrightarrow H_n(y, B).$$

Proof. Refer to Algebraic Topology by Hatcher □

We can also see the LES of a triple. Suppose $B \subseteq A \subseteq X$, then we write (X, A, B) as a triple, and we have SES

$$0 \to C_n(A, B) \xrightarrow{i} C_n(X, B) \xrightarrow{q} C_n(X, A) \to 0$$

This gives us a long, exact sequence.

$$\to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \dots$$

2.5 Excision

:- Now we will discuss some topics on **Excision**, which is an important tool for calculating Homology. Let X be a space and $U = \{U_\alpha \mid U_\alpha \subseteq X, \alpha \in A\}$ such that $X = \bigcup_{\alpha \in A} \operatorname{Int}(U_\alpha)$. Let $C_n^u(X) \subseteq C_n(X)$ be the subgroup generated by singular n-simplices $\sigma : \Delta^n \to X$ such that $\sigma(\Delta^n) \subseteq U_\alpha$ for some α. Then we have a chain complex $C^u.(x)$

$$\dots \to C_n^u(x) \to C_{n-1}^u(x) \to \dots \to C_0^u(x) \to 0.$$

Let $H_n^u(X)$ denote the homology groups of this chain complex. The inclusion maps $i : C_n^u(x)$ give us a chain map $i : C^u.(x) \to C.(x)$.

Theorem 2.29.

There is a chain map $\rho : C.(x) \to C^u.(x)$ such that $\rho \circ i$ and $i \circ \rho$ are chain homotopic to identity. Hence $i_* : H_n^u(x) \to H_n(x)$ are isomorphisms for all n.

Proof. Refer to Algebraic Topology by Hatcher. □

Theorem 2.30 (Excision).

Let $Z \subseteq A \subseteq X$ such that $\bar{Z} \subseteq \text{Int}(A)$ then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X, A) \cong H_n(X - Z, A - Z)$ for all n.

Proof. Refer to Algebraic Topology by Hatcher. □

2.6 CW-Homology

Definition 2.31.

A CW complex X is a space along with sub-spaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_n \subseteq \ldots$$

such that

$$X = \bigcup_{n=0}^{\infty} X_n.$$

The subspace X_n is called the n-skeleton of X and these satisfy the following:

(a) X_0 is a discrete subspace of X.

(b) X_n is obtained from X_{n-1} by attaching n-cell, through attaching maps $\varphi_\alpha^n : S^{n-1} \to X_{n-1}, \alpha \in A_n$, and

$$X_n = \left(X_{n-1} \bigsqcup \left(\bigsqcup_{\alpha \in A} D_\alpha^n \right) \right) / \sim$$

where $x \sim \varphi_\alpha^n(x)$ if $x \in \partial D_\alpha^n$. Here D_α^n are all homeomorphic to D^n, and A_n could be empty. The topology on X is the weak topology that is $U \subseteq X$ is open $\Leftrightarrow U \cap X_n$ is open in X_n for all n.

Remark 2.32. • There might not be any n-cell in X, which means $X_n = X_{n-1}$.

- If $X = X_n$ for some $n \geqslant 0$; then we say X is finite dimensional and

$$\begin{aligned}\dim(x) &= \inf\{n \mid X = X_n\}\\ &= \sup\{n \mid X \text{ has an } n\text{-cell }\}.\end{aligned}$$

  In this case the topology of X_i the same as topology of X_n.

- If X is a finite dimensioned CW complex and there are only finitely many cells in each dimension, then we say that X is a finite CW complex, and X is compact.

- For each $\alpha_0 \in A_n$, there is a continuous map $C^n_{\alpha_0} : D^n_{\alpha_0} \to X$ which is the composition of

$$D^n_{\alpha_0} \hookrightarrow \left(X_{n-1} \bigsqcup \left(\bigsqcup_{\alpha} D^n\right)\right) \twoheadrightarrow X_n \hookrightarrow X.$$

  The subspace $E^n_{\alpha_0} = e^n_{\alpha_0}\left(D^n_{\alpha_0}\right) \subseteq X$ is celled an n-cell of X, with characteristic map $e^n_{\alpha_0}$ and attaching map $\varphi^n_{\alpha_0}$. Note that $e^n_{\alpha_0}$ restricted to the interior of $D^n_{\alpha_0}$ is a homeomorphism onto $E^n_{\alpha_0} \backslash X_{n-1}$.

- A CW complex is a Hausdorff topological space.

Definition 2.33.

If Y is a topological space and X is a CW complex, such that $X \cong Y$, then we say that X is a CW structure on Y

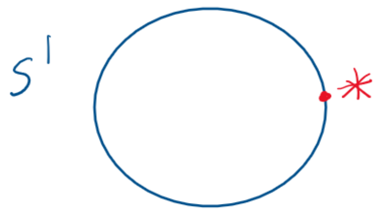
Now let's Discuss some Examples.

Example 2.34 (Sphere)**.**

There is a CW structure X on S^1 with one 0-cell $*$ and one 1 -cell E^1. $X_0 = *, X_1 = X = S^1$, and the attaching map for E^1 will be

$$\varphi : \partial D^1 = S^0 = \{-1, 1\} \to X_0 = \{*\} \text{ is given by } \varphi(\pm 1) = *.$$

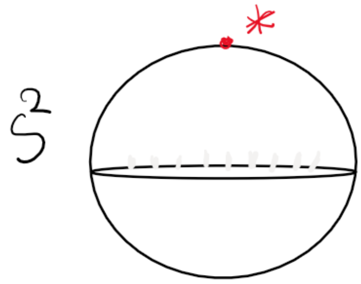
Then $X \cong [-1, 1]/\sim$ where $-1 \sim 1 \Rightarrow X \cong S^1$. Similarly there is a CW structure X on S^n, where X has one O-cell $*$ and one n-cell E^n,



$$X_0 = \{*\}, \quad X_0 = X_1 = X_2 = \ldots = X_{n-1}, \quad X = X_n,$$

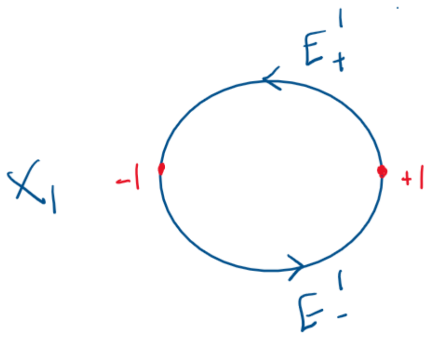
the attaching map for E^n is the constant may $\varphi^n : S^{n-1} \to *$.

Again $X \cong D^n/\partial D^n \cong S^n$.



Example 2.35.

(Sphere) We give a CW structure X on S^n with 2 cells in each dimension $k = 0, \ldots, n$. Here $X_k \cong S^k$, and $X = X_n \cong S^n$. $X_0 = \{\pm 1\}$, then we have two 1 -cells E^1_- and E^1_+



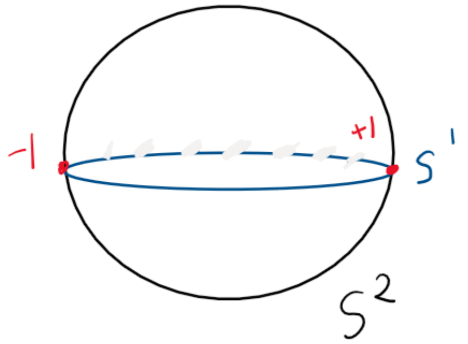
The attaching maps for $E^1_\pm$ are $\phi^1_\pm : \{-1, 1\} \to \{-1, 1\}$

$$\begin{aligned} \phi^1_+(-1) &= 1, & \phi^1_+(1) &= -1 \\ \phi^1_-(-1) &= -1, & \phi^1_-(1) &= 1. \end{aligned}$$

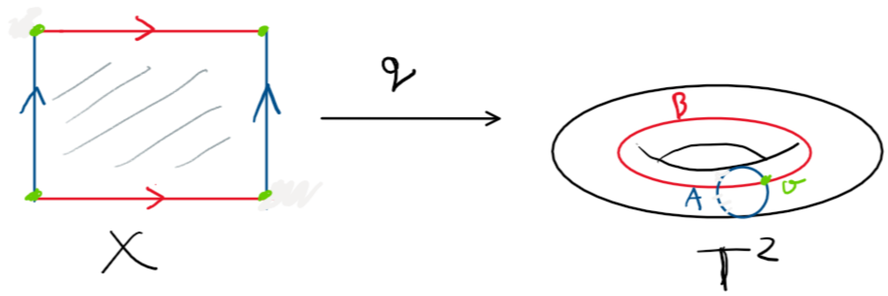
Check that $X_1 \cong S^1$. Now we attach two 2-cells E^2_- and E^2_+ and the attaching maps $\phi^2_\pm$ are homeomorphisms $\phi^2_+, \phi^2_- : S^1 \to X_1$, such that

$$\phi^2_+(z) = \phi^2_-(\bar{z}).$$

Then $X_2 \cong S^2$. Continuing in this way, we get $X_n = X \cong S^n$.



Example 2.36 (Torus).



There is a CW structure X on T with one 0 -cell, θ two 1-cells, A and B and one 2-cell, F.

$$\begin{array}{l|l} X_0 = \{v\} & v = q(0,0) = q(0,1) = q(1,0) = q(1,1), \\ X_1 = A \cup B & A = q(\{0\} \times [0,1]) = q(\{1\} \times [0,1]) \\ X_2 = T & B = q([0,1] \times \{0\}) = q([0,1] \times \{1\}). \\ & F = q([0,1] \times [0,1]). \end{array}$$

The attaching maps for A and B are constant maps. The attaching map for F is $\phi : S^1 \to A \cup B$, is the Composition

$$S^1 \cong \partial([0,1] \times [0,1]) \xrightarrow{q} A \cup B.$$

Example 2.37 (Subcomplex).

If X is a CW complex, then a subcomplex of X is a subspace $A \subseteq X$ such that A is a union of cells of X. In particular A is also a CW complex, and its n-skeleton

$$A_n = A \cap X_n.$$

If X is a CW complex and $A \subseteq X$ is a subcomplex, then we say that (X, A) is a *CW* pair.

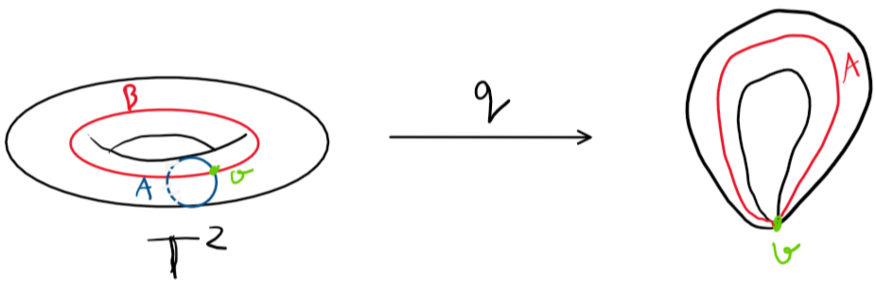
Now we will see some constructions of the new CW-Complex from the old one.

Product:-If X and Y are CW complexes then $X \times Y$ has a natural CW structure on it. The n-cells of $X \times Y$ are the products $E_\alpha^k \times F_\beta^{n-k}$ where E_α^k are the k-cells of X, F_β^{n-k} are the $n-k$ cells of Y $k = 0, \ldots, n$. We are using the homeomorphism

$$D^k \times D^{n-k} \cong D^n.$$

We have $(X \times Y)_n = \bigcup_{k=0}^n (X_k \times Y_{n-k})$.

Quotient:-If (X, A) is a CW pair, then X/A this has a natural CW structure.This CW structure has cells corresponding to the cell in X not contained in A, along with an extra 0 -cell corresponding to A. Then $(X/A)_n = (X_n/A_n)$. For example if we take the CW pair (T, A), then T/A consists of the cells $v = A/A$,

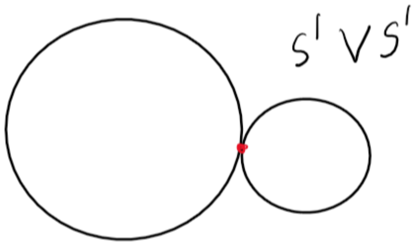


$q(B)$, and $q(F)$ where F is the 2 cell of T. If E_α^n is an n-cell of X, st. $E_\alpha^n \nsubseteq A$, then the corresponding n-cell in X/A will have attaching map the composition of

$$S^{n-1} \xrightarrow{\varphi_\alpha^n} X_{n-1} \xrightarrow{q} X_{n-1}/A_{n-1} = (X/A)_{n-1}$$

Where φ_α^n is the attaching map of E_α^n.

Wedge Sum:-A based space (X, x_0) is a space X and a point $x_0 \in X$.



If (X, x_0) and (Y, y_0) are two based spaces, then their wedge sum is the space $X \vee Y = (X \bigsqcup Y)/\{x_0, y_0\}$.If X and Y are CW complexes and $x_0 \in X_0, y_0 \in Y_0$, then $(X \bigsqcup Y, \{x_0, y_0\})$ is a CW pair. Then $X \vee Y = (X \bigsqcup Y)/\{x_0, y_0\}$ has a natural CW structure.

Theorem 2.38.

If (X, A) is a CW pair, and A is contractible then $X/A \simeq X$.

Proof. Refer to Algebraic Topology by Hatcher

□

Example 2.39.

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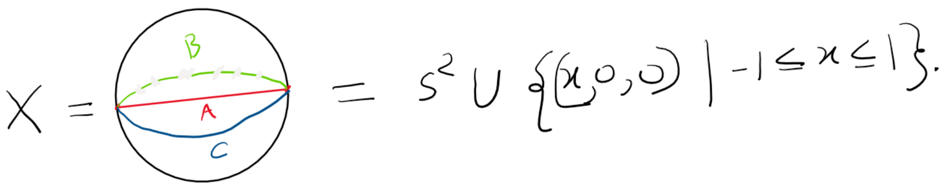


$$X = S^2 \cup \{(x, 0, 0) \mid -1 \leqslant x \leqslant 1\}.$$

Then X has a CW structure where

$$X_0 = \{n = (0,0,1), s = (0,0,-1)\}$$

$$X_1 = A \cup B \cup C \text{ where}$$

$$A = \{(x,0,0) \mid -1 \leqslant x \leqslant 1\},$$

$$B = \{(\cos\theta, \sin\theta, 0) \mid 0 \leqslant \theta \leqslant \pi\},$$

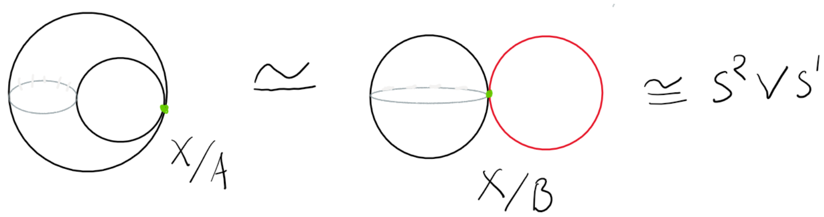
$$C = \{(\cos\theta, \sin\theta, 0) \mid \pi \leq \theta \leq 2\pi\}.$$

$X_2 = X$ is obtained by attaching two 2 -cells

$$E = \left\{(x,y,z) \in S^2 \mid y \geqslant 0\right\}$$

$$F = \left\{(x,y,z) \in S^2 \mid y \leq 0\right\}.$$

Then $A \subseteq X$ is a contractible sub complex. Hence $X/A \simeq X$. Similarly B is a contractible sub complex $\Rightarrow X/B \simeq X$. Thus $\quad X/A \simeq X/B$



Now we will try to see Relative Homology as Reduced Homology of quotient.

Definition 2.40.

Relative Homology as Reduced Homology of quotient. If $A \subseteq X$ is closed, (X, A) is called a good pair if there is $V \subseteq X$ open which deformation retracts to A.

Example 2.41.

A CW pair (X, A) is a good pair [Refer to Proposition A5 of Algebraic Topology by Hatcher]

Proposition 2.42.

If $A \subseteq V \subseteq X$ such that Y deformation retracts onto A, then the inclusion $(X, A) \hookrightarrow$

(X, V) induces isomorphism

$$H_n(X, A) \cong H_n(X, V).$$

Proof. We have a long exact sequence of the triple (X, V, A)

$$\cdots \to H_n(V, A) \to H_n(X, A) \xrightarrow{\cong} H_n(X, V) \to H_{n-1}(V, A) \to \cdots$$

There is a homotopy equivalence of pairs (Y, A) and (A, A), hence

$$H_n(V, A) \cong H_n(A, A) = 0 \text{ for all } n.$$

Therefore we get isomorphisms $H_n(X, A) \to H_n(X, V)$. □

Theorem 2.43.

For good pairs (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphisms

$$H_n(X, A) \cong H_n(X/A, A/A) \cong \widetilde{H}_n(X/A) \text{ for all } n.$$

Proof. Let $V \subseteq X$ be an open set which deformation retracts to A. Then we have a commutative diagram

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{\cong} & H_n(X, V) & \xrightarrow{\cong (Excision)} & H_n(X - A, V - A) \\ \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_* \cong} \\ H_n(X/A, A/A) & \xrightarrow{\cong (Proposition\, 2.42)} & H_n(X/A, V/A) & \xrightarrow{\cong (Excision)} & H_n(X/A - A/A, V/A - A/A) \end{array}$$

Note that $X - A \xrightarrow{q} X/A - A/A$ is a homeomorphism and takes $V - A$ to $V/A - A/A$. Hence

$$q_* : H_n(X - A, V - A) \to H_n(X/A - A/A, V/A - A/A)$$

is an isomorphism. Then all the vertical arrows are isomorphisms. □

Corollary 2.44.

$\widetilde{H}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{H}_k(S^n) = 0$ for $k \neq n$.

Proof. Note that $(D^n, \partial D^n)$ is a good pain and $\partial D^n = S^{n-1}$. More over $D^n/\partial D^n \cong S^n$. Thew from the long exact sequence of $(D^n, \partial D^n)$ we get

$$\ldots \to \widetilde{H}_k(D^n) \to \widetilde{H}_k(S^n) \to \widetilde{H}_{k-1}\left(S^{n-1}\right) \to \widetilde{H}_{k-1}(D^n) \to \ldots$$

Thus $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$. Since $S^0 = \{\pm 1\}$,

$$\widetilde{H}_0\left(S^0\right) \cong \mathbb{Z} \text{ and } \widetilde{H}_k\left(S^0\right) = 0 \text{ for } k > 0.$$

The result then follows by induction on n. □

Corollary 2.44 and Excision both gives us $\mathbb{R}^m \cong \mathbb{R}^n$ iff $m = n$.From this Excision we also get a nice result.

Theorem 2.45 (Invariance of Domain)**.**
If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are open and $U \cong V$, then $m = n$.

Proof. Refer to Algebraic Topology by Hatcher. □

Definition 2.46 (Local Homology Group)**.**
If X is a space and $x_0 \in X$, then $H_k(X, X - x_0)$ is cell the local homology group of X at x_0.

If U is a ngbd of x_0 then

$$H_k(U, U - x_0) \cong H_k(X, X - x_0).$$

If $x_0 \in X, y_0 \in Y$, and U is a ngbd of x_0, V is a ngbd of y_0, and these is a homeomorphism

$$f : U \to V \text{ st. } f(x_0) = y_0$$

then

$$H_k(X, X - x_0) \cong H_k(U, U - x_0) \xrightarrow[f_*]{\cong} H_k(V, V - y_0) \cong H_k(Y, Y - y_0).$$

Definition 2.47.

If $X_\alpha, \alpha \in A$, are spaces and $x_\alpha \in X_\alpha$ such that (X_α, x_α) is a good pair for all $\alpha \in A$, and

$$\bigvee_\alpha X_\alpha = \left(\bigsqcup_\alpha X_\alpha\right) / \{x_\alpha \mid \alpha \in A\}.$$

Theorem 2.48.

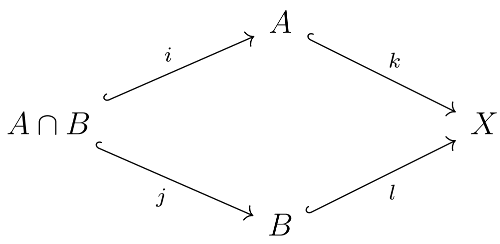
$\widetilde{H}_k\left(\bigvee_\alpha X_\alpha\right) \cong \oplus_\alpha \widetilde{H}_k\left(X_\alpha\right)$ for all k.

Proof. Refer to Algebraic Topology by Hatcher. □

Theorem 2.49 (Mayer-Vietoris Sequence)**.**

X is a space, $A, B \subseteq X$ such that

$$X = \operatorname{Int} A \cup \operatorname{Int} B,$$



We have long exact sequences

$$\cdots \to H_n(A \cap B) \xrightarrow{(i_*, j_*)} H_n(A) \oplus H_n(B) \xrightarrow{k_* - l_*} H_n(X) \to H_{n-1}(A \cap B) \to \cdots$$

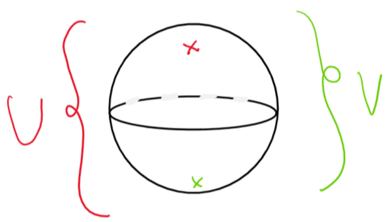
$$\ldots \to \widetilde{H}_n(A \cap B) \longrightarrow \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \longrightarrow \widetilde{H}_n(X) \to \widetilde{H}_{n-1}(A \cap B) \to \cdots$$

Proof. Refer to Algebraic Topology by Hatcher. □

Let's Calculate some example to see the power of the above result.

Example 2.50 (Sphere)**.**

Let $U = S^n - \{(0, \ldots, 1)\}$ and $V = S^n - \{(0, \ldots, -1)\}$ $S^n = U \cup V, \quad U, V$ are open,



$$\widetilde{H}_n(U) = 0 \quad \forall n,$$

$$\widetilde{H}_n(V) = 0 \quad \forall n,$$

$$U \cap V \simeq S^{n-1}, \quad \widetilde{H}_k(U \cap V) = \begin{cases} \mathbb{Z}, & k = n-1, \\ 0, & k \neq n-1. \end{cases}$$

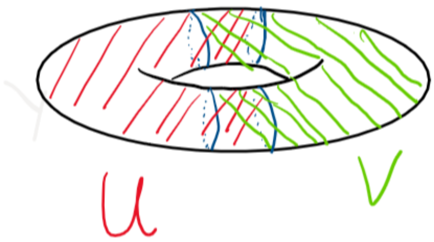
$$\cdots \to \widetilde{H}_k(U) \oplus \widetilde{H}_k(V) \to \widetilde{H}_k\left(S^n\right) \to \widetilde{H}_{k-1}\left(S^{n-1}\right) \to\to \widetilde{H}_{k-1}(U) \oplus \widetilde{H}^0_{k-1}(V) \to \cdots$$

So $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$, and thus we can compute the homology groups of S^n by induction.

Example 2.51 (Torus)**.**

$$T = U \cup V, \quad U \cap V \simeq S^1 \sqcup S^1$$
$$U \simeq S^1, V \simeq S^1$$

$\widetilde{H}_n(U \cap V) \cong \widetilde{H}_n(S^1 \vee S^1) \cong \widetilde{H}_n(S^1) \oplus \widetilde{H}_n(S^1) \cong \widetilde{H}_n(U) \oplus \widetilde{H}_n(V)$



For $n > 2$,

$$\cdots \to \widetilde{H}_n(V) \oplus \widetilde{H}_n(U) \to \widetilde{H}_n(T) \to \widetilde{H}_{n-1}(U \cap V) \to \cdots$$

$\widetilde{H}_n(T) = 0$ for $n > 2$. (From Example 2.50$\widetilde{H}_n(V) \cong \widetilde{H}_n(S^1) \cong \widetilde{H}_n(U) \cong 0$)

From Example 2.50 $\widetilde{H}_2(V) \cong \widetilde{H}_2(S^1) \cong \widetilde{H}_2(U) \cong 0 \cong \widetilde{H}_2(U) \oplus \widetilde{H}_2(V)$

$$0 \to \widetilde{H}_2(T) \to \widetilde{H}_1(U \cap V) \to \widetilde{H}_1(U) \oplus \widetilde{H}_1(V) \to \widetilde{H}_1(T) \to \widetilde{H}_0(U \cap V) \to \widetilde{H}_0(U) \oplus \widetilde{H}_0(X) \to \cdots$$

Hence we have

$$0 \to \widetilde{H}_2(T) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \widetilde{H}_1(T) \to \mathbb{Z} \to 0.$$

Now we need to check the maps

$$\cdots \longrightarrow 0 \longrightarrow \widetilde{H}_2(\mathbb{T}^2) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(i_*, j_*)} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \ldots$$

Despite the two last groups being isomorphic, it doesn't mean that the morphism (i_*, j_*) between them in the LES, induced by the inclusions $i : U \cap V \longrightarrow U$ and $j : U \cap V \longrightarrow V$, is an isomorphism. And actually it is not. (If it was, then $\widetilde{H}_2(\mathbb{T}^2)$ would be zero.) At this stage, we need to compute what this (i_*, j_*) is. For this, we choose 1-cicles generating the homologies of U, V and $U \cap V$ as follows: for each cylinder of the intersection $U \cap V$, take an equatorial circumference. Name their homology classes α and β. So, actually, those $\mathbb{Z}$ in the piece of LES depicted above are the free abelian groups generated by α and β :

$$(i_*, j_*) : \mathbb{Z}\langle\alpha\rangle \oplus \mathbb{Z}\langle\beta\rangle \longrightarrow \mathbb{Z}\langle\alpha\rangle \oplus \mathbb{Z}\langle\beta\rangle$$

And now we compute:

$$(i_*, j_*)\,(\alpha, 0) = (i_*, j_*)\,(0, \beta) = (\alpha, \beta)$$

since $\alpha = \beta$ in $\widetilde{H}_1(U)$ and $\widetilde{H}_1(V)$. Hence, in terms of these basis, our morphism (i_*, j_*) can be represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

Hence,

$$\widetilde{H}_2\left(\mathbb{T}^2\right) = \operatorname{im} \partial = \ker \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \mathbb{Z}\langle\alpha - \beta\rangle = \mathbb{Z}$$

As for $\widetilde{H}_1\left(\mathbb{T}^2\right)$, let's focus on the following piece of the LES:

$$\cdots \longrightarrow \widetilde{H}_1(U \cap V) \xrightarrow{(i_*, j_.)} \widetilde{H}_1(U) \oplus \widetilde{H}_1(V) \xrightarrow{k_* - l_*} \widetilde{H}_1\left(\mathbb{T}^2\right) \xrightarrow{\partial} \widetilde{H}_0(U \cap V) \xrightarrow{(i_*, j_*)} \widetilde{H}_0(U) \oplus \widetilde{H}_0(V)$$

where $k_* - l_*$ is the morphism induced by the inclusions $k : U \longrightarrow \mathbb{T}^2$ and $l : V \longrightarrow \mathbb{T}^2$. Again, we know all the groups except $\widetilde{H}_1\left(\mathbb{T}^2\right)$:

$$\cdots \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(i_*, j_*)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{k_* - l_*} \widetilde{H}_1\left(\mathbb{T}^2\right) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(i_*, j_*)} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \ldots$$

We claim that, taking as generators for $\widetilde{H}_0(U)$ and $\widetilde{H}_0(V)$ two points p, q, one in each component of $U \cap V$, we have also generators for $\widetilde{H}_0(U \cap V)$ and, with these generators and similar computations, the second morphism (i_*, j_*) can also be represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

Now, we get a SES from that piece of the LES:

$$0 \longrightarrow \ker \partial \longrightarrow \widetilde{H}_1\left(\mathbb{T}^2\right) \longrightarrow \operatorname{im} \partial \longrightarrow 0$$

But, we can see that

$$\operatorname{im}\partial = \ker(i_*, j_*) = \ker\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \mathbb{Z}$$

Differently,

$$\ker\partial = \operatorname{im}(k_* - l_*) = (\mathbb{Z}\oplus\mathbb{Z})/\ker(k_* - l_*) = (\mathbb{Z}\oplus\mathbb{Z})/\operatorname{im}(i_*, j_*) = \mathbb{Z}$$

Hence, we have the following short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{H}_1\left(\mathbb{T}^2\right) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Thus

$$\widetilde{H}_1\left(\mathbb{T}^2\right) = \mathbb{Z}\oplus\mathbb{Z}$$

So,

$$H_k(\mathbb{T}^2) = \begin{cases} \mathbb{Z}^2 & \text{if } k = 1, \\ \mathbb{Z} & \text{if } k = 2, \\ 0 & \text{else.} \end{cases}$$

Proposition 2.52.

$H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ and is generated by the identity map $i_n : \Delta^n \to \Delta^n$.

Proof. Refer to Elements of Algebraic Topology by Munkres. □

Corollary 2.53.

Consider $S^n \cong (\Delta_1^n \sqcup \Delta_0^n)\,/\sim$ where $d_i\Delta_1^n$ is identified with $d_i\Delta_2^n$ using the identity may Let $\sigma_i : \Delta_i^n \to S^n$ be restriction of the quotient map to Δ_i^n, then $\widetilde{H}_n(S^n)$ is generated by $[\sigma_1 - \sigma_2]$.

Proof. Refer to Elements of Algebraic Topology by Munkres. □

2.7 Degree

Definition 2.54 (degree)**.**

Let $f : S^n \to S^n$ be a continuous map then we have an isomorphism $\widetilde{H}_n(S^n) \cong \mathbb{Z}$, hence

$f_* : \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$ is given by $f_*(n) = kn$, where $k = f_*(1)$. We define the degree of f to be

$$\deg(f) = k.$$

Properties:

- $\deg(Id_{S^n}) = 1$.
- $\deg(f \circ g) = \deg(f)\deg(g)$.
- If $f \simeq g$ then $\deg(f) = \deg(g)$.
- If $f : S^n \to S^n$ is not surjective then $\deg(f) = 0$.

$$S^n \xrightarrow{f} S^n \backslash \{x_0\} \xrightarrow{i} S^n$$

  f_* is the composition

$$H_n(S^n) \xrightarrow{f_*} H_n(S^n - \{x_0\}) \xrightarrow{i_*} H_n(S^n).$$

  as $H_n(S^n - \{x_0\}) = 0$ because $S^n - \{x_0\} \cong \mathbb{R}^n$

- If f is a reflection then $\deg(f) = -1$.

$$f_*([\sigma_1 - \sigma_2]) = [\sigma_2 - \sigma_1] = -[\sigma_1 - \sigma_2].$$

  when $f(x_0, \ldots, x_n) = f(x_0, \ldots, x_{n-1}, -x_n)$.

- If f is the antipodal map, then $\deg(f) = (-1)^{n+1}$, $f(x_0, \ldots, x_n) = (-x_0, \ldots, -x_n)$ which is the composition of $n+1$ reflections.
- Let $V \subseteq \mathbb{R}^{n+1}$ be a vector subspace of dimension n, and $\rho_V : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the reflection with respect to origin then ρ_V maps S^n to S^n fixing $S^{n-1} \cong V \cap S^n$, and interchanging the two hemispheres, with boundary $V \cap S^n$. Clearly,

$$\deg(\rho_V) = -1.$$

  To see this we can give a Δ-structure on S^n, such that the two hemispheres are the two n-simplices

$$\Delta_1^n, \Delta_2^n \text{ and } \sigma_i : \Delta_i^n \to S^n$$

are the inclusions. Then $[\sigma_1 - \sigma_2]$ generates $\widetilde{H}_n(S^n)$ and

$$(\rho_V)_*([\sigma_1 - \sigma_2]) = [\sigma_2 - \sigma_1] = -1([\sigma_1 - \sigma_2]).$$

Corollary 2.55.

S^n has a non-vanishing continuous vector field iff n is odd.

Proof. Refer to Algebraic Topology by Hatcher. □

Proposition 2.56.

If $f : S^n \to S^n$ has no fined point, then

$$\deg(f) = (-1)^{n+1}$$

Proof. Refer to Algebraic Topology by Hatcher. □

Corollary 2.57.

If n is even, the only groups that act on S^n freely are $\mathbb{Z}/2\mathbb{Z}$ and 0 .

Proof. Refer to Algebraic Topology by Hatcher. □

Now we will try to compute the degree locally which will help us to get a degree of map efficiently.

Let $f : S^n \to S^n$ be continuous and $y \in S^n$ such that $f^{-1}(y)$ is finite. Let V be ngbd of y, then $\exists\ U \subseteq f^{-1}(v)$ open such that $U \cap f^{-1}(y) = \{x\}$, then $f(U - x) \subseteq V - y$, also

$$H_n(U, U - x) \cong H_n(S^n, S^n - x) \cong \widetilde{H}_n(S^n) \cong \mathbb{Z},$$

similarly

$$H_n(V, V - y) \cong H_n(S^n, S^n - y) \cong \widetilde{H}_n(S^n) \cong \mathbb{Z},$$

thus $f_* : H_n(U, U - x) \to H_n(V, V - y)$ is determined by $f_*(1)$.In this setting we define the local degree of f at x as

$$\deg f|_x = f_*(1).$$

Proposition 2.58.

If $f : S^n \to S^n$ is continuous and

$$f^{-1}(y) = \{x_1, \dots, x_k\} \text{ then}$$
$$\deg f = \sum_{i=1}^{k} \deg f|_{x_i}.$$

Proof. Refer to Algebraic Topology by Hatcher. □

2.8 CW-Homology Theorem

Proposition 2.59.

Let X be a CW complex, $X_n \subseteq X$ be its n-skeleton, then

(a) $X_n/X_{n-1} \cong \vee_\alpha S^n_\alpha, \alpha$ ranging over n-cells of X and

$$H_k\left(X_n, X_{n-1}\right) \cong \begin{cases} \oplus_\alpha \mathbb{Z}\left\langle e^n_\alpha\right\rangle, & k = n \\ 0, & \text{otherwise.} \end{cases}$$

(b) If $k > n$, then $H_k\left(X_n\right) = 0$, hence if X is finite dimensional

$$H_k(X) = 0 \text{ when } k > \dim X.$$

(c) If $k < n$, then $H_k(x) \cong H_k\left(X_n\right)$.

Proof. (a)$X_n = \left(\left(\sqcup D^n_\alpha\right) \sqcup X_{n-1}\right) / \sim$

$$\begin{array}{ccccc} & & \sqcup D^n_\alpha & \xrightarrow{\sqcup e^n_\alpha} & X_n \\ & & \downarrow & \searrow & \downarrow \\ \vee_\alpha S^n_\alpha & \xrightarrow{\cong} & \left(\sqcup D^n_\alpha\right) / \left(\sqcup \partial D^n_\alpha\right) & \xrightarrow{\cong} & X_n/X_{n-1} \end{array}$$

$$H_k\left(X_n, X_{n-1}\right) \cong \begin{cases} \oplus_\alpha \mathbb{Z}\left\langle e^n_\alpha\right\rangle, & k = n \\ 0, & \text{otherwise.} \end{cases}$$

(b) Consider the LES of the pair (X_n, X_{n-1}).

$$\dots \to H_{k+1}\left(X_n, X_{n-1}\right) \xrightarrow{\delta} H_k\left(X_{n-1}\right) \xrightarrow{\cong} H_k\left(X_n\right) \to H_k\left(X_n, X_{n-1}\right) \to \dots$$

By part (a) we get the isomorphism from the LES.

Since $k > n$, $\quad H_k\left(X_n\right) \cong H_k\left(X_{n-1}\right)$.

Continuing we get $H_k(X_n) \cong H_k(X_{n-1}) \cong H_k(X_{n-2}) \cong \ldots \cong H_k(X_0) = 0$.

(c) Consider the LES of the pair (X_{n+1}, X_n),

$$\ldots \to H_{k+1}(X_{n+1}, X_n) \to H_k(X_n) \to H_k(X_{n+1}) \to H_k(X_{n+1}, X_n) \to \ldots$$

By part (a) we get the isomorphism from the LES.

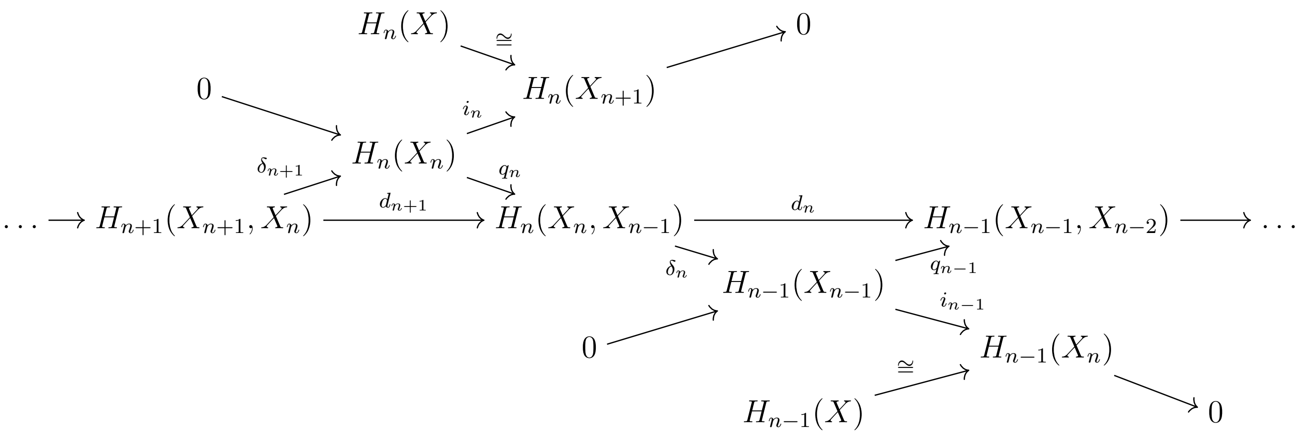
Since $k < n$, $\quad H_k(X_n) \cong H_k(X_{n+1})$. In this way we get

$$H_k(X_n) \cong H_k(X_m) \text{ for any } m > n > k.$$

If X is finite dimensional, then $X = X_m$ for some m and we are done. [For general X Refer Lemma 2.34 of Hatcher.] $\square$

Cellular Chain Complex:-

When X is a CW complex, we have an associated chain complex $C.^{CW}(X)$, where $C_n^{CW}(X) = H_n(X_n, X_{n-1}) = \underset{\alpha}{\oplus}\mathbb{Z}\langle e_\alpha^n \rangle$.



Let $d_n = q_{n-1} \circ \delta_n : H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$, then

$$d_n \circ d_{n+1} = (q_{n-1} \circ \delta_n) \circ (q_n \circ \delta_{n-1}) = q_{n-1} \circ (\delta_n \circ q_n) \circ \delta_{n+1} = 0.$$

Hence we get a chain complex $C.^{CW}(X)$,

$$\cdots \to \underset{\alpha}{\oplus}\mathbb{Z}\left\langle e_\alpha^{n+1} \right\rangle \xrightarrow{d_{n+1}} \underset{\beta}{\oplus}\mathbb{Z}\left\langle e_\beta^{n} \right\rangle \xrightarrow{d_n} \underset{\gamma}{\oplus}\mathbb{Z}\left\langle e_\gamma^{n-1} \right\rangle \longrightarrow \cdots$$

Define the cellular homology groups $H_n^{CW}(X) = H_n\left(C^{CW}(X)\right) = \frac{\mathrm{Ker}_2(d_n)}{I_m(d_{n+1})}$.

Theorem 2.60.

For any CW complex $X, H_n^{CW}(X) \cong H_n(X)$.

Proof. Since q_{n-1} is injective $\operatorname{Ker}(d_n) = \operatorname{Ker}(q_{n-1} \circ \delta_n) = \operatorname{Ker}(\delta_n) = \operatorname{Im}(q_n)$.Now q_n is injective, hence $q_n : H_n(X_n) \to \operatorname{Im}(q_n)$ is isomorphism. Also $\operatorname{Im}(d_{n+1}) = \operatorname{Im}(q_n \cdot \delta_{n+1}) = q_n(\operatorname{Im}(\delta_{n+1}))$, thus q_n takes $\operatorname{Im}(\delta_{n+1})$ to $\operatorname{Im}(d_{n+1})$ bijectively.

$$H_n^{CW}(x) = \frac{K_{en}(d_n)}{\operatorname{Im}(d_{n+1})} \cong \frac{\operatorname{Im}(q_n)}{q_n(\operatorname{Im}(\delta_{n+1}))} \cong \frac{H_n(X_n)}{\operatorname{Im}(\delta_{n+1})}.$$

Moreover $\operatorname{Im}(\delta_{n+1}) = \operatorname{Ker}(i_n) \Rightarrow H_n^{CW}(X) \cong \frac{H_n(X_n)}{\operatorname{Ker}(i_n)}$.

Now $i_n : H_n(X_n) \longrightarrow H_n(X_{n+1})$ is surjective hence

$$H_n^{CW}(X) \cong \frac{H_n(X_n)}{\operatorname{Ker}(i_n)} \cong H_n(X_{n+1}) \cong H_n(X).$$

□

2.9 Application

- If X does not have any n-cells then $H_n(X) = 0$.
- If X has k n-cells then $H_n(X)$ can be generated by at most k-elements.
- If X is a finite CW complex, then $H_n(X)$ is finitely generated for all n.

Let's discuss some example.

Example 2.61 ($\mathbb{C}P^n$)**.**

$\mathbb{C}P^n$ has CW structure with 1, k-cell for every k even, $0 \le k \le 2n$. Thus cellular chain complex is

$$0 \to \underset{2n}{\mathbb{Z}} \to \underset{2n-1}{0} \to \underset{2n-2}{\mathbb{Z}} \to \ldots \to \mathbb{Z} \to 0$$

Hence

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & 0 \le k \le 2n, \text{ k is even;} \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.62 (Sphere)**.**

Let $n > 1$, then S^n has CW structure with one 0 -cell and one n-cell. So $S^n \times S^n$, has one 0 -cell, two n-cells and one $2n$ cell.

$$H_k(S^n \times S^n) \cong C_k^{CW}(S^n \times S^n) = \begin{cases} \mathbb{Z} & k = 0, 2n \\ \mathbb{Z} \oplus \mathbb{Z} & k = n \\ 0 & \text{otherwise.} \end{cases}$$

Now we will try to find what are these boundary maps.

Theorem 2.63.

The map $d_n : \bigoplus_{\alpha} \mathbb{Z} \langle e^n_\alpha \rangle \longrightarrow \bigoplus_{\beta} \mathbb{Z} \left\langle e^{n-1}_\beta \right\rangle$ is given by

$$d_n \left(e^n_\alpha\right) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$$

where $d_{\alpha\beta}$ is the degree of $\Delta_{\alpha\beta} = q_\beta \circ \varphi^n_\alpha$, where

$$\partial D^n_\alpha = S^{n-1}_\alpha \xrightarrow{\varphi^n_\alpha} X_{n-1} \xrightarrow{q_\beta} X_{n-1}/\left(X_{n-1} - \mathring{e}^{n-1}_\beta\right) \cong S^{n-1}_\beta.$$

φ^n_α is the attaching map.

Proof. There is a commutative diagram

$$\Delta_{\alpha\beta} : S^{n-1}_\alpha \xrightarrow{\varphi^n_\alpha} X_{n-1} \xrightarrow{q_\beta} X_{n-1}/\left(X_{n-1} - \mathring{e}^{n-1}_\beta\right) \cong S^{n-1}_\beta$$

$$\begin{array}{ccccc}
\sigma^n_\alpha & & [\partial\sigma^n_\alpha] & & d_{\alpha\beta} \\
H_n(D^n_\alpha, \partial D^n_\alpha) & \xrightarrow{\delta} & \widetilde{H}_{n-1}(\partial D^n_\alpha) & \xrightarrow{\Delta_{\alpha\beta *}} & \widetilde{H}_{n-1}(S^{n-1}_\beta) \\
{\scriptstyle e^n_{\alpha *}}\downarrow & & \varphi^n_{\alpha *}[\partial\sigma^n_\alpha] \quad \downarrow{\scriptstyle \varphi^n_{\alpha *}} & & q_{n-1}\circ\varphi^n_{\alpha *}[\partial\sigma^n_\alpha] \quad \uparrow{\scriptstyle p_{\beta *}} \\
H_n(X_n, X_{n-1}) & \xrightarrow{\delta_n} & \widetilde{H}_{n-1}(X_{n-1}) & \xrightarrow{q_*} & \widetilde{H}_{n-1}(X_{n-1}/Xn-2) \\
e^n_\alpha & \searrow{\scriptstyle d_n} & \downarrow{\scriptstyle q_{n-1}} \quad q_{n-1}\circ\varphi^n_{\alpha *}[\partial\sigma^n_\alpha] & & \uparrow{\scriptstyle \cong} \\
d_n(e^n_\alpha) & & H_{n-1}(X_{n-1}, X_{n-2}) & \xrightarrow{\cong} & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{array}$$

Where p_β is given by the following commutative diagram.

$$\begin{array}{ccc}
X_{n-1} & \xrightarrow{q} & X_{n-1}/X_{n-2} \\
& \searrow{\scriptstyle q_\beta} & \downarrow{\scriptstyle p_\eta} \\
& & X_{n-1}/(X_{n-1} - \mathring{e}^n_\beta)
\end{array}$$

Let σ^n_α be a generator of $H_n\left(D^n_\alpha, \partial D^n_\alpha\right) \cong \mathbb{Z}$

$$e^n_{\alpha *}\left(\sigma^n_\alpha\right) = e^n_\alpha$$

By commutativity of the diagram,

$$d_n\left(e^n_\alpha\right) = q_{n-1}\varphi^n_{\alpha *}\left[\partial\sigma^n_\alpha\right].$$

Suppose $d_n\left(e_\alpha^n\right)=\sum d_{\alpha\beta}e_\beta^{n-1}$, we need to compute $d_{\alpha\beta}$.

Finally $p_\beta^* : \widetilde{H}_{n-1}\left(X_{n-1}/X_{n-2}\right) \longrightarrow H_n\left(S_\beta^{n-1}\right)$ is given by $p_\beta^*\left(\sum d_{\alpha\gamma}e_\gamma^{n-1}\right) = d_{\alpha\beta}$, in particular

$$p_\beta^*\left(e_\gamma^{n-1}\right)=\begin{cases}1 & \text{if } \beta=\gamma\\ 0 & \text{otherwise.}\end{cases}$$

We have

$$S_\gamma^{n-1} \xhookrightarrow{i_\gamma} X_{n-1}/X_{n-2} \xrightarrow{p_\beta} S_\beta^{n-1}$$

If $\gamma\neq\beta,\quad p_\beta\circ i_\gamma$ is a constant map where as

$$p_\beta\circ i_\beta \text{ is } Id_{S_\beta^{n-1}}.$$

If σ_γ^{n-1} is a generator for $\widetilde{H}_{n-1}\left(S_\gamma^{n-1}\right)$ then

$$i_{\gamma_*}\left(\sigma_\gamma^{n-1}\right)=e_\gamma^{n-1}$$

$$p_{\beta_*}\left(e_\gamma^{n-1}\right)=p_{\beta_*}\circ i_{\gamma_*}\left(\sigma_\gamma^{n-1}\right)=\begin{cases}0 & \text{if } \gamma\neq\beta\\ 1 & \text{if } \quad \gamma=\beta\end{cases}$$

$$p_{\beta_*}\left(\sum d_{\alpha\gamma}e_\gamma^{n-1}\right)=d_{\alpha\beta}=\Delta_{\alpha\beta_*}\left(\left[\partial\sigma_\alpha^n\right]\right)=\deg\left(\Delta_{\alpha\beta}\right),$$ □

Example 2.64 (**Torus** $\mathbb{T}^2=S^1\times S^1$).

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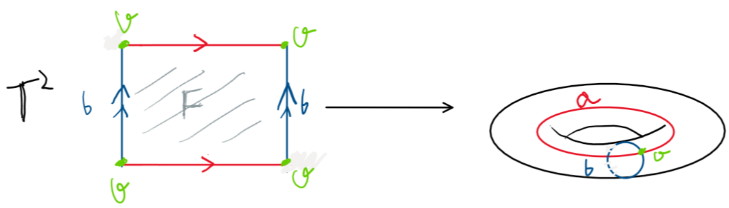


The chain complex $C.^{CW}(T)$ is

$$0\to\mathbb{Z}\xrightarrow{d_2=0}\mathbb{Z}^2\xrightarrow{d_1=0}\mathbb{Z}\longrightarrow 0$$

$d_1=0$ clearly because there is just 1, 0 -cell.

$$d_1(a)=v-v=0,\quad d_1(b)=v-v=0.$$

We claim that d_2 is also 0.

$$d_2(F)=d_{Fa}a+d_{Fb}b, \text{ where}$$

d_{Fa} is the degree of the map $\partial D_F^2 \longrightarrow a \cup b \longrightarrow a$ which is homotopic to the constant map because ∂D_F^2 maps to $aba^{-1}b^{-1}$, hence $d_{Fa} = 0$. Similarly $d_{Fb} = 0$. Hence,

$$H_k(T) \cong \begin{cases} \mathbb{Z}, & k = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z}, & k = 1 \\ 0, & k > 2. \end{cases}$$

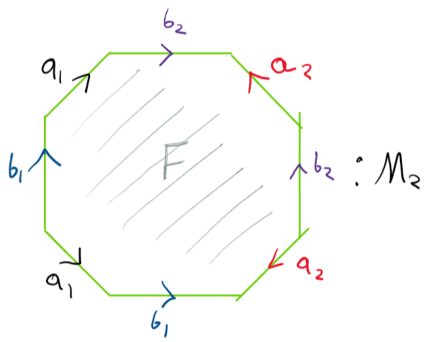
Example 2.65 (**Surface of genus** g, M_g)**.**

M_g has a CW structure with one 2-cell, $2g$, 1-cells and one 0 -cell. So the cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0.$$

Similar to the Example 2.64 d_1 and d_2 are both 0 maps. Hence

$$H_k\left(M_g\right) \cong \begin{cases} \mathbb{Z}, & k = 0, 2 \\ \mathbb{Z}^{2g}, & k = 1 \\ 0, & k > 2. \end{cases}$$



Example 2.66 ($\mathbb{R}P^n$)**.**

Recall that there is a quotient map.

$$\begin{array}{ccc} D^n & \hookrightarrow & S^n \\ \downarrow & \searrow & \downarrow \varphi^n \\ D^n/\{x \sim -x | x \in \partial D\} & \xrightarrow{\cong} & \mathbb{R}P^n = S^n/\{x \sim -x\} \end{array}$$

Hence $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell by the attaching map which is the quotient map

$$\varphi^{n-1} : S^{n-1} \longrightarrow \mathbb{R}P^{n-1}.$$

Thus $\mathbb{R}P^n$ has a CW structure with one k-cell for each $k \in \{0, \dots, n\}$. Hence the cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \to \dots \xrightarrow{d_1} \mathbb{Z} \to 0.$$

$$q : S^{k-1} \xrightarrow{\varphi^{k-1}} \mathbb{R}P^{k-1} \to \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \cong S^{k-1}$$

Let U and V be the two components of $S^{k-1} - S^{k-2}$ If $x_0 = \mathbb{R}P^{k-2}/\mathbb{R}P^{k-2} \in S^{k-1}$, then we have homeomorphisms

$$q : U \longrightarrow S^{k-1} - \{x_0\} \text{ and } q : V \longrightarrow S^{k-1} - \{x_0\}.$$

If $y \in S^{k-1} - \{x_0\}$ then $q^{-1}(y) = \{\pm x\}$ where $x \in U$. $\quad -Id$ takes U to V,

$$\begin{array}{ccc} U & \xrightarrow{q} & S^{k-1} - \{x_0\} \\ {\scriptstyle -Id}\downarrow & \nearrow_{q^{-1}} & \\ V & & \end{array}$$

$q|_U = q|_V \circ (-Id)$

$$\begin{aligned} \deg q &= \deg q|_x + \deg q|_{-x} \\ &= \deg(Id) + \deg(-Id) \\ &= 1 + (-1)^k = \begin{cases} 2 & \text{if k even} \\ 0 & \text{if } k \text{ odd.} \end{cases} \end{aligned}$$

For even n, the cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

and

$$H_k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/2, & 0 < k < n, \quad k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

For odd n, the chain complex is

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$$H_k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & k = 0, n \\ \mathbb{Z}/2, & 0 < k < n, k \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.67.

A continuous map $f : (X, A) \to (Y, B)$ between CW-pairs is called cellular if $f\left(X^{(n)}\right) \subseteq Y^{(n)}$ for all n where $X^{(n)}$ and $Y^{(n)}$ denote the n-skeletons of X and Y respectively.

Theorem 2.68 (Cellular Approximation Theorem).

Let X and Y be CW complexes, and let A be a subcomplex of X. If $f : X \to Y$ is a continuous map such that $f|_A$ is cellular, then f is homotopic rel A to a cellular map $g : X \to Y$.

Proof. Refer to Topology and Geometry by Bredon. $\square$

Theorem 2.69.

If $g : (X, A) \to (Y, B)$ is a cellular map of CW-pairs, then there is an induced map of CW-chain complexes

$$g_* : \underline{C}_*(X, A) \to \underline{C}_*(Y, B)$$

given by

$$g_*(\sigma) = \sum_{\tau} \deg\left(g_{\tau,\sigma}\right) \tau$$

where $g_{\tau,\sigma} : S^n_\sigma \to S^n_\tau$ is defined as the composite $g_{\tau,\sigma} = \bar{p}_\tau \circ \bar{g} \circ \bar{f}_\sigma$ from the following commutative diagram.

$$\begin{array}{ccccccc}
D^n_\sigma & \xrightarrow{f_\sigma} & X^{(n)} & \xrightarrow{g} & Y^{(n)} & \xrightarrow{p_\tau} & S^n_\tau \\
\downarrow & & \downarrow & & \downarrow & & \downarrow = \\
S^n_\sigma & \xrightarrow{\bar{f}_\sigma} & X^{(n)}/X^{(n-1)} & \xrightarrow{\bar{g}} & Y^{(n)}/Y^{(n-1)} & \xrightarrow{\bar{p}_\tau} & S^n_\tau
\end{array}$$

Proof. Refer to Topology and Geometry by Bredon. $\square$

Chapter 3

De-Rham Cohomology

3.1 de Rham Cohomology

Let's delve into the concept of de Rham cohomology in a smooth manifold M, possibly with boundary. We'll start by defining some key terms and then explore the construction of de Rham cohomology groups.

In the realm of differential forms on M, the exterior derivative $d : \Omega^p(M) \to \Omega^{p+1}(M)$ plays a central role. Here, $\Omega^p(M)$ represents the set of all p-forms on M. This operator, being linear, naturally gives rise to two important subspaces: the kernel and the image.

Let's introduce these subspaces formally:

$$\mathcal{Z}^p(M) = \operatorname{Ker}\left(d : \Omega^p(M) \to \Omega^{p+1}(M)\right) = \{ \text{ closed } p\text{-forms on } M\},$$
$$\mathcal{B}^p(M) = \operatorname{Im}\left(d : \Omega^{p-1}(M) \to \Omega^p(M)\right) = \{ \text{ exact } p\text{-forms on } M\}$$

Here, $\mathcal{Z}^p(M)$ denotes the space of closed p-forms, those that are annihilated by the exterior derivative, while $\mathcal{B}^p(M)$ represents the space of exact p-forms, those that are in the image of the exterior derivative.

It's essential to note that we extend our conventions to account for boundary cases.

We also establish conventions for the cases where $p < 0$ or $p > n = \dim M$. In these instances, $\Omega^p(M)$ is considered the zero vector space. Consequently, we have $\mathcal{B}^0(M) = 0$ and $\mathcal{Z}^n(M) = \Omega^n(M)$.

The inclusion of exact forms within the set of closed forms implies that $\mathcal{B}^p(M) \subseteq \mathcal{Z}^p(M)$. This observation leads us to define the **de Rham cohomology group in degree** p (or the p**th de Rham group**) of M as the quotient vector space:

$$H_{\mathrm{dR}}^{p}(M) = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)}$$

This group, formed as the quotient of closed forms modulo exact forms, is a real vector space and, therefore, a group under vector addition. Although a more precise term might be "de Rham cohomology space," the traditional terminology aligns with other cohomology theories, which typically yield groups.

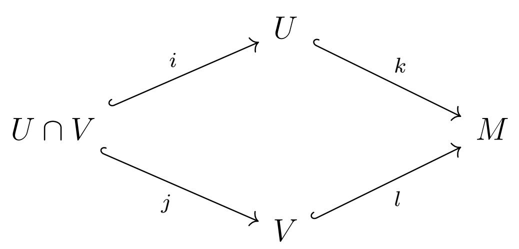
It's worth highlighting that $H_{\mathrm{dR}}^{p}(M) = 0$ for $p < 0$ or $p > \dim M$, as $\Omega^p(M) = 0$ in those cases. For $0 \leq p \leq n$, $H_{\mathrm{dR}}^{p}(M) = 0$ if and only if every closed p-form on M is exact. This condition reflects a fundamental aspect of the cohomology group's structure and its relationship with the underlying manifold M.

Proposition 3.1 (**Cohomology of Disjoint Unions**).
Let $\{M_j\}$ be a countable collection of smooth n-manifolds with or without boundary, and let $M = \coprod_j M_j$. For each p, the inclusion maps $\iota_j : M_j \hookrightarrow M$ induce an isomorphism from $H_{\mathrm{dR}}^{p}(M)$ to the direct product space $\prod_j H_{\mathrm{dR}}^{p}(M_j)$.

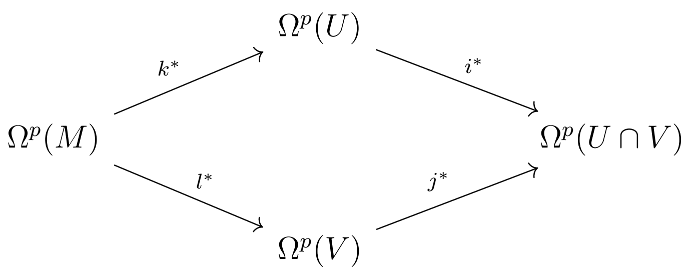
Proof. Refer to Chapter 17 of Introduction to Smooth Manifolds by Lee. □

Certainly! Let's elaborate on the setup for the Mayer-Vietoris Theorem for de Rham Cohomology.

Consider a smooth manifold M, which may or may not have a boundary. Let U and V be open subsets of M such that $M = U \cup V$. In this setup, we have four inclusion maps:



These inclusions induce pullback maps on differential forms:



These pullback maps, essentially restrictions, are denoted as k^*, l^*, i^*, and j^*. For instance, $k^*\omega = \omega|_U$. Now, let's construct a sequence of maps as follows:

$$0 \to \Omega^p(M) \xrightarrow{k^* \oplus l^*} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{i^* - j^*} \Omega^p(U \cap V) \to 0$$

Here,

$$(k^* \oplus l^*)\omega = (k^*\omega, l^*\omega)$$
$$(i^* - j^*)(\omega, \eta) = i^*\omega - j^*\eta$$

Since pullbacks commute with the exterior derivative d, these maps extend to linear maps on the corresponding de Rham cohomology groups.

Now, in the statement of the Mayer-Vietoris theorem, we will adopt standard algebraic terminology. Suppose we have a sequence of vector spaces and linear maps:

$$\cdots \to V^{p-1} \xrightarrow{F_{p-1}} V^p \xrightarrow{F_p} V^{p+1} \xrightarrow{F_{p+1}} V^{p+2} \to \cdots$$

This sequence is exact if the image of each map is equal to the kernel of the next. In other words, for each p,

$$\operatorname{Im} F_{p-1} = \operatorname{Ker} F_p$$

This exactness condition is crucial for the Mayer-Vietoris theorem.

Theorem 3.2 (**Mayer-Vietoris for de Rham Cohomology**).

Let M be a smooth manifold with or without boundary, and let U, V be open subsets of M whose union is M. For each p, there is a linear map $\delta : H^p_{\mathrm{dR}}(U \cap V) \to H^{p+1}_{\mathrm{dR}}(M)$ such that the following sequence, called the Mayer-Vietoris sequence for the open cover $\{U, V\}$, is exact:

$$\cdots \xrightarrow{\delta} H^p_{\mathrm{dR}}(M) \xrightarrow{k^* \oplus l^*} H^p_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V) \xrightarrow{i^* - j^*} H^p_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta} H^{p+1}_{\mathrm{dR}}(M) \xrightarrow{k^* \oplus l^*} \cdots .$$

Proof. Refer to Chapter 17 of Introduction to Smooth Manifolds by Lee. □

Corollary 3.3.

The connecting homomorphism in the Mayer-Vietoris sequence, $\delta : H^p_{\mathrm{dR}}(U \cap V) \to H^{p+1}_{\mathrm{dR}}(M)$, is defined as follows. For each $\omega \in \mathcal{Z}^p(U \cap V)$, there are p-forms $\eta \in \Omega^p(U)$ and $\eta' \in \Omega^p(V)$ such that $\omega = \eta|_{U \cap V} - \eta'|_{U \cap V}$; and then $\delta[\omega] = [\sigma]$, where σ is the $(p+1)$-form on M that is equal to $d\eta$ on U and to $d\eta'$ on V. If $\{\varphi, \psi\}$ is a smooth partition of unity subordinate to $\{U, V\}$, we can take $\eta = \psi\omega$ and $\eta' = -\varphi\omega$, both extended by zero outside the supports of ψ and φ.

Proof. Refer to Chapter 17 of Introduction to Smooth Manifolds by Lee. □

Certainly! Let's delve into the concepts of Singular Cohomology and its properties in a more elaborate manner.

3.2 Singular Cohomology

Before diving into Singular Cohomology, it's imperative to refresh our understanding of Singular Homology. Here are some key points to recall:

- For a one-point space $\{q\}$, the singular homology group $H_0(\{q\})$ is the infinite cyclic group generated by the homology class of the unique singular 0-simplex mapping Δ_0 to q. Additionally, $H_p(\{q\}) = 0$ for all $p \neq 0$.
- Given a collection of topological spaces $\{M_j\}$ and their disjoint union $M = \bigsqcup_j M_j$, the inclusion maps $\iota_j : M_j \hookrightarrow M$ induce an isomorphism $\bigoplus_j H_p(M_j) \cong H_p(M)$.
- Homotopy equivalent spaces exhibit isomorphic singular homology groups.

Definition of Singular Cohomology:- Now, let's introduce Singular Cohomology. In addition to singular homology groups, which are based on the study of cycles and boundaries, Singular Cohomology introduces a sequence of groups $H^p(M;G)$ for any topological space M and abelian group G.

We'll focus on the case where $G=\mathbb{R}$. In this scenario, $H^p(M;\mathbb{R})$ is shown to be a real vector space naturally isomorphic to $\operatorname{Hom}(H_p(M),\mathbb{R})$. This means that each element of $H^p(M;\mathbb{R})$ corresponds to a homomorphism from the singular homology group $H_p(M)$ to $\mathbb{R}$.

Functorial Properties and Universal Coefficient Theorem:- Any continuous map $F: M \to N$ induces a linear map $F^*: H^p(N;\mathbb{R}) \to H^p(M;\mathbb{R})$. This map is defined by applying $\gamma\left(F_*[c]\right)$ for each $\gamma \in H^p(N;\mathbb{R})$ and each singular p-chain c in M.

These properties are functorial, meaning they preserve the structure under mappings. For example, $(G \circ F)^* = F^* \circ G^*$ and $\left(\operatorname{Id}_M\right)^* = \operatorname{Id}_{H^p(M;\mathbb{R})}$. Consequently, singular cohomology with coefficients in $\mathbb{R}$ defines a contravariant functor from the topological category to the category of real vector spaces and linear maps.

The Universal Coefficient Theorem demonstrates how singular cohomology groups with coefficients in any group can be obtained from the singular homology groups. This implies that cohomology groups don't offer new information beyond what's encoded in the homology groups, but they present it in a more convenient manner for certain purposes.

Now, let's explore some properties of singular cohomology, building upon its foundational concepts and functorial properties.

Proposition 3.4 (**Properties of Singular Cohomology**).
(a) For any one-point space $\{q\}$, $H^p(\{q\};\mathbb{R})$ is trivial except when $p=0$, in which case it is 1-dimensional.
(b) If $\{M_j\}$ is any collection of topological spaces and $M=\coprod_j M_j$, then the inclusion maps $\iota_j: M_j \hookrightarrow M$ induce an isomorphism from $H^p(M;\mathbb{R})$ to $\prod_j H^p\left(M_j;\mathbb{R}\right)$.
(c) Homotopy equivalent spaces have isomorphic singular cohomology groups.

Proof. From the above mentioned properties of Singular Homology and from the definition the proof will follow. □

Theorem 3.5 (**Mayer-Vietoris for Singular Cohomology**).
Suppose M, U, and V satisfy the hypotheses of Theorem 2.49 where $U, V \overset{\circ}{\subseteq} M$ (M) is a smooth manifold. The following sequence is exact:

$$\cdots \xrightarrow{\partial^*} H^p(M;\mathbb{R}) \xrightarrow{k^* \oplus l^*} H^p(U;\mathbb{R}) \oplus H^p(V;\mathbb{R}) \xrightarrow{i^* - j^*} H^p(U \cap V;\mathbb{R}) \xrightarrow{\partial^*} H^{p+1}(M;\mathbb{R}) \xrightarrow{k^* \oplus l^*} \cdots,$$

where the maps $k^* \oplus l^*$ and $i^* - j^*$ are defined as in Theorem 3.2, and ∂^* is defined by $\partial^*(\gamma) = \gamma \circ \partial_*$, with ∂_* as in Theorem 3.2.

Let's now define **Smooth Singular Cohomology**.

To bridge the gap between singular and de Rham cohomology groups, we will utilize the concept of integrating differential forms over singular chains. Specifically, given a singular p-simplex σ in a manifold M and a p-form ω on M, our aim is to pull ω back via σ and integrate the resulting form over Δ_p. However, there is an immediate hurdle with this approach: forms can only be pulled back by smooth maps, whereas singular simplices are generally only continuous. (In fact, since the formula for the pullback only involves first derivatives of the map, considering C^1 maps would suffice, but purely continuous maps will not suffice at all.) Now, we will attempt to overcome this obstacle by demonstrating that singular homology can be just as effectively computed using smooth simplices.

Let M be a smooth manifold. A cornerstone of manifold theory is the concept of a smooth p-simplex in M. Formally, a smooth p-simplex is a smooth map $\sigma : \Delta_p \to M$, where Δ_p denotes the standard p-simplex in $\mathbb{R}^{p+1}$, such that this map extends smoothly to a neighborhood of each point in Δ_p. In simpler terms, it's a smooth way of mapping the standard geometric shape of a simplex into our manifold M.

Now, considering the collection of all such smooth simplices, we form a subgroup of the singular chain group $C_p(M)$, denoted by $C_p^\infty(M)$, representing smooth chains in degree p.

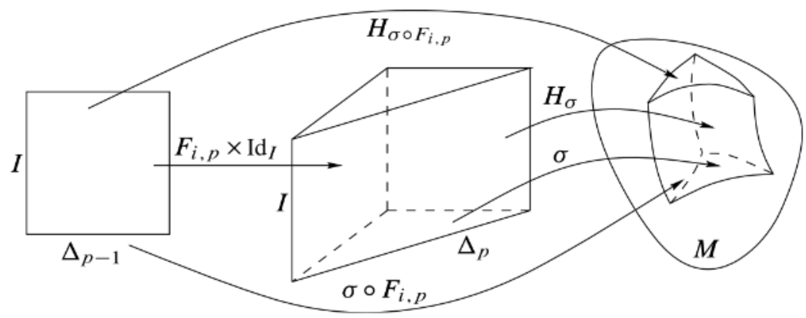
These smooth chains are essentially finite formal linear combinations of smooth simplices, akin to how we'd sum up vectors with coefficients.

Since the boundary of a smooth simplex is itself a smooth chain, we can naturally define the boundary operator ∂ on $C_p^\infty(M)$, which maps a smooth p-simplex to its boundary $(p-1)$-chains in a smooth manner. Consequently, we arrive at the definition of the pth smooth singular homology group $H_p^\infty(M)$ of M as the quotient group:

$$H_p^\infty(M) = \frac{\mathrm{Ker}(\partial : C_p^\infty(M) \to C_{p-1}^\infty(M))}{\mathrm{Im}(\partial : C_{p+1}^\infty(M) \to C_p^\infty(M))}$$

Here, the kernel of ∂ represents smooth chains that form cycles, meaning they have no boundary, while the image of ∂ represents boundaries of smooth chains, indicating the closure of higher-dimensional chains. The quotient group captures the equivalence classes of smooth chains modulo the boundaries, providing insight into the topological structure of the manifold M at the p-dimensional level, but with the added smoothness condition.

The inclusion map $\iota : C_p^\infty(M) \hookrightarrow C_p(M)$ commutes with the boundary operator, thus inducing a map on homology: $\iota_* : H_p^\infty(M) \to H_p(M)$, given by $\iota_*[c] = [\iota(c)]$.



Theorem 3.6 (**Smooth Singular vs. Singular Homology**).
For any smooth manifold M, the map $\iota_* : H_p^\infty(M) \to H_p(M)$ induced by inclusion is an isomorphism.

Proof. Refer to Chapter 18 of Introduction to Smooth Manifolds by Lee. □

Lemma 3.7 (**The Five Lemma**).

Consider the following commutative diagram of modules and linear maps:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow{f_1} & & \downarrow{f_2} & & \downarrow{f_3} & & \downarrow{f_4} & & \downarrow{f_5} \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

If the horizontal rows are exact and f_1, f_2, f_4, and f_5 are isomorphisms, then f_3 is also an isomorphism.

Proof. Refer to Algebraic Topology by Hatcher. □

3.3 Stokes's Theorem & de Rham Theorem

Let M be a smooth manifold, ω a closed p-form on M, and σ a smooth p-simplex in M.

The notion of integrating ω over σ is introduced as follows:

$$\int_\sigma \omega = \int_{\Delta_p} \sigma^* \omega$$

Here, Δ_p represents a smooth p-submanifold with corners embedded in $\mathbb{R}^p$, inheriting the orientation of $\mathbb{R}^p$. Alternatively, we can view Δ_p as a domain of integration within $\mathbb{R}^p$.

This integral concept aligns neatly with the notion of line integrals when $p = 1$, where it corresponds to integrating ω along a smooth curve segment $\sigma : [0, 1] \to M$.

Now, extending this notion to a smooth p-chain $c = \sum_{i=1}^k c_i \sigma_i$, we define the integral of ω over c as:

$$\int_c \omega = \sum_{i=1}^k c_i \int_{\sigma_i} \omega$$

Moving forward, assuming familiarity with the proof of Stokes's Theorem, we aim to establish the de Rham Theorem by stating its essence.

Theorem 3.8 (Stokes's Theorem for Chains)**.**

If c is a smooth p-chain in a smooth manifold M, and ω is a smooth $(\ p - 1 \ )$-form on M, then

$$\int_{\partial c} \omega = \int_c d\omega$$

Proof. Refer to Chapter 18 of Introduction to Smooth Manifolds by Lee. □

Utilizing this theorem, we establish a significant linear mapping denoted as $\ell : H^p_{\mathrm{dR}}(M) \to H^p(M;\mathbb{R})$, often termed the **de Rham homomorphism**. Its operation is delineated as follows: Given any $[\omega] \in H^p_{\mathrm{dR}}(M)$ and $[c] \in H_p(M) \cong H^\infty_p(M)$, we define

$$\ell[\omega][c] = \int_{\tilde{c}} \omega$$

where $\tilde{c}$ represents any smooth p-cycle that embodies the homology class $[c]$. The coherence of this definition stems from the theorem's assurance that if $\tilde{c}$ and $\tilde{c}'$ are smooth cycles representing identical homology classes, then $\widetilde{c} - \widetilde{c'} = \partial\widetilde{b}$ holds true for a smooth $(p+1)$-chain $\tilde{b}$. Consequently,

$$\int_{\tilde{c}} \omega - \int_{\tilde{c}'} \omega = \int_{\partial\tilde{b}} \omega = \int_{\tilde{b}} d\omega = 0$$

Moreover, if $\omega = d\eta$ is exact, then

$$\int_{\tilde{c}} \omega = \int_{\tilde{c}} d\eta = \int_{\partial\tilde{c}} \eta = 0$$

(It's worth noting that $\partial\widetilde{c} = 0$ since $\tilde{c}$ denotes a homology class, and $d\omega = 0$ as ω signifies a cohomology class.) Evidently, $\ell[\omega]\,[c + c'] = \ell[\omega][c] + \ell[\omega]\,[c']$, and the resultant homomorphism $\ell[\omega] : H_p(M) \to \mathbb{R}$ exhibits linearity with respect to ω. Hence, $\ell[\omega]$ emerges as a well-defined member of $\mathrm{Hom}\,(H_p(M), \mathbb{R}) \cong H^p(M;\mathbb{R})$.

Theorem 3.9 (Naturality of the de Rham homomorphism)**.**
For a smooth manifold M and nonnegative integer p, let $\ell : H^p_{\mathrm{dR}}(M) \to H^p(M;\mathbb{R})$ denote the de Rham homomorphism.

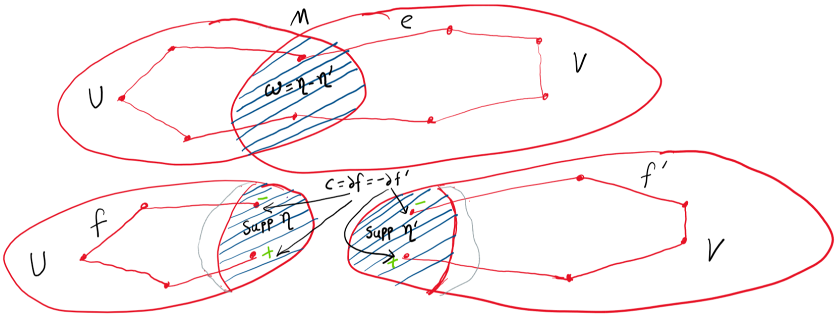
(a) If $F : M \to N$ is a smooth map, then the following diagram commutes:

$$\begin{array}{ccc} H^p_{\mathrm{dR}}(N) & \xrightarrow{F^*} & H^p_{\mathrm{dR}}(M) \\ \ell\downarrow & & \downarrow\ell \\ H^p(N;\mathbb{R}) & \xrightarrow[F^*]{} & H^p(M;\mathbb{R}) \end{array}$$

(b) If M is a smooth manifold and U, V are open subsets of M whose union is M, then the following diagram commutes:

$$\begin{array}{ccc} H^{p-1}_{\mathrm{dR}}(U \cap V) & \xrightarrow{\delta} & H^p_{\mathrm{dR}}(M) \\ \ell\downarrow & & \downarrow\ell \\ H^{p-1}(U \cap V;\mathbb{R}) & \xrightarrow[\partial^*]{} & H^p(M;\mathbb{R}) \end{array}$$

where δ and ∂^* are the connecting homomorphisms of the Mayer-Vietoris sequences for de Rham and singular cohomology, respectively.



Proof. Starting with the given definitions, let's consider a smooth p-simplex σ in M and a smooth p-form ω on N. We have the integral equation:

$$\int_{\sigma} F^*\omega = \int_{\Delta_p} \sigma^* F^*\omega = \int_{\Delta_p} (F \circ \sigma)^*\omega = \int_{F\circ\sigma} \omega.$$

This equation suggests that:

1. Integrating the pullback of ω along σ is equivalent to integrating the pullback of ω along the composition of F and σ.
2. The integral of ω over $F \circ \sigma$ yields the same result.

This leads to the conclusion:

$$\ell(F^*[\omega])[\sigma] = \ell[\omega][F \circ \sigma] = \ell[\omega](F_*[\sigma]) = F^*(\ell[\omega])[\sigma],$$

which establishes property (a).

Now, let's focus on property (b). For the diagram to commute, we need:

$$\ell(\delta[\omega])[e] = (\partial^* \ell[\omega])[e]$$

for any $[\omega] \in H_{\mathrm{dR}}^{p-1}(U \cap V)$ and any $[e] \in H_p(M)$. With the identification of $H^p(M;\mathbb{R})$ as $\operatorname{Hom}(H_p(M), \mathbb{R})$, we rewrite this as:

$$\ell(\delta[\omega])[e] = \ell([\omega])(\partial_*[e]).$$

Expanding further, if σ represents $\delta[\omega]$ and c represents $\partial_*[e]$, we aim to show $\int_e \sigma = \int_c \omega$.

According to Theorem 2.49, we can express c as ∂f, where f and f' are smooth p-chains in U and V respectively, representing the same homology class as e.

Similarly, by Corollary 3.3, we can select $\eta \in \Omega^{p-1}(U)$ and $\eta' \in \Omega^{p-1}(V)$ such that $\omega = \eta|_{U\cap V} - \eta'|_{U\cap V}$. Then, σ can be defined as the p-form equal to $d\eta$ on U and $d\eta'$ on V.

Considering $\partial f + \partial f' = \partial e = 0$ and $d\eta|_{U\cap V} - d\eta'|_{U\cap V} = d\omega = 0$, we arrive at:

$$\begin{aligned}\int_c \omega = \int_{\partial f} \omega &= \int_{\partial f} \eta - \int_{\partial f} \eta' \\ &= \int_{\partial f} \eta + \int_{\partial f'} \eta' = \int_f d\eta + \int_{f'} d\eta' \\ &= \int_f \sigma + \int_{f'} \sigma = \int_e \sigma.\end{aligned}$$

Thus, the diagram commutes as required. □

To establish the de Rham theorem, an additional definition and property of a manifold are indispensable.

Firstly, we introduce the concept of an exhaustion function for a topological space M. An exhaustion function, denoted by $f : M \to \mathbb{R}$, is a continuous function possessing the crucial property that the preimage of each closed interval $(-\infty, c]$ under f, denoted as $f^{-1}((-\infty, c])$, is compact for every $c \in \mathbb{R}$. This property is fundamental as it ensures that the subsets $f^{-1}((-\infty, n])$, where n varies over the positive integers, serve as an exhaustion of M by compact sets. Consequently, an exhaustion function provides a continuous analogue of an exhaustion by compact sets. For instance, consider the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{B}^n \to \mathbb{R}$ defined as:

$$f(x) = |x|^2, \quad g(x) = \frac{1}{1 - |x|^2}$$

These functions exemplify smooth exhaustion functions. Notably, for compact manifolds, any continuous real-valued function suffices as an exhaustion function, rendering the notion particularly relevant for noncompact manifolds.

Proposition 3.10 (Existence of Smooth Exhaustion Functions)**.**
Every smooth manifold with or without boundary admits a smooth positive exhaustion function.

Proof. Consider a smooth manifold M, which may or may not have a boundary. Let's denote $\{V_j\}_{j=1}^{\infty}$ as any countable open cover of M by precompact open subsets, and $\{\psi_j\}$ as a smooth partition of unity subordinate to this cover.

Now, we define a function f belonging to the set of smooth functions on M as follows:

$$f(p) = \sum_{j=1}^{\infty} j\psi_j(p)$$

Firstly, to establish the smoothness of f, note that for any point p in M, only finitely many terms in the summation are non-zero in any given neighborhood around p. Hence, f is indeed smooth.

Furthermore, f is positive. This is evident since $f(p)$ is greater than or equal to $\sum_j \psi_j(p) = 1$, as each $\psi_j(p)$ is non-negative, ensuring the positivity of f.

Now, let's examine why f is an exhaustion function. Take any arbitrary real number c, and select a positive integer N greater than c. If a point p does not belong to the closure of the union of the first N sets in the cover, i.e., $p \notin \bigcup_{j=1}^{N} \bar{V}_j$, then for each j from 1 to N, $\psi_j(p) = 0$. Consequently,

$$f(p) = \sum_{j=N+1}^{\infty} j\psi_j(p) \geq \sum_{j=N+1}^{\infty} N\psi_j(p) = N\sum_{j=1}^{\infty} \psi_j(p) = N > c$$

This inequality holds as $\psi_j(p)$ is non-negative, and each term in the summation is multiplied by j, which is always greater than or equal to N.

Conversely, if $f(p) \leq c$, then p must belong to the closure of the union of the first N sets in the cover, i.e., $p \in \bigcup_{j=1}^{N} \bar{V}_j$. Hence, $f^{-1}((-\infty, c])$ is a closed subset of the compact set $\bigcup_{j=1}^{N} \bar{V}_j$ and is therefore compact. This completes the demonstration that f is indeed an exhaustion function on M. □

Theorem 3.11 (de Rham Theorem).

For every smooth manifold M and nonnegative integer p, the de Rham homomorphism $\ell : H^p_{\mathrm{dR}}(M) \to H^p(M; \mathbb{R})$ is an isomorphism.

Sketch. We will prove this in six steps. Before that we need to define some terminology.

de Rham Manifold:-A smooth manifold M is a de Rham manifold if the homomorphism $\ell : H^p_{\mathrm{dR}}(M) \to H^p(M; \mathbb{R})$ is an isomorphism for each p.

de Rham Cover:-If M is any smooth manifold, an open cover $\{U_i\}$ of M is a de Rham cover if each subset U_i is a de Rham manifold, and every finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is also de Rham.

de Rham Basis:-A de Rham cover that is also a basis for the topology of M is called a de Rham basis for M.

Now as ℓ commutes with the cohomology maps induced by smooth maps so any manifold diffeomorphic to a de Rham Manifold is also de Rham [Using 3.9 (a)].

Step 1:- By Proposition 3.1 and Proposition 3.4 (b) both de Rham and Singular Cohomology the inclusions $\iota_j : M_j \hookrightarrow \coprod_j M_j$ induce isomorphisms between the cohomology groups of the disjoint union and the direct product of the cohomology groups of the manifolds M_j. By Theorem 3.9, ℓ commutes with these isomorphisms.

Step 2:- We will try to prove every convex subset of $\mathbb{R}^n$ is de Rham. Now if $U \mathring{\subseteq} \mathbb{R}^n$ then it is homotopy equivalent to a one-point space.

So by **poincare lemma** $H^p_{dR}(U) \cong 0$ when $p \neq 0$ & also $H_p(U) \cong 0$ when $p \neq 0$ [we know this by properties of singular homology]

Now for $p = 0$ case $H^0_{dR}(U)$ is 1-dim sphere consisting of constant functions & $H^0(U; \mathbb{R}) = H_m(H_0(U), \mathbb{R})$ is also 1-dim because. $H_0(U)$ is generated by any 0 -simplex.

Now if $\sigma : \Delta_0 \to M$ is a singular 0− simplex, σ is smooth as 0-manifold is smooth and f

is the constant function equal to 1 , then

$$\ell[f][\sigma] = \int_{\sigma^0} \sigma^* f = (f \circ \sigma)(0) = 1$$

so $\ell : H^0_{dR}(U) \longrightarrow H^0(U, R)$ is not the zero map so ℓ is an isomorphism.

Step 3:-Now our goal is to prove If M has a finite de Rham Cover then M is de Rham. Suppose $M = \cup_{i=1}^k U_i$ and U_i, $\underset{i \in A \subseteq [k]}{\cap} U_i$ is de Rham. We will use induction on k to prove this now for two sets we will try to show this first this way induction will follow.

Now we will write the Mayer- Vietoris LES of de Rham Cohomology and Singular Cohomology sequentially in two lines and then we will get the commutative diagram by Theorem 3.9.

$$\begin{array}{ccccccccc}
H^{p-1}_{\mathrm{dR}}(U) \oplus H^{p-1}_{\mathrm{dR}}(V) & \longrightarrow & H^{p-1}_{\mathrm{dR}}(U \cap V) & \longrightarrow & H^p_{\mathrm{dR}}(M) & \longrightarrow & H^p_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V) & \longrightarrow & H^p_{\mathrm{dR}}(U \cap V) \\
\cong [Assumption] \downarrow & & \cong [Assumption] \downarrow & & \cong [5 - Lemma] \downarrow & & \cong [Assumption] \downarrow & & \cong [Assumption] \downarrow \\
H^{p-1}(U;\mathbb{R}) \oplus H^{p-1}(V;\mathbb{R}) & \longrightarrow & H^{p-1}(U \cap V;\mathbb{R}) & \longrightarrow & H^p(M;\mathbb{R}) & \longrightarrow & H^p(U;\mathbb{R}) \oplus H^p(V;\mathbb{R}) & \longrightarrow & H^p(U \cap V;\mathbb{R})
\end{array}$$

Assuming that the claim holds true for smooth manifolds that admit a de Rham cover with at least $k \geq 2$ sets. Now, consider a de Rham cover $\{U_1, \ldots, U_{k+1}\}$ of the manifold M. We define two sets: $U = U_1 \cup \cdots \cup U_k$ and $V = U_{k+1}$. The hypothesis assures us that both U and V are de Rham sets. Additionally, the intersection $U \cap V$ inherits this property since it can be covered by k sets, namely $\{U_1 \cap U_{k+1}, \ldots, U_k \cap U_{k+1}\}$, each of which is de Rham.

With these considerations, we conclude that M can be expressed as the union of U and V, i.e., $M = U \cup V$. By the argument presented above, M is also de Rham. **Step 4:-** This step aims to establish the assertion that if a manifold M has a de Rham basis, then M itself is de Rham.

Let's delve into the argument:

Firstly, consider $\{U_\alpha\}$ as a de Rham basis for M. An "exhaustion function" $f : M \to \mathbb{R}$ is introduced, satisfying the conditions specified in Proposition 3.10.

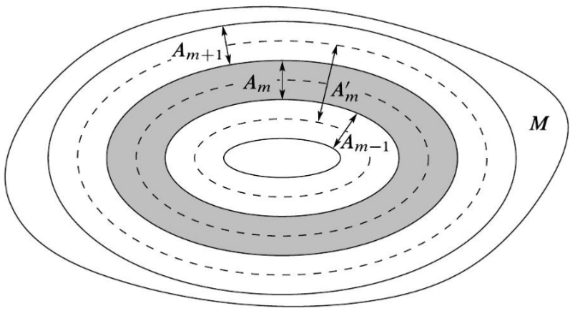
Now, for every integer m, we define subsets A_m and A'_m of M as follows:

$$A_m = \{q \in M : m \leq f(q) \leq m+1\},$$
$$A'_m = \left\{q \in M : m - \frac{1}{2} < f(q) < m + \frac{3}{2}\right\}.$$

Next, it's observed that $\{U_j | q \in U_j,\, U_j \subseteq A'_m\}$ forms an open cover of A_m. Since A_m is compact and f is an exhaustion function, this cover admits a finite subcover. Denote the union of these finite sets as $B_m = \cup_{i=1}^{k} U_i$.

Utilizing the argument from Step 3, it's noted that B_m is de Rham, and moreover, $B_m \subseteq A'_m$.

Here's a visual representation of the setup:



We can infer that B_m can only intersect non-trivially with $B_{\tilde{m}}$ when $\tilde{m} = m-1, m$, or $m+1$.

With this insight, we define:

$$U = \bigcup_{m \text{ odd}} B_m, \quad V = \bigcup_{m \text{ even}} B_m$$

Both U and V are disjoint unions of de Rham manifolds, hence they are themselves de Rham by Step 1. Furthermore, $U \cap V$ is de Rham since it's the disjoint union of sets $B_m \cap B_{m+1}$ for $m \in \mathbb{Z}$. Each of these sets has a finite de Rham cover, as illustrated, establishing that $M = U \cup V$ is de Rham by the reasoning provided in Step 3.

Step 5:- This step asserts that every open subset of $\mathbb{R}^n$ is de Rham.

Consider an open subset $U \subseteq \mathbb{R}^n$. Such a subset possesses a basis comprising Euclidean balls. Since each ball is convex, it is de Rham. Moreover, any finite intersection of balls remains convex, ensuring that these intersections are also de Rham. Consequently, U has a de Rham basis, making it de Rham by the argument presented in Step 4.

Step 6:- This step concludes that every smooth manifold is de Rham.

Any smooth manifold can be endowed with a basis consisting of smooth coordinate domains. Given that each smooth coordinate domain is diffeomorphic to an open subset of $\mathbb{R}^n$, and considering their finite intersections, this constitutes a de Rham basis. Thus, the claim follows from Step 4. $\square$

Chapter 4

Basic Morse Theory

4.1 Morse functions

Let's consider M to be a smooth manifold of dimension m. If $\phi(x) = (x_1, ..., x_m)$ is a local coordinate system near critical point p, then $\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_m}|_p$ is a basis for T_pM.

Definition 4.1 (Critical point)**.**

A critical point of a smooth function $f : M \longrightarrow \mathbb{R}$ is a point p at which the differential

$$df_p : T_pM \longrightarrow T_{f(p)}\mathbb{R} \approx \mathbb{R}$$

vanishes, i.e., all the partial derivatives. $\frac{\partial(f\circ\phi^{-1})}{\partial x_1}|_p, ..., \frac{\partial(f\circ\phi^{-1})}{\partial x_m}|_p$ vanishes.

***df_p doesn't depend on the basis, so this is well defined.

Definition 4.2 (Hessian)**.**

The Hessian of f at p is the bilinear map:

$$H_p(f) =: T_pM \times T_pM \longrightarrow H_p(f)(V, W) = V_p(\tilde{W}(f))$$

$\tilde{W}$ is a vector field extension of $W \in T_pM$ locally

The matrix of $H_p(f)$ with respect to this basis is expressed by the $m \times m$ matrix of second partial derivatives:

$$H_p(f) = \left(\frac{\partial^2(f \circ \phi^{-1})}{\partial x_i \partial x_j} \phi(p) \right)$$

*** Using a suitable bump function defined inside a local chart of $p \in M$, we can show that $\exists$ a vector field $\tilde{W}$ for $W \in T_pM$ such that $\tilde{W}(p) = W$.

Proposition 4.3.

Hessian is a well-defined, bilinear, and symmetric map

Proof. First, we will try to show that it is symmetric.

$$H_p(f)(V,W) - H_p(f)(W,V) = V_p(\tilde{W}(f)) - W_p(\tilde{V}(f)) = [\tilde{V},\tilde{W}]_p = df_p\dot{[}\tilde{V},\tilde{W}] = 0$$

The last term is 0 because p is a critical point of f.So, it is symmetric. Looking again at the Definition 4.2,

$$H_p(f)(V,W) = V_p(\tilde{W}(f))$$

it is obvious that this does not depend on the extension $\tilde{V}$ that we choose for V, as the expression only depends on $\tilde{W}(p) = W$ regardless of the extension. As we just proved, $H_p(f)(V,W) = H_p(f)(W,V)$, so, applying a similar argument, the Hessian does not depend on the chosen extension for W. This shows that the Hessian is well-defined. Finally, the Hessian is bilinear because

$$H_p(f)(\alpha V + \beta U, W) = (\alpha V + \beta U)_p(\tilde{W}(f)) = \alpha V_p(\tilde{W}(f)) + \beta U_p(\tilde{W}(f))$$

$$\text{So, } H_p(f)(\alpha V + \beta U, W) = \alpha H_p(f)(V,W) + \beta H_p(f)(U,W)$$

□

Definition 4.4.

Let p be a critical point of a smooth function $f : M \longrightarrow \mathbb{R}$

1. The number of negative eigenvalues of $H_p(f)$ is called the **index** denoted by λ_p.
2. The critical point p is said to be **non-degenerate** if and only if the determinant of $H_p(f)$ is non-zero.
3. A **Morse function** on a smooth manifold is a smooth function whose critical points are all non-degenerate.

***The nondegeneracy and the index of a function f at a critical point p does not depend on the choice of local coordinates. We can see this by applying Sylvester's law, which

says that the number of negative eigenvalues of the Hessian is independent of how it is diagonalized. Since diagonalization of a matrix corresponds to changing the basis of the given vector space so that the basis vectors are the eigenvectors of the matrix, this means that. The number of negative eigenvalues of the Hessian is invariant under coordinate transformation. If we change the basis then Hessian H_p^{ψ} with respect to new basis will be $J^t H_p^{\phi} J$ where J is a symmetric matrix, i.e. Jacobian of $\psi \circ \phi^{-1}$
We can see this in a different way than above.

Proposition 4.5.

When p is a critical point for $f : M \longrightarrow \mathbb{R}$, the Hessian at p is independent of the coordinate chart.

Proof. Now suppose we had a different coordinate chart around p, $(V, \psi : V \longrightarrow \mathbb{R}^n)$, with $\psi(p) = 0$. Write ψ as $(y_1, \ldots, y_n)$. Then $Q = \psi \circ \phi^{-1}$: $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism, and $dQ\left(e_i\right) = \sum_{j=1}^n \frac{\partial x_i}{\partial y_j} e_j$ (where $e_1, \ldots, e_n$ is the standard basis in $\mathbb{R}^n$), then the Hessian defined for this new coordinate chart is

$$
\begin{aligned}
\mathrm{H}_p(f)(v, w) &= \sum_{i,j=1}^n \frac{\partial}{\partial y_i}\left(\frac{\partial}{\partial y_j}(f)\right) v_i w_j \\
&= \sum_{i,j=1}^n \frac{\partial x_k}{\partial y_i} \frac{\partial}{\partial x_k}\left(\frac{\partial x_m}{\partial y_j} \frac{\partial}{\partial x_m}(f)\right) v_i w_j \\
&= \sum_{i,j=1}^n \frac{\partial x_k}{\partial y_i} \frac{\partial}{\partial x_k}\left(\frac{\partial x_m}{\partial y_j}\right) \frac{\partial}{\partial x_m}(f) v_i w_j + \sum_{i,j=1}^n \frac{\partial x_k}{\partial y_i} \frac{\partial x_m}{\partial y_j} \frac{\partial^2}{\partial x_k \partial x_m}(f) v_i w_j \\
&= \sum_{i,j=1}^n \frac{\partial x_k}{\partial y_i} \frac{\partial}{\partial x_k}\left(\frac{\partial x_m}{\partial y_j}\right) \frac{\partial}{\partial x_m}(f) v_i w_j + \sum_{i,j=1}^n \frac{\partial^2}{\partial x_k \partial x_m}(f) dQ(v)_k dQ(w)_m
\end{aligned}
$$

Now note that the first term is zero when p is a critical point, so the Hessian is well-defined as a bilinear form on T_pM. □

4.2 Morse Lemma

Now we will try to see the **Morse Lemma**, which is a gateway theorem of Morse Theory and allows us to directly analyze the neighborhood of a non-degenerate critical point on a manifold in a valuable and intuitive manner, akin to the slopes around a hole in m-dimensional space.

Proposition 4.6.

Let $f : U \to \mathbb{R}$ be a C^∞ function on a convex neighborhood U of $0 \in \mathbb{R}^m$ such that $f(0) = 0$. Then there exist functions g_i with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$ and $f(x_1, \ldots, x_m) = \sum_{i=1}^m x_i g_i(x_1, \ldots, x_m)$.

Proof. We have,

$$\frac{d}{dt} f(tx) = \sum_{i=1}^m x_i \frac{\partial f}{\partial x_i}(tx_i)$$

Hence, setting

$$g_i(x_1, \ldots, x_m) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$$

we get

$$\begin{aligned} f(x) = f(x) - f(0) &= \int_0^1 \frac{d}{dt} f(tx)dt \\ &= \int_0^1 \sum_{i=1}^m x_i \frac{\partial f}{\partial x_i}(tx)dt \\ &= \sum_{i=1}^m x_i g_i(x_1, \ldots, x_m) \end{aligned}$$

and $g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(0)dt = \frac{\partial f}{\partial x_i}(0)$. □

Remark 4.7.

If $g_i(0) = 0$ then we can apply the Proposition 4.6 to g_i, and we have

$$g_i(x_1, \ldots, x_m) = \sum_{j=1}^m x_j h_{ij}(x_1, \ldots, x_m)$$

where the h_{ij} are C^∞ functions with $h_{ij}(0) = \frac{\partial g_i}{\partial x_j}(0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0)$. We have

$$\begin{aligned} f(x_1, \ldots, x_m) &= \sum_{i=1}^m x_i \left(\sum_{j=1}^m x_j h_{ij}(x_1, \ldots, x_m) \right) \\ &= \sum_{i,j} x_i x_j h_{ij}(x_1, \ldots, x_m) \\ &= {}^t x S_x x \end{aligned}$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ and $S_x = (s_{ij}(x))$ is the symmetric matrix with entries

$$s_{ij}(x) = \frac{1}{2}\left(h_{ij}(x) + h_{ji}(x)\right).$$

The expression $f(x) = {}^t x S_x x$ is a Taylor formula of order 2 for the function $f : U \to \mathbb{R}$.

Proposition 4.8.

Let $A = \operatorname{diag}(a_1, a_2, \ldots, a_m)$ be a diagonal matrix with diagonal entries $a_j = \pm 1$ for all $j = 1, \ldots, m$. Then there is a neighborhood U of A in the vector space of symmetric matrices $\left(\approx \mathbb{R}^{m(m+1)/2}\right)$ and a C^∞ map $P : U \to GL_m(\mathbb{R})$ that satisfies

1. $P(A) = I_{m\times m}$
2. If $P(S) = Q$, then $Q^t S Q = A$.

Proof. We embark on a proof by induction concerning the dimensionality denoted by m. Initially, let's consider the base case when $m = 1$, and we have $A = (a)$ with $a = \pm 1$. If $S = (s)$ is any 1×1 matrix sufficiently close to A, it necessarily follows that s will be non-zero with the same sign as a. To proceed, we define a transformation $P(S)$ as follows:

$$P(S) = Q = \left(\frac{1}{\sqrt{|s|}} \right)$$

Now, let's extend our consideration to the case where A is a diagonal matrix represented as $\operatorname{diag}(a_1, a_2, \ldots, a_m)$, with each a_j being ± 1 for $j = 1, \ldots, m$. For the induction step, let's assume the existence of a neighborhood U_1 of $A_1 = \operatorname{diag}(a_2, \ldots, a_m)$ in the vector space of $(m-1)\times(m-1)$ symmetric matrices. Within this neighborhood, for every $S_1 \in U_1$, we assert the existence of a smooth mapping $P_1 : U_1 \to GL_{m-1}(\mathbb{R})$. This mapping satisfies $P_1(A_1) = I_{(m-1)\times(m-1)}$ and $Q_1^t S_1 Q_1 = A_1$, where $Q_1 = P_1(S_1) \in GL_{m-1}(\mathbb{R})$.

Now, let $S = (s_{ij})$ be a symmetric $m\times m$ matrix sufficiently close to $A = \operatorname{diag}(a_1, a_2, \ldots, a_m)$ such that s_{11} is non-zero and has the same sign as a_1. This matrix S defines a symmetric bilinear form $B : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ given by $B(x, y) = x^t S y$ for all $x, y \in \mathbb{R}^m$.

Following the initial step in the Gram-Schmidt orthogonalization process, we transition from the standard basis $e_1, \ldots, e_m$ of $\mathbb{R}^m$ to a new basis $v_1, \ldots, v_m$ defined as:

$$v_1 = \frac{e_1}{\sqrt{|s_{11}|}}$$

and

$$v_j = e_j - B(v_1, v_1)\, B(v_1, e_j)\, v_1 = e_j - \frac{s_{1j}}{s_{11}} e_1$$

for all $j = 2, \ldots, m$. The corresponding change of basis matrix $C \in GL_m(\mathbb{R})$ is given by:

$$C = \left(\begin{array}{c|ccc} \frac{1}{\sqrt{|s_{11}|}} & -\frac{s_{12}}{s_{11}} & \cdots & -\frac{s_{1m}}{s_{11}} \\ \hline 0 & & & \\ \vdots & & I & \\ 0 & & & \end{array}\right)$$

where I denotes the $(m-1)\times(m-1)$ identity matrix. This new basis satisfies $B(v_1, v_j) = 0$ for all $j = 2, \ldots, m$. Consequently, it can be observed that:

$$C^t S C = \left(\begin{array}{c|ccc} a_1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & S_1 & \\ 0 & & & \end{array}\right)$$

where S_1 is an $(m-1)\times(m-1)$ symmetric matrix that varies smoothly with S. If S is sufficiently close to A, then $S_1 \in U_1$, enabling the application of the induction hypothesis. This implies the existence of $Q_1 \in GL_{m-1}(\mathbb{R})$, dependent smoothly on S_1, such that $Q_1^t S_1 Q_1 = A_1 = \operatorname{diag}(a_2, \ldots, a_m)$.

Finally, define $P(S) = Q = CR$ where:

$$R = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & Q_1 & \\ 0 & & & \end{array}\right)$$

This yields $Q^t S Q = A$, where $P(S) = Q \in GL_m(\mathbb{R})$, depending smoothly on S, and $P(A) = I_{m\times m}$. □

We will now use this proposition to prove the Morse lemma.

Theorem 4.9 (Morse Lemma)**.**

Let $p \in M$ be a non-degenerate critical point of a smooth function $f : M \longrightarrow \mathbb{R}$ of index k. There exists a smooth chart $\phi : U \longrightarrow \mathbb{R}^m$, where U is an open neighborhood of p, with $\phi(p) = 0$ such that if $\phi(x) = (x_1, ..., x_m)$ for $x \in U$, then

$$(f \circ \phi^{-1})(x_1, ..., x_m) = f(p) - x_1^2 - x_2^2 - ... - x_k^2 + x_{k+1}^2 + x_{k+2}^2 + ... + x_m^2$$

Proof. As this is a local property let the function f is defined on a convex neighborhood of the origin 0 in $\mathbb{R}^m$, with certain properties: $p = 0$, $f(0) = 0$, $df(0) = 0$, and the second partial derivatives of f at 0 form a diagonal matrix A with entries ± 1 on the diagonal.

Remark 4.7 provides an important insight: it expresses $f(x)$ in terms of a symmetric matrix S_x which varies smoothly with x, leading to $f(x) = {}^t x S_x x$. This matrix S_x satisfies $S_0 = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right) = A$.

This implies the existence of a neighborhood U_0 of 0 such that for x in U_0, S_x also lies in a neighborhood U (as per Proposition 4.8). If we denote the mapping in Proposition 4.8 as P, then $P(S_x) = Q_x$ satisfies ${}^t Q_x S_x Q_x = A$, and $Q_0 = I_{m \times m}$.

Now, consider the mapping $\phi : U \to \mathbb{R}^m$ defined by $\phi(x) = Q_x^{-1} x$. It's evident that $\phi(0) = Q_0^{-1}(0) = 0$, and for any $v \in \mathbb{R}^m$, we can analyze its derivative:

$$\begin{aligned}
(d\phi_0)(v) &= \lim_{t \to 0} \frac{\phi(tv) - \phi(0)}{t} \\
&= \lim_{t \to 0} \frac{\phi(tv)}{v} \\
&= \lim_{t \to 0} \frac{Q_{(tv)}^{-1}(tv)}{t} \\
&= \lim_{t \to 0} \frac{t Q_{(tv)}^{-1}(v)}{t} \\
&= \lim_{t \to 0} Q_{(tv)}^{-1}(v) \\
&= Q_0^{-1}(v) \\
&= v.
\end{aligned}$$

Thus, $d\phi_0$ is the identity map, and by the Inverse Function Theorem, we conclude that ϕ, when restricted to a smaller neighborhood, serves as a coordinate system near 0.

Now, let $y = \phi(x) = Q_x^{-1}(x)$. We can manipulate $f(x)$ using y:

$$\begin{aligned}\sum_{i=1}^{m} a_i y_i^2 &= {}^t y A y \\ &= {}^t(Q_x^{-1}x)A(Q_x^{-1}x) \\ &= {}^t x {}^t Q_x^{-1}[{}^t Q_x S_x Q_x] Q_x^{-1} x \\ &= {}^t x S_x x \\ &= f(x).\end{aligned}$$

□

Corollary 4.10.

The non-degenerate critical points of a Morse function f are isolated.

Proof. In some neighborhood around p, we have $f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$, Thus $\frac{\partial f}{\partial x_i} = 2x_i$, and so $\frac{\partial f}{\partial x_i} = 0$ iff $x_1 = x_2 = \cdots = x_n = 0$. □

Corollary 4.11.

On a closed (compact) manifold M, a Morse function has only finitely many critical points.

Proof. Let there be an infinite number of critical points. Then, by Corollary 4.10, they are all isolated so we can get $\{U_i\}_{i\in\Lambda}$. $x_i \in U_i$ is critical point & $U_i \subset M$ so from this are can get $\{U_i\}_{i\in\Lambda} \cup \{M \setminus \cup_{i\in\Lambda} U_i\}$ an infinite cover of M but M is compact so $\exists$ a finite cover $\{U_i\}_{i=1}^n$ of M so $\exists j \in \Lambda$ such that $i \neq j$ but $x_j \in U_i$ but that is a contradiction. □

4.3 Existence of Morse functions

Showing that there exist Morse functions on all manifolds is more challenging, but it is true. We will try to show that on a compact manifold, there exist many (in a precise sense) Morse functions (in fact, we will show this for any manifold that embeds as a submanifold into $\mathbb{R}^n$ for some n).

Proposition 4.12.

Let $V \subset \mathbf{R}^n$ be a submanifold. For almost every point p of $\mathbf{R}^n$, the function

$$f_p : V \longrightarrow \mathbf{R}$$
$$x \longmapsto \|x - p\|^2$$

is a Morse function.

Proof. Firstly, consider a smooth function f_p defined on a smooth manifold V and a point p in this manifold. The differential of f_p at a point x in V is given by $T_x f_p(\xi) = 2(x-p)\cdot\xi$, where ξ is a tangent vector at x.

We establish that x is a critical point of f_p if and only if the vector $(x-p)$ is orthogonal to the tangent space T_xV at x.

To analyze this further, let's introduce local coordinates $(u_1, \ldots, u_d)$ for the manifold V, such that x can be parametrized as $x(u_1, \ldots, u_d)$.

In these local coordinates, the partial derivatives of f_p with respect to u_i are given by $\frac{\partial f_p}{\partial u_i} = 2(x-p) \cdot \frac{\partial x}{\partial u_i}$.

Similarly, the second partial derivatives of f_p with respect to u_i and u_j are expressed as $\frac{\partial^2 f_p}{\partial u_i \partial u_j} = 2\left(\frac{\partial x}{\partial u_j} \cdot \frac{\partial x}{\partial u_i} + (x-p) \cdot \frac{\partial^2 x}{\partial u_i \partial u_j}\right)$.

Thus, we conclude that x is a non-degenerate critical point if and only if the vector $x-p$ is orthogonal to T_xV, and the matrix formed by the second partial derivatives has full rank d.

To establish that f_p is a Morse function for almost all p, it is sufficient to demonstrate that the p values not satisfying the aforementioned condition are the critical values of a smooth map, and then invoke Sard's theorem.

We then focus on the "normal fiber bundle" of V in $\mathbf{R}^n$, denoted by N, defined as the set $\{(x, v) \in V \times \mathbf{R}^n \mid v \perp T_xV\}$. Additionally, consider the map $E : N \to \mathbf{R}^n$ defined by $E(x, v) = x + v$.

The proposition mentioned is a consequence of the following lemma □

Lemma 4.13.

The normal vector bundle N is a submanifold of $V \times \mathbf{R}^n$. The point $p = x + v \in \mathbf{R}^n$ is a critical value of E if and only if the matrix

$$\frac{\partial^2 f}{\partial u_i \partial u_j} = 2\left(\frac{\partial x}{\partial u_j} \cdot \frac{\partial x}{\partial u_i} - v \cdot \frac{\partial^2 x}{\partial u_i \partial u_j}\right)$$

is not invertible.

Proof. Since V is a submanifold of $\mathbf{R}^n$, there is a (local) chart that sends $\mathbf{R}^n$ onto an open subset of $\mathbf{R}^n$, and $\mathbf{R}^d$ onto an open subset of V. The tangent map of the chart sends the canonical basis of $\mathbf{R}^n$ onto a basis of vectors tangent to V followed by vectors generating a complement. It then suffices to make this basis orthonormal in order to obtain $n-d$ vectors $v_1, \ldots, v_{n-d}$ that at every point of V form an orthonormal basis of $(T_x V)^\perp$. The map

$$(u_1, \ldots, u_d, t_1, \ldots, t_{n-d}) \longmapsto \left(x\left(u_1, \ldots, u_d\right), \sum_{i=1}^{n-d} t_i v_i\left(u_1, \ldots, u_{n-d}\right)\right)$$

is then a local parametrization of N, which is therefore a submanifold of $V \times \mathbf{R}^n$. In these coordinates, the partial derivatives of E are

$$\begin{cases} \frac{\partial E}{\partial u_i} = \frac{\partial x}{\partial u_i} + \sum_{k=1}^{n-d} t_k \frac{\partial v_k}{\partial u_i} \\ \frac{\partial E}{\partial t_j} = v_j \end{cases}$$

Computing the inner products of these n vectors with the n independent vectors

$$\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_d}, v_1, \ldots, v_{n-d}$$

gives a matrix that has the same rank as the Jacobian of E and that is of the form

$$\begin{pmatrix} \left(\frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} + \sum_k t_k \frac{\partial v_k}{\partial u_i} \cdot \frac{\partial x}{\partial u_j}\right) & \left(\sum_k \frac{\partial v_k}{\partial u_i} \cdot v_\ell\right) \\ 0 & \mathrm{Id} \end{pmatrix}$$

Now v_k is orthogonal to $\partial x / \partial u_j$, so

$$\frac{\partial}{\partial u_i}\left(v_k \cdot \frac{\partial x}{\partial u_j}\right) = \frac{\partial v_k}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} + v_k \cdot \frac{\partial^2 x}{\partial u_i \partial u_j} = 0.$$

This completes the proof of the lemma □

4.4 Examples

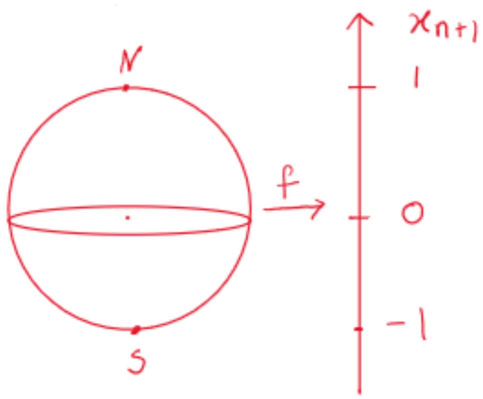
Example 4.14 (The height function on n-Sphere).

Let the n-Sphere be

$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1\}$. The chart is only the projection onto the first n coordinates. We are taking only two charts of two hemispheres as we will only get all our critical points in these two charts. Now define $f : s^n \to \mathbb{R}$ by $f(x_1, \ldots, x_{n+1}) = x_{n+1}$ This function is a smooth Morse function on S^n with only two critical points, the northpole $N = (0, \ldots, 0, +1)$ (the maximum) and the south pole $S = (0, \ldots, -1)$ (the minimum)

$$
\begin{aligned}
&\left(\frac{\partial}{\partial x_1} f(x), \frac{\partial}{\partial x_2} f(x), \ldots, \frac{\partial}{\partial x_n} f(x)\right) = 0 \\
&\Rightarrow \left(\frac{1}{2} \frac{-2x_1}{x_{n+1}}, \frac{1}{2} \frac{-2x_2}{x_{n+1}}, \ldots, \frac{1}{2} \frac{-2x_n}{x_{n+1}}\right) = 0 \\
&\Rightarrow x_i = 0 \quad \forall i \in [n] \\
&f(x) = x_{n+1} = \sqrt{1 - (x_1^2 + x_2^2 + \ldots + x_n^2)}
\end{aligned}
$$

So $x_{n+1} = \pm 1$ so it has two critical points. now



$$
M_p(f) = \begin{pmatrix}
-\frac{1}{x_{n+1}} + \frac{x_1^2}{x_{n+1}^{1/2}} & \frac{x_1 x_2}{x_{n+1}^{1/2}} & \cdots & \frac{x_1 x_n}{x_{n+1}^{3/2}} \\
\vdots & -\frac{1}{x_{n+1}} + \frac{x_1^2}{x_{n+1}^{3/2}} & \vdots & \\
\vdots & & \ddots & \vdots \\
\vdots & & \ddots & \vdots \\
\frac{x_1 x_n}{x_{n+1}^{1/2}} & \cdots & \cdots & -\frac{-1}{x_{n+1}} + \frac{x_n^2}{x_{n+1}^{1/2}}
\end{pmatrix}_{n \times n}
$$

$$M_N(f) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ \vdots & -1 & \vdots & \\ \vdots & & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}_{n\times n}$$

So, the index is n.

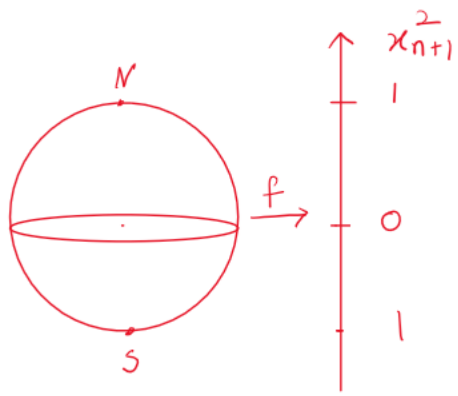
$$M_S(f) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & 1 & \vdots & \\ \vdots & & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}$$

So, the index is 0.

Example 4.15 (A function with non-degenerate critical points)**.**

The function $g = f^2 : S^3 \to [0,1]$ is not Morse function because it has infinitely many critical points by polar coordinate we get $p = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

$\theta \in (-\pi, \pi), \varphi \in (0, 2\pi)$



Now $f^2(p) = r^2\cos^2\varphi$.

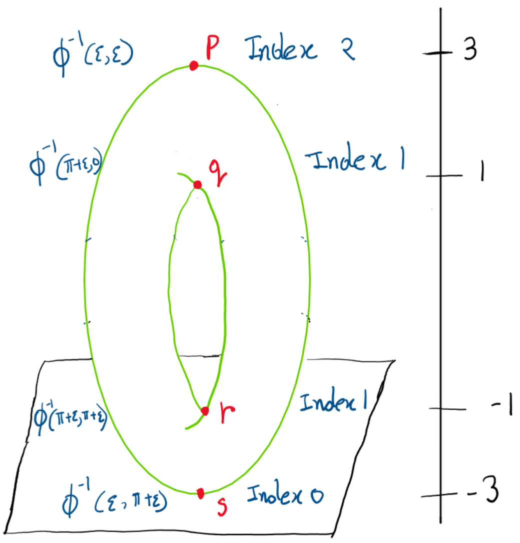
$\left(\frac{\partial}{\partial\theta} r^2\cos^2\varphi, \frac{\partial}{\partial\varphi} r^2\cos^2\varphi\right) = 0 \Rightarrow (0, -2r^2\cos\varphi\sin\varphi) = 0$ so $\varphi = \pi/2, -\pi/2$ or $\varphi = 0$

so for $\varphi = 0$, the whole equator is a critical point, so it is not a Morse function by Corollary 4.10. & $\varphi = \pi/2, -\pi/2$ it is maxima these are the points we get from the 2-dimensional version of the Example 4.14.

Example 4.16 (The height function on $\mathbb{T}^2$).

If we parameterize Torus, we get (choosing $1 > \epsilon > 0$):

$$\Phi(\theta, \phi) = (sin(\theta - \epsilon), (2 + cos(\theta - \epsilon))sin(\phi - \epsilon), (2 + cos(\theta - \epsilon))cos(\phi - \epsilon))\,; \theta \in (0, 2\pi), \phi \in (0, 2\pi)$$



$$(f \circ \Phi)(\theta, \phi) = (2 + cos(\theta - \epsilon))cos(\phi - \epsilon)$$

$$\frac{\partial}{\partial \theta}(f \circ \Phi)(\theta, \phi) = -sin(\theta - \epsilon)cos(\phi - \epsilon)$$

$$\frac{\partial}{\partial \phi}(f \circ \Phi)(\theta, \phi) = -(2 + cos(\theta - \epsilon))sin(\phi - \epsilon)$$

So the critical points we get by taking both the partial derivatives 0 are $(\epsilon, \epsilon), (\pi + \epsilon, \epsilon), (\pi + \epsilon, \pi + \epsilon), (\epsilon, \pi + \epsilon)$

$$H_p(f) = \begin{pmatrix} -cos(\theta - \epsilon)cos(\phi - \epsilon) & sin(\theta - \epsilon)sin(\phi - \epsilon) \\ sin(\theta - \epsilon)sin(\phi - \epsilon) & -(2 + cos(\theta - \epsilon))cos(\phi - \epsilon) \end{pmatrix}|_{(\epsilon,\epsilon)} = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$$

So index of p is 2.

$$H_q(f) = \begin{pmatrix} -cos(\theta - \epsilon)cos(\phi - \epsilon) & sin(\theta - \epsilon)sin(\phi - \epsilon) \\ sin(\theta - \epsilon)sin(\phi - \epsilon) & -(2 + cos(\theta - \epsilon))cos(\phi - \epsilon) \end{pmatrix}|_{(\pi+\epsilon,\epsilon)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So index of q is 1.

$$H_r(f) = \begin{pmatrix} -cos(\theta - \epsilon)cos(\phi - \epsilon) & sin(\theta - \epsilon)sin(\phi - \epsilon) \\ sin(\theta - \epsilon)sin(\phi - \epsilon) & -(2 + cos(\theta - \epsilon))cos(\phi - \epsilon) \end{pmatrix}|_{(\pi+\epsilon,\pi+\epsilon)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

So index of r is 1.

$$H_s(f) = \begin{pmatrix} -cos(\theta - \epsilon)cos(\phi - \epsilon) & sin(\theta - \epsilon)sin(\phi - \epsilon) \\ sin(\theta - \epsilon)sin(\phi - \epsilon) & -(2 + cos(\theta - \epsilon))cos(\phi - \epsilon) \end{pmatrix}|_{(\epsilon,\pi+\epsilon)} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

So, the index of s is 0.

Example 4.17 (Bott's perfect Morse functions).

Consider

$$S^{2n+1} = \left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \left| \left| \sum_{k=0}^{n} \right| z_k \right|^2 = 1 \right\}$$

and define $f : S^{2n+1} \to \mathbb{R}$ by

$$f(z) = \sum_{k=1}^{n} k \left| z_k \right|^2$$

where $z = (z_0, \ldots, z_n)$. This function is invariant under the natural action of S^1 on S^{2n+1} given by $s \cdot z = (sz_0, sz_1, \ldots, sz_n)$. Hence, it descends to the quotient $S^{2n+1}/S^1 = \mathbb{C}P^n$. We will still denote by f the induced function $f : \mathbb{C}P^n \to \mathbb{R}$. The projective space $\mathbb{C}P^n$ is covered by the $n+1$ open sets

$$U_j = \{[z_0, \ldots, z_n] \mid z_j \neq 0\}$$

which are the domains of the charts $\phi_j : U_j \to \mathbb{R}^{2n}$ given by

$$\phi_j \left([z_0, \ldots, z_n]\right) = (x_0, \ldots, \widehat{x}_j, \ldots, x_n, y_0, \ldots, \widehat{y}_j, \ldots, y_n)$$

where $|z_j| \frac{z_k}{z_j} = x_k + iy_k$ and $\widehat{x_j}$ and $\widehat{y_j}$ denote deleted coordinates. Clearly we have $\left|z_k\right|^2 = x_k^2 + y_k^2$ for $k \neq j$ and

$$\left|z_j\right|^2 = 1 - \left(\sum_{k \neq j} \left|z_k\right|^2 \right) = 1 - \left(\sum_{k \neq j} x_k^2 + y_k^2 \right)$$

Example 4.18 (Morse function on RP^2 and generalization)**.**

Charts of $\mathbb{RP}^2$ are the equivalence class of charts of Example 4.14 as the northern hemisphere is in the same equivalence class as the southern hemisphere and so on, so it has $n+1$ charts

$$\mathbb{RP}^2 = S^2/\{\pm 1\} \quad x_1^2 + x_2^2 + x_2^2 = 1$$

$f : \mathbb{RP}^2 \to \mathbb{R} \quad f([x_1, x_2, x_3]) = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 \quad a_1 < a_2 < a_3$

$\pi : S^3 \to \mathbb{RP}^2 \quad \pi(x_1, x_2, x_3) = [x_1, x_2, x_3]$ Now, we can use the charts from Example 4.14 to calculate this. Case $U_1 : \left[\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right] = 0$

$$\Rightarrow \left[\frac{\partial}{\partial x_2} \left\{ a_1 \left(1 - x_2^2 - x_3^2\right) + a_2 x_2^2 + a_3 x_3^2 \right\}, \frac{\partial}{\partial x_3} \left\{ a_1 \left(1 - x_2^2 - x_3^2\right) + a_2 x^2 + a_3 x_3^2 \right\} \right] = 0$$
$$\Rightarrow \left[2x_2 \left(a_2 - a_1\right), 2x_3 \left(a_3 - a_1\right) \right] = 0$$

So $x_2 = 0 \quad \& \quad x_3 = 0$.

So critical point is $[1, 0, 0]$.

Index:- as $a_1 < a_2 < a_3$

$$M_p f = 2 \begin{pmatrix} a_2 - a_1 & \\ & a_3 - a_1 \end{pmatrix}$$

so the index is 0.

Case $U_2 : \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_3}\right] = 0$

$$\Rightarrow \left[\frac{\partial}{\partial x_1}\left\{a_2\left(1 - x_1^2 - x_3^2\right) + a_1 x_1^2 + a_3 x_3^2\right\}, \frac{\partial}{\partial x_3}\left\{a_2\left(1 - x_1^2 - x_3^2\right) + a_1 x_1^2 + a_3 x_3^2\right\}\right] = 0.$$

$$\Rightarrow \left[2x_1\left(a_1 - a_2\right), 2x_3\left(a_3 - a_2\right)\right] = 0$$

so $x_1 = 0 \quad \& \quad x_3 = 0$.

so critical point is $[0, 1, 0]$.

Index:- as $a_1 < a_2 < a_3$

$$M_p f = 2 \begin{pmatrix} a_1 - a_2 & \\ & a_3 - a_2 \end{pmatrix}$$

so index is 1.

Case $U_3 : \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right] = 0$

$$\Rightarrow \left[\frac{\partial}{\partial x_1}\left\{a_3\left(1 - x_1^2 - x_2^2\right) + a_1 x_1^2 + a_2 x_2^2\right\}, \frac{\partial}{\partial x_2}\left\{a_3\left(1 - x_1^2 - x_2^2\right) + a_1 x_1^2 + a_2 x_2^2\right\}\right] = 0$$

$$\Rightarrow \left[2x_1\left(a_1 - a_3\right), 2x_2\left(a_2 - a_3\right)\right] = 0$$

so $x_1 = 0 \quad \& \quad x_2 = 0$ So critical point is $[0, 0, 1]$.

Index:- as $a_1 < a_2 < a_3$

$$M_p f = 2 \begin{pmatrix} a_1 - a_3 & \\ & a_2 - a_3 \end{pmatrix}$$

so index is 2.

For $\mathbb{RP}^n$ Critical points are $[1, \ldots, 0], [0, 1, \ldots 0], \ldots, [0, \ldots 0, 1]$ with indices 0,1,2,3,...,n+1

for U_j

$$M_p f = 2 \begin{pmatrix} a_1 - a_j & & \\ & a_i - a_j & \\ & & a_{n+1} - a_j \end{pmatrix}_{(n)\times(n)}$$

$i \neq j \quad i \in [n]$

Example 4.19 (Morse function on $\mathrm{SL}(2,\mathbb{R})$).

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mid \quad x_{11}x_{22} - x_{12}x_{21} = 1 \right\} U_{i,j} = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mid \quad x_{ij} \neq 0 \right\}$$

$$\phi_{i,j}\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right) = (x_{11}, \cdots, \hat{x}_{ij}, \cdots, x_{22})$$

So the basis of T_pU_1 is $\left\{ \frac{\partial}{\partial x_{11}}, \cdots, \frac{\hat{\partial}}{\partial x_{12}}, \cdots \frac{\partial}{\partial x_{22}} \right\}$

Now our function is $cx_{11} + dx_{22} \quad 0 < c < d$

Case $U_{1,2}$: $-$

$$\left(\frac{\partial}{\partial x_{11}} (cx_{11} + dx_{22}), \frac{\partial}{\partial x_{21}} (cx_{11} + dx_{22}), \frac{\partial}{\partial x_{22}} (cx_{11} + dx_{22}) \right) = 0 \Rightarrow (c, 0, d) = 0$$

as $0 < c < d$ so there is no critical point in $U_{1,2}$.

Case $U_{2,1}$: $-$ Similar to Case $U_{1,2}$.

Case $U_{1,1}$: $-$

$$\left[\frac{\partial}{\partial x_{12}} c \left(\frac{1 + x_{12}x_{21}}{x_{22}} + dx_{22} \right) \frac{\partial}{\partial x_{21}} c \left(\frac{1 + x_{12}x_{21}}{x_{22}} \right) + dx_{22}, \frac{\partial}{\partial x_{22}} c \left(\frac{1 + x_{12}x_{22}}{x_{22}} \right) + dx_{22} \right] = 0$$

$$\Rightarrow \left[c\frac{x_{21}}{x_{22}}, c\frac{x_{12}}{x_{22}}, d - \frac{c\,(1 + x_{12}x_{21})}{x_{22}^2} \right] = 0$$

So $x_{21} = 0, x_{12} = 0 \quad \&$

$$dx_{22}^2 - c\,(1 + x_{12}x_{21}) = 0$$

$$\Rightarrow dx_{22}^2 - c = 0$$

$$\Rightarrow x_{22} = \pm\sqrt{\frac{c}{d}}$$

so critical points are $p_1 = \begin{pmatrix} \sqrt{\frac{d}{c}} & 0 \\ 0 & \sqrt{\frac{c}{d}} \end{pmatrix}, p_2 = \begin{pmatrix} -\sqrt{\frac{d}{c}} & 0 \\ 0 & -\sqrt{\frac{c}{d}} \end{pmatrix}$ Index

$$M_{p_1}f = \begin{pmatrix} 0 & \frac{c}{x_{22}} & -\frac{cx_{11}}{x_{22}^2} \\ \frac{c}{x_{22}} & 0 & -\frac{cx_{12}}{x_{22}^2} \\ -\frac{cx_{21}}{x_{22}} & -\frac{cx_{12}}{x_{22}^2} & \frac{c(1+x_{12}x_{22}).}{x_{22}^3} \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\frac{d}{c}} & 0 \\ \sqrt{\frac{d}{c}} & 0 & 0 \\ 0 & 0 & \frac{2d\sqrt{d}}{\sqrt{c}} \end{pmatrix}$$

Now, we need to find the eigenvalues.

Eigenvalues

$$\begin{vmatrix} -\lambda & \sqrt{\frac{d}{c}} & 0 \\ \sqrt{\frac{d}{c}} & -\lambda & 0 \\ 0 & 0 & \frac{2d\sqrt{d}}{\sqrt{c}} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda\left[-\lambda\left(\frac{2d\sqrt{d}}{\sqrt{c}} - \lambda\right)\right] - \sqrt{\frac{d}{c}}\left[\sqrt{\frac{d}{c}}\left(\frac{2d\sqrt{\alpha}}{\sqrt{c}} - \lambda\right)\right] = 0$$

$$\Rightarrow -\lambda^3 + \frac{2d\sqrt{d}}{\sqrt{c}}\lambda^2 + \frac{d}{c}\lambda - \frac{2d}{c}\sqrt{\frac{d}{c}} = 0$$

$$\Rightarrow -\lambda^2\left[\lambda - \frac{2d\sqrt{d}}{\sqrt{c}}\right] + \frac{d}{c}\left[\lambda - \frac{2d\sqrt{\alpha}}{\sqrt{c}}\right] = 0$$

$$\Rightarrow \left[-\lambda^2 + \frac{d}{c}\right]\left[\lambda - \frac{2\lambda\sqrt{d}}{\sqrt{c}}\right] = 0$$

$$\text{so } \lambda = \pm\sqrt{\frac{d}{c}}, \frac{2d\sqrt{d}}{\sqrt{c}} = 0$$

So index is 1.

Similarly we can show $M_{p_2}f = \begin{pmatrix} 0 & -\sqrt{\frac{d}{c}} & 0 \\ -\sqrt{d/2} & 0 & 0 \\ 0 & 0 & -\frac{2d\sqrt{d}}{\sqrt{c}} \end{pmatrix}$ has eigenvalues $\lambda = \mp\sqrt{\frac{d}{c}}$

So index is 2.

Similarly critical points on $U_{2,2}$ is $p_3 = \begin{pmatrix} \sqrt{\frac{d}{c}} & 0 \\ 0 & \sqrt{\frac{c}{d}} \end{pmatrix}, p_4 = \begin{pmatrix} -\sqrt{\frac{d}{c}} & 0 \\ 0 & -\sqrt{\frac{c}{d}} \end{pmatrix}$

This two are in fact $p_1 \& p_2$

So f on $SL(2, R)$ has two critical points.

Example 4.20 (Morse function on Klein's Bottle)**.**

φ is the coordinate chart on Torus and P is the covering map (double cover) of Klein's bottle, and $f([z, w]) = Im(z)^2 + Im(w)^2$ is the Morse function on Klein's bottle. As Torus can be covered using four charts, Klein's bottle can be covered using two charts. Now for $x \in \left[0, \frac{1}{2}\right]$

$$f\left(P\left(\varphi^{-1}(x, y)\right)\right) = \sin^2(4\pi x) + \sin^2(2\pi y)$$

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] f\left(P\left(\varphi^{-1}(x, y)\right)\right) = 0$$

$$\Rightarrow [8\pi \sin(4\pi x)(\cos 4\pi x), 4\pi(\sin 2\pi y)\cos(2\pi y)] = 0$$

$$\Rightarrow [8\pi \sin(8\pi x), 4\pi \sin(4\pi y)] = 0$$

So $8\pi x = 0, \pi \Rightarrow x = 0, \frac{1}{8}$

$4\pi y = 0, \pi \Rightarrow y = 0, \frac{1}{4}$

for $x \in \left[\frac{1}{2}, 1\right]$

$$f\left(P\left(\varphi^{-1}(x,y)\right)\right) = \sin^2(2\pi(2x-1)) + \sin^2(2\pi(1-y))$$

Similarly

$$[8\pi \sin(4\pi(2x-1)), -4\pi\sin(4\pi(1-y))] = 0$$

So

$$4\pi(2x-1) = 0, \pi \Rightarrow x = \frac{1}{2}, \frac{5}{8}$$
$$4\pi(1-y) = 0, \pi \Rightarrow y = 1, \frac{3}{4}$$

So critical points are $\left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{5}{8}, 1\right), \left(\frac{5}{8}, \frac{3}{4}\right), (0,0), \left(0, \frac{1}{4}\right), \left(\frac{1}{8}, 0\right), \left(\frac{1}{8}, \frac{1}{4}\right)$ So actually the points are $(1,1), (1,i), (i,1), (i,i)$

Index:-

$$H_p(f) = \begin{pmatrix} 64\pi^2(\cos 8\pi x) & 0 \\ 0 & 16\pi^2\cos(4\pi y) \end{pmatrix}$$

$$H_{(1,1)}f = \begin{pmatrix} 64\pi^2 & 0 \\ 0 & 16\pi^2 \end{pmatrix} \text{ Index } -0$$
$$H_{(1,i)}f \begin{pmatrix} 64\pi^2 & 0 \\ 0 & -16\pi^2 \end{pmatrix} \text{ Index } -1$$
$$H_{(i,1)}f = \begin{pmatrix} -64\pi^2 & 0 \\ 0 & 16\pi^2 \end{pmatrix} \text{ Index } -1$$
$$H_{(i,i)}f = \begin{pmatrix} -64\pi^2 & 0 \\ 0 & -16\pi^2 \end{pmatrix} \text{ Index } -2$$

Example 4.21 (Morse function on $S^1 \times S^2$).

We are taking the function to be $f : S^1 \times S^2 \to \mathbb{R} \quad f(x,y,z,u,v) = z + v$ we take the charts to be the Cartesian product of the charts of S^1 and the charts of S^2. So we need only to look up the charts (Northern hemisphere $\times$ Southern hemisphere of S^2, Northern hemisphere $\times$ Northern hemisphere of S^2, Southern hemisphere $\times$ Southern hemisphere of S^2, Southern hemisphere $\times$ Northern hemisphere of S^2).

Charts which contain critical points are

$$U_1 = \{(x,y,z,u,v) \mid z>0, v>0\}, \varphi_1^{-1}(x,y,u) = \left(x, y, \sqrt{1-x^2-y^2}, u, \sqrt{1-u^2}\right)$$
$$U_2 = \{(x,y,z,u,v) \mid z>0, v<0\}, \varphi_2^{-1}(x,y,u) = \left(x, y, \sqrt{1-x^2-y^2}, u, \sqrt{1-u^2}\right)$$
$$U_3 = \{(x,y,z,u,v) \mid z<0, v>0\}, \varphi_3^{-1}(x,y,u) = \left(x, y, \sqrt{1-x^2-y^2}, u, \sqrt{1-u^2}\right)$$
$$U_4 = \{(x,y,z,u,v) \mid z<0, v<0\}, \varphi_4^{-1}(x,y,u) = \left(x, y, \sqrt{1-x^2-y^2}, u, \sqrt{1-u^2}\right)$$

So $f \circ \varphi_i^{-1}(x,y,u) = \sqrt{1-x^2-y^2} + \sqrt{1-u^2} \quad i \in [4]$.

Now $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial u}\right]\left(\sqrt{1-x^2-y^2} + \sqrt{1-u^2}\right) = 0$

$$\Rightarrow \left(\frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}}, \frac{-u}{\sqrt{1-u^2}}\right) = 0$$
$$\Rightarrow x = 0 \quad \& y = 0 \quad \& u = 0$$

so critical points are $(0,0,1,0,1), (0,0,-1,0,1), (0,0,1,0,-1), (0,0,-1,0,-1)$

$$M_p f = \begin{pmatrix} -\frac{1}{\sqrt{1-x^2-y^2}} - \frac{x^2}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{-xy}{(1-x^2-y^2)^{\frac{3}{2}}} & 0 \\ \frac{-xy}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{-1}{\sqrt{1-x^2-y^2}} - \frac{-y^2}{(1-x^2-y^2)^{\frac{3}{2}}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{1-u^2}} - \frac{u^2}{(1-u^2)^{\frac{3}{2}}} \end{pmatrix}$$

$$M_{(0,0,1,0,1)} f = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

so the index is 3

$$M_{(0,0,-1,0,-1)} f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the index is 0

$$M_{(0,0,-1,0,1)} f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So the index is 1

$$M_{(0,0,1,0,-1)} f = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the index is 2 This way, we can get a generalized version of this product space of S^n, which we have mentioned below.

So the summary of the calculated examples are:-

Manifolds	Morse Function	Critical Points$_{\text{index}}$
S^n	$f(x_0,...,x_n)=x_n$	$(0,..,-1_{n+1})_0,(0,..,1_{n+1})_n$
$\mathbb{C}P^n$	$f(z_0,...,z_n)=\frac{\sum_{k=1}^n k\lvert z_k\rvert^2}{\sum_{k=1}^n \lvert z_k\rvert^2}$	$[0,...,1_j,..,0_{n+1}]_{2(j-1)}$ $j\in[n+1]$
$\mathbb{R}P^n$	$f(x_1,...,x_{n+1})=\frac{\sum_{k=1}^n k(x_k)^2}{\sum_{k=1}^n (x_k)^2}$	$[0,...,1_j,..,0_{n+1}]_{j-1}$ $j\in[n+1]$
$S^n\times S^m$	$f(x_0,...,x_{n+m+1})=$ x_n+x_{n+m+1}	$(0,...1_n,..,-1_{n+m+1})_m$, $(0,...,1_n,..,1_{n+m+1})_{m+n}$, $(0,...-1_n,..,-1_{n+m+1})_0$, $(0,...,-1_n,..,1_{n+m+1})_m$
K^2(Klein's Bottle)	$f(z,w)=Im(z)^2+Im(w)^2$	$(1,1)_0,(1,i)_1,(i,1)_1,(i,i)_2$
$\mathbb{T}^2$(vertical Torus)	$f(x,y,z)=z$	$(0,0,3)_2$,$(0,0,1)_1$, $(0,0,-1)_1$,$(0,0,-3)_0$
SL$(2,\mathbb{R})$	$f\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix}\right)=$ $cx_{11}+dx_{22}$	$\begin{pmatrix}\sqrt{\frac{d}{c}} & 0 \\ 0 & \sqrt{\frac{c}{d}}\end{pmatrix}_1$, $\begin{pmatrix}-\sqrt{\frac{d}{c}} & 0 \\ 0 & -\sqrt{\frac{c}{d}}\end{pmatrix}_2$

4.5 Gradient Flow

Let's delve into key points regarding Riemannian Metric:

1. **Symmetric Bilinear Inner Product:-** The Riemannian Metric, denoted as $\langle\cdot,\cdot\rangle: V\times V\to\mathbb{R}$, is characterized by being a symmetric bilinear function, meaning $\langle v,w\rangle=\langle w,v\rangle$ for all vectors $v,w\in V$.

2. **Positive Definiteness:-** If the inner product $\langle v,v\rangle$ is strictly greater than zero for all nonzero vectors $v\in V$, the metric is termed positive definite.

3. **Non-Degeneracy:-** A metric is considered non-degenerate if for every nonzero vector v, there exists another nonzero vector ω such that their inner product $\langle v,\omega\rangle$ is nonzero.

4. **Tangent Bundle:-** The tangent bundle TM over a smooth manifold M is a smooth vector bundle. Its fiber at each point $x\in M$ is the tangent space T_xM.

5. **Vector Bundle Structure:-** A vector bundle of rank k is a tuple $(M,V,\pi,\cdot,+)$ where:

- M and V are smooth manifolds,

- $\pi : V \to M$ is a smooth map,
- $\cdot : \mathbb{R} \times V \to V$ and $+ : V \times_M V \longrightarrow V$ are maps satisfying certain conditions, ensuring compatibility with the manifold structure.
- Locally, the bundle resembles the product manifold $U \times \mathbb{R}^k$ via diffeomorphisms.

  1. $\pi_1 \circ h = \pi$ on $V|_u$ &
  2. The map $h|_{V_x} : V_x \to x \times \mathbb{R}^k$ is an isomorphism of vector spaces for all $x \in U$.

$$\begin{array}{ccc} V_u & \xrightarrow{\ h\ } & U \times \mathbb{R}^k \\ {\scriptstyle\pi}\big\downarrow & \swarrow{\scriptstyle \pi_1} & \\ x \in U \overset{\circ}{\subseteq} M & & \end{array}$$

$\{U_i, h_i\}$ is a local Trivialization.

6. **Riemannian Metric Function:-** A Riemannian metric g on T_xM is a smooth function assigning to each point $x \in M$ a positive definite inner product $\langle \cdot, \rangle_x$ on the tangent space T_xM.

7. **Isomorphism with Cotangent Bundle:-** A non-degenerate inner product on a vector space induces an isomorphism between the vector space and its dual. Similarly, a Riemannian metric g on a smooth manifold M establishes an isomorphism $\tilde{g} : T_*M \to T^*M$ between the tangent and cotangent bundles.

8. **Action on Vector Fields:-** For any vector field w, $\tilde{g}(w)$ is a unique 1-form defined such that for any vector field V,

$$\tilde{g}(w)(v) = g(w, v).$$

Definition 4.22.

If $f : M \to \mathbb{R}$ is a smooth function on a Riemannian manifold (M, g), then the gradient vector field of f with respect to the metric g is the unique smooth vector field ∇f such that

$$g(\nabla f, V) = df(V) = V \cdot f$$

for all smooth vector fields V on M, i.e. $\nabla f = \tilde{g}^{-1}(df)$. In particular,

$$(\nabla f) \cdot f = g(\nabla f, \nabla f) = \|\nabla f\|^2$$

Let $\varphi_t : M \to M$ be the local 1-parameter group of diffeomorphisms generated by $-\nabla f$ (the negative gradient), i.e.

$$\frac{d}{dt}\varphi_t(x) = -(\nabla f)(\varphi_t(x))$$
$$\varphi_0(x) = x.$$

The integral curve $\gamma_x : (a,b) \to M$ given by

$$\gamma_x(t) = \varphi_t(x)$$

is called a gradient flow line.

Now, we will see some important results from this definition.

Proposition 4.23.

Every smooth function $f : M \to \mathbb{R}$ on a finite-dimensional smooth Riemannian manifold (M,g) decreases along its gradient flow lines.

Proof.

$$\begin{aligned}\frac{d}{dt}f(\gamma_x(t)) &= \frac{d}{dt}(f \circ \varphi_t(x)) \\ &= df_{\varphi_t(x)} \circ \frac{d}{dt}\varphi_t(x) \\ &= df_{\varphi_t(x)}(-(\nabla f)(\varphi_t(x))) \\ &= -\|(\nabla f)(\varphi_t(x))\|^2 \leq 0\end{aligned}$$

□

Proposition 4.24.

Let $f : M \to \mathbb{R}$ be a Morse function on a finite-dimensional compact smooth Riemannian manifold (M,g). Then every gradient flow line of f begins and ends at a critical point, i.e., for any $x \in M, \lim_{t\to+\infty}\gamma_x(t)$ and $\lim_{t\to-\infty}\gamma_x(t)$ exist, and they are both critical points of f.

Proof. We start by considering a smooth manifold M and a smooth function $f : M \to \mathbb{R}$ on M. Given $x \in M$, we define $\gamma_x(t)$ as the gradient flow line through x, governed by the equation:

$$\frac{d}{dt}\gamma_x(t) = -\nabla f(\gamma_x(t)),$$

where ∇f denotes the gradient vector field of f.

Now, due to the compactness of M, by Proposition 4.23, the gradient flow line $\gamma_x(t)$ is well-defined for all $t \in \mathbb{R}$. Furthermore, since f is continuous and M is compact, the image of $f \circ \gamma_x$, denoted as $(f \circ \gamma_x)(\mathbb{R})$, is bounded in $\mathbb{R}$.

Thus, according to Proposition 4.23, we have:

$$\lim_{t\to\pm\infty} \frac{d}{dt} f(\gamma_x(t)) = \lim_{t\to\pm\infty} -\|\nabla f(\varphi_t(x))\|^2 = 0.$$

Consider a sequence $t_n \in \mathbb{R}$ such that $\lim_{n\to\infty} t_n = -\infty$. Then, $\{\gamma_x(t_n)\}$ forms an infinite set of points in the compact manifold M, implying it has an accumulation point, denoted as q.

Since $\|\nabla f(\gamma_x(t_n))\| \to 0$ as $n \to \infty$, point q is a critical point of f. By Corollary 4.10, we can select a closed neighborhood U of q, where q is the only critical point in U.

If $\lim_{t\to-\infty} \gamma_x(t) \neq q$, then there exists an open neighborhood $V \subset U$ of q, and a sequence $\tilde{t}_n \in \mathbb{R}$ such that $\lim_{n\to\infty} \tilde{t}_n = -\infty$ and $\gamma_x(\tilde{t}_n) \in U - V$. Consequently, the sequence $\{\gamma_x(\tilde{t}_n)\}$ has an accumulation point in the compact set $U - V$, which, by the argument above, must be a critical point of f. This contradicts the choice of U.

Therefore, we conclude that $\lim_{t\to-\infty} \gamma_x(t) = q$. Similarly, a similar argument shows that $\lim_{t\to+\infty} \gamma_x(t) = p \in M$ for some critical point p. □

4.6 First Fundamental Theorem of Classical Morse Theory and Reeb's theorem

Theorem 4.25 (First Fundamental Theorem of Classical Morse Theory).
Let $f : M \longrightarrow \mathbb{R}$ be a smooth function on a finite-dimensional smooth manifold with boundaries. For all $a \in \mathbb{R}$, let

$$M^a = f^{-1}((-\infty, a]) = \{x \in M | f(x) \leq a\}$$

Let $a < b$ and assume that $f^{-1}([a,b])$ is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b, and M^a is a deformation retract of M^b.Moreover, there is a smooth diffeomorphism $F : f^{-1}(a) \times [a,b] \longrightarrow f^{-1}([a,b])$ such that the diagram

$$\begin{array}{ccc} f^{-1}(a) \times [a,b] & \xrightarrow{F} & f^{-1}([a,b]) \\ & \searrow^{\pi_2} & \downarrow f \\ & & [a,b] \end{array}$$

commutes. In particular, all the level surfaces of f between a and b are diffeomorphic.

Proof. We start with $W \subset M$, the open set comprising non-critical points of the function f, and we assume a Riemannian metric g on M. Our objective is to utilize this metric to construct a vector field, enabling the formulation of a suitable flow crucial for our desired diffeomorphism.

We define $X = \frac{1}{\|\nabla f\|^2} \nabla f$, a vector field on W, and let $\gamma : I \to M$ be a maximal integral curve of X. From the definitions, it follows that:

$$\begin{aligned} \frac{d}{dt} f(\gamma(t)) &= df(\gamma(t)) \cdot \gamma'(t) \\ &= df(\gamma(t)) \cdot X(\gamma(t)) \\ &= g(\nabla f(\gamma(t)), X(\gamma(t))) \\ &= \frac{1}{\|\nabla f(\gamma(t))\|^2} g(\nabla f(\gamma(t)), \nabla f(\gamma(t))) \\ &= 1. \end{aligned}$$

Thus, assuming $0 \in I$, we deduce that $f(\gamma(t)) = f(\gamma(0)) + t$. Now, let $K = f^{-1}([a,b]) \subset W$, which, by hypothesis, is compact. We initiate the integral curve at $\gamma(0) \in f^{-1}(a)$. Two scenarios emerge:

1. If $\gamma(t) \in K$ for all $t > 0$, then the solution remains confined within a compact set, ensuring its definition for all positive time, i.e., $[0, +\infty) \subset I$. Specifically, the solution exists within $[0, b-a]$.

2. If there exists $s \in I, s > 0$, such that $\gamma(s) \notin K$, then $b < f(\gamma(s)) = f(\gamma(0)) + s = a + s$, implying $s > b - a$. Consequently, $[0, b-a] \subset I$.

Moreover, we extend X to the entire manifold without sacrificing the established properties. Utilizing a bump function $\psi : M \to \mathbb{R}$ with certain properties:

1. $\psi|_K = 1$.

2. Its support is contained in W.

We construct the vector field Y on the entire manifold as follows:

$$Y = \begin{cases} \psi(x)X(x) & \text{if } x \in W \\ 0 & \text{otherwise} \end{cases}.$$

This extension retains the characteristics of X within K. Let φ^t represent the flow of Y. If necessary, we adjust the support of ψ to ensure the flow is defined up to $b-a$ time, thus φ^{b-a} serves as a well-defined diffeomorphism on M mapping M_a onto M_b.

Moving on to the proof that M_a is a deformation retract of M_b, we introduce the collection of maps:

$$r : M_b \times [0,1] \to M_b$$

defined by:

$$r(x,t) = \begin{cases} x & \text{if } f(x) \leq a \\ \varphi^{t(a-f(x))}(x) & \text{if } a \leq f(x) \leq b \end{cases}.$$

This definition yields the desired retraction.

For the second part of the theorem, we define $F : f^{-1}(a) \times [a,b] \to f^{-1}([a,b])$ by $F(x,t) = \varphi^{t-a}(x)$. It's observed that for any $x \in f^{-1}([a,b])$, $\varphi^{a-f(x)}(x) \in f^{-1}(a)$. Thus, $F(\varphi^{a-f(x)}(x), f(x)) = x$, and F is surjective. Since f increases along its gradient flow lines, F also increases along the flow lines of X. Hence, F is injective. Moreover, F is an immersion since gradient lines are transverse to level sets. Thus, F is a diffeomorphism. □

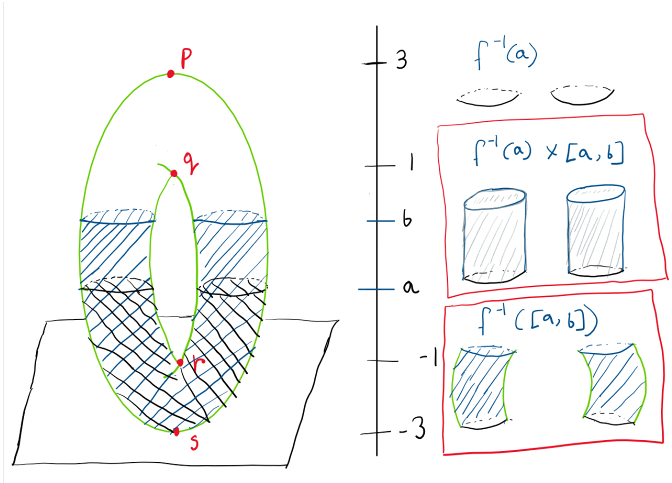
Corollary 4.26.

Let M be a compact, smooth manifold with boundary $\partial M = A \sqcup B, i.e., A \cap B = \phi$. Suppose there exists a C^∞ function $f : M \longrightarrow [0,1]$ with no critical points such that $f(A) = 0$ and $f(B) = 1$. Then M is diffeomorphic to $A \times [0,1] \approx B \times [0,1]$.

Proof. This follows from the second part of the First Fundamental Theorem of Morse Theory since $f^{-1}([0,1]) = M$ □

Remark 4.27.
- $f^{-1}([a,b])$ is compact and contains no critical point. So from the figure, we can easily verify that M^a is the region colored with black and M^b is the region colored blue.
- $M^a \subseteq M^b$
- Now the First Fundamental Theorem of Morse Theory says that M^b is diffeomorphic to M^a and M^a is the deformation retract of M^b.
- Also it says that all the level surfaces of f between a and b are diffeomorphic.

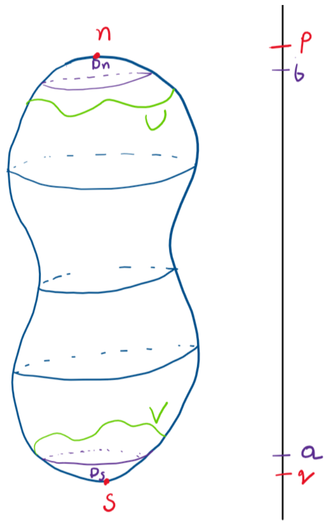
Here $f^{-1}(a)$ is a disjoint union of two circles. In this case, now, we can easily see that $f^{-1}(a) \times [a,b]$ is a disjoint union of two cylinders and $f^{-1}([a,b])$ is also a disjoint union of two cylinders. So, we can vaguely see that there is a diffeomorphism for which the diagram commutes. So, the final statement is also true.



Theorem 4.28 (Reeb's Theorem)**.**
If M is a compact smooth manifold without boundary of dimension m admitting a Morse function $f : M \longrightarrow \mathbb{R}$ with only two critical points, then M is homeomorphic to the m-sphere S^m.

Proof. M has only two critical points and is compact so that it will attain maximum and minimumminimum at some pointpoint.

Let n be maximum and s be minimum, and the critical values be $f(n) = p$ and $f(s) = q$.



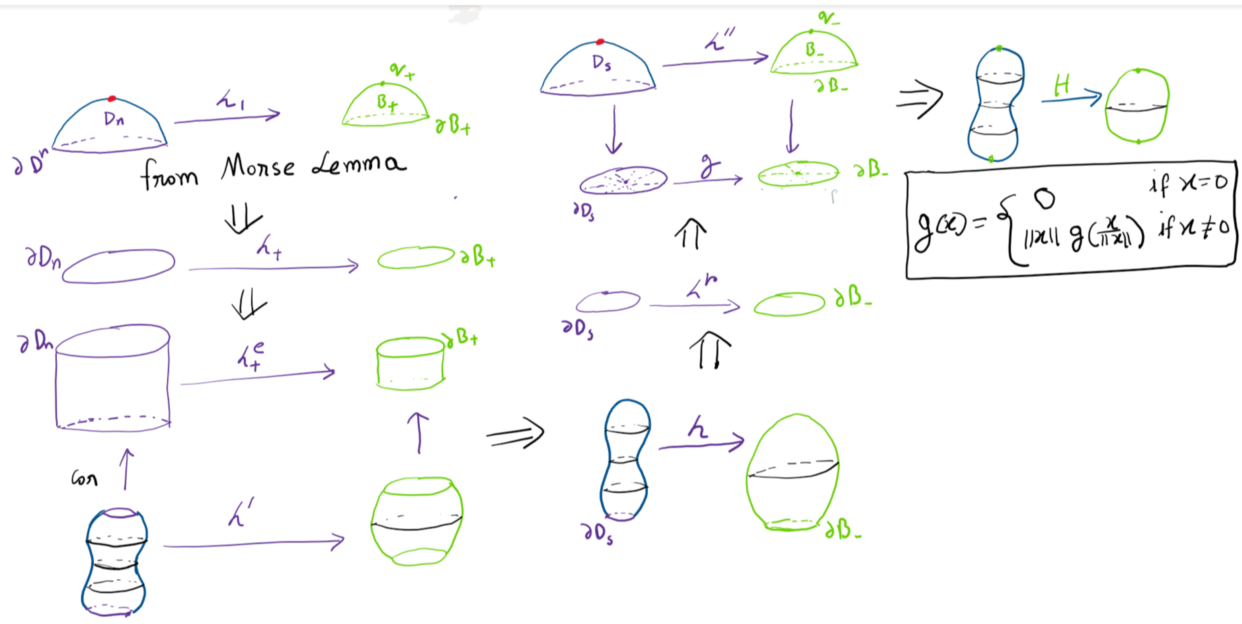
Now by **Morse Lemma** There exists an open neighbourhood U and coordinates $(u_1, ..., u_m)$ around n on which $f(u) = p - u_1^2 - u_2^2 - ... - u_m^2$ and an open neighbourhood V and coordinates $(v_1, ...v_m)$ around s on which $f(v) = q + v_1^2 + v_2^2 + ... + v_m^2$.

We can get a disk D_n around n inside U such that $f(\partial D_n) = b$ and we can write $D_n = \{(u_1, u_2, ..., u_m) | u_1^2 + u_2^2 + ... + u_m^2 \leq p - b\}$ similarly we get D_s around s such that $f(\partial D_s) = a$. They both are diffeomorphic to D^m.

Now by Corollary 4.26 $f^{-1}([a, b])$ is diffeomorphic to $S^{m-1} \times [0, 1]$.

Let $B_\pm$ be a disjoint neighborhood of $q_\pm$, i.e., north and south poles of S^m. So $S^m = B_+ \cup B_- \cup C$, Where $C = S^m - (B_- \cup B_+)^\circ \approx S^{m-1} \times [0, 1]$.

We can first consider the diffeomorphism from D_n to B_+ from Morse Lemma. Then, after restricting it to the boundary, if we extend that to the cylinder we, we, wecylinder, we get an extension of the previous map again again. Again, after restricting the new map on the boundary, if we take an extension radially $g(x) = \begin{cases} 0 & x = 0 \\ ||x|| \dot{h}^r(\frac{x}{||x||}) & x \neq 0 \end{cases}$ Where h^r is a restriction of homeomorphism $h : D^n \cup f^{-1}([a, b]) \longrightarrow B_+ \cup C$ onto ∂D_s.Then, our map extends to a homeomorphism from M to S^m. We can see this using another figure.



□

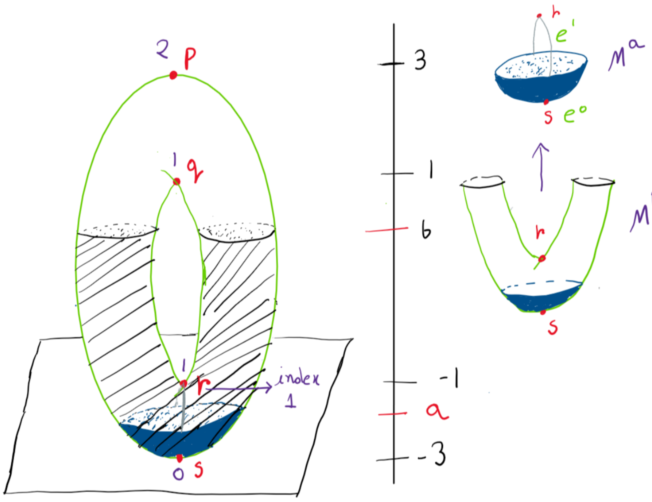
4.7 Second Fundamental Theorem of Morse Theory

Theorem 4.29 (Second Fundamental Theorem of Morse Theory).
Let $f : M \longrightarrow \mathbf{R}$ be a smooth function. Suppose that for $a < b$, $f^{-1}([a,b])$ is compact and inside $f^{-1}([a,b])$ there is exactly one critical point. Assume that this critical point is non-degenerate and of index k. Then M^b has the homotopy type of M^a with one k-cell attached. In fact, there exists a set $e^k \subseteq M^b$ diffeomorphic to the closed k-disk $D_k = \{x \in \mathbf{R}^k| \ ||x|| < 1\}$ such that $M^a \cup e^k \subseteq M^b$ is a deformation retract of M^b.

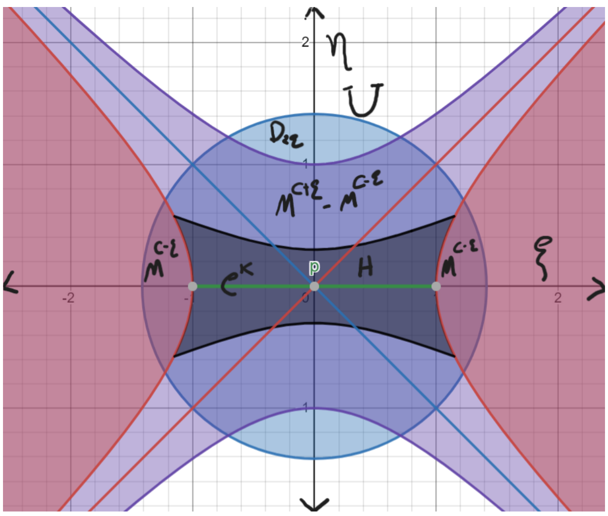
Proof. Let $p \in (f^{-1}([a,b]))^\circ$ be the unique critical point of index k and let $c = f(p)$. So from **Morse Lemma** we get a smooth chart $\phi : U \longrightarrow \mathbb{R}^m$ around p where $\phi(p) = 0$ and if $\phi(x) = (x_1, x_2, ..., x_m)$, then $(f \circ \phi^{-1})(x_1, x_2, ..., x_m) = c - x_1^2 - x_2^2 - ... - x_k^2 + x_{k+1}^2 + x_{k+2}^2 + ... + x_m^2$.
Choose $0 < \epsilon < 1$ small enough so that $f([c-\epsilon, c+\epsilon])$ is compact and $\phi(U)$ contains the closed ball of radius 2ϵ, i.e.

$$\{(x_1, ..., x_m) \in \phi(U)|x_1^2 + x_2^2 + ... + x_m^2 \leq 2\epsilon\}$$

By **First Fundamental Theorem of Classical Morse Theory**, M^a and $M^{c-\epsilon}$ have the same homotopy type, and the same is true for M^b and $M^{c+\epsilon}$. Thus, to prove the theorem, we need only show that $M^{c+\epsilon}$ has the same homotopy type as $M^{c-\epsilon} \cup e^k$. We may do this working inside $f^{-1}([c-\epsilon, c+\epsilon]) \cap U$ since **First Fundamental Theorem of Classical Morse Theory** implies that the homotopy type of $M^t - U$ is the same for all $a \leq t \leq b$.



Denoting local coordinates of $x \in U$ by $\phi(x) = (x_1, x_2, ..., x_m)$ and denoting $\xi^2 = x_1^2 + x_2^2 + ... + x_k^2$, $\eta^2 = x_{k+1}^2 + x_{k+2}^2 + ... + x_m^2$, We get



$$f(x_1, x_2, ..., x_m) = c - \xi^2 + \eta^2$$

$$M^{c-\epsilon} \cap U = \{x \in U | \eta^2 - \xi^2 \leq \epsilon\}$$

$$M^{c+\epsilon} \cap U = \{x \in U | \xi^2 - \eta^2 \geq \epsilon\}$$

$$e^k = \{x \in U | \xi^2 \leq \epsilon \text{ and } \eta^2 = 0\}$$

Now let $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function such that $\mu(0) \geq \epsilon$, $\mu(r) = 0$ for $r \geq 2\epsilon$, and $-1 < \mu'(r) \leq 0$. Define a new function $F : M \longrightarrow \mathbb{R}$ by: $F(x) = \begin{cases} f(x) & x \notin U \\ f(x) - \mu(\xi^2 + 2\eta^2) & x \in U \end{cases}$

We can observe some points here:-

(1) $F(p) = f(p) - \mu(0) = c - \mu(0) < c - \epsilon$.

(2) Notice that F and f coincide outside of the region $E := \{\xi^2 + \eta^2 \leq 2\epsilon\}$ so it sufficies to show that $F^{-1}(-\infty, c+\epsilon) \cap E = M^{c+\epsilon} \cap E$. But notice that if $q \in E$,

$$F(q) \leq f(q) = c - \xi^2 + \eta^2 \leq c + \frac{1}{2}\xi^2 + \eta^2 = c + \frac{1}{2}(\xi^2 + 2\eta^2) \leq c + \epsilon$$

so $E \subseteq F^{-1}(-\infty, c+\epsilon)$ and $F^{-1}(-\infty, c+\epsilon) \subseteq M^{c+\epsilon}$ the other side is trivial so we can say that $F^{-1}(-\infty, c+\epsilon) = M^{c+\epsilon}$.

(3) F and f has same critical points as $DF = 0$ if $d\xi^2 = d\eta^2 = 0$ because $DF = \frac{\partial F}{\partial \xi^2} d\xi^2 + \frac{\partial F}{\partial \eta^2} d\eta^2$ and from the properties of μ we can say that $\frac{\partial F}{\partial \xi^2} = -1 - \mu^{'}(\xi^2 + 2\eta^2) < 0$, $\frac{\partial F}{\partial \eta^2} = 1 - 2\mu^{'}(\xi^2 + 2\eta^2) \geq 1$.

(4) $F^{-1}([c-\epsilon, c+\epsilon]) = F^{-1}((-\infty, c+\epsilon]) - F^{-1}((-\infty, c-\epsilon])$
$\subseteq M^{c+\epsilon} - F^{-1}((-\infty, c-\epsilon])$[by (1)] $\subseteq M^{c+\epsilon} - f^{-1}((-\infty, c-\epsilon]) \subseteq f^{-1}([c-\epsilon, c+\epsilon])$.

So from this we can say that $F^{-1}([c-\epsilon, c+\epsilon])$ has no critical point so by using **First Fundamental Theorem of Classical Morse Theory** we can say that $F^{-1}(-\infty, c-\epsilon)$ is a deformation retract of $F^{-1}(-\infty, c+\epsilon) = M^{c+\epsilon}$ and we can define:

$H = \overline{F^{-1}(-\infty, c-\epsilon) - M^{c-\epsilon}}$ In the variables (ξ^2, η^2) the disk e^k in M can be expressed as $e^k = \{q \in U \mid \xi(q) \leq \varepsilon \text{ and } \eta(q) = 0\}$. We claim that $e^k \subset H$. First of all, $e^k \subset F^{-1}((-\infty, c-\varepsilon])$. This can be seen because if $q \in e^k$,

$$F(q) = c - \xi^2(q) - \mu(\xi^2(q)) \leq c - \mu(0) \leq c - \varepsilon.$$

In the first inequality, we used the fact that $\xi \geq 0$ and that μ is a decreasing function. We used that $\mu(0) \geq \varepsilon$ in the second inequality.

On the other hand, $f(q) = c - \xi^2 \geq c - \varepsilon$, with an equality only when $\xi^2 = \varepsilon$, this means, at ∂e^k. Therefore, as we claimed, $e^k \subset H$.

Now we can construct a retraction of $M_{c-\varepsilon} \cup H$ onto $M_{c-\varepsilon} \cup e^k$. Let us call it r_t. Let r_t be the identity outside of U for all t, and separate $U \cap (M_{c-\varepsilon} \cup H)$ in three regions:

$$C_1 = \{q \mid \xi^2(q) \leq \varepsilon\},$$

$$C_2 = \{q \mid \varepsilon \leq \xi^2(q) \leq \eta^2(q) + \varepsilon\},$$

$$C_3 = \{q \mid \eta^2(q) + \varepsilon \leq \xi^2(q) \Leftrightarrow f(q) \leq c - \varepsilon\}.$$

We will construct r_t separately on these three regions and prove that it is the desired retraction.

- r_t on C_1. We define

  $$r_t\left(u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\right) = \left(u_1, \ldots, u_k, tu_{k+1}, \ldots, tu_n\right),$$

  or, equivalently, $r_t(\xi^2, \eta^2) = (\xi^2, t^2\eta^2)$. It is clear that r_1 is the identity and r_0 is a projection onto e^k. Moreover, $F\left(r_t(q)\right) \leq c - \varepsilon$, because $\frac{\partial F}{\partial \eta^2} > 0$.

- r_t on C_2. We define

  $$r_t\left(u_1, \ldots, u_n\right) = \left(u_1, \ldots, u_k, s_t u_{k+1}, \ldots, s_t u_n\right),$$

  or, as before, $r_t(\xi^2, \eta^2) = (\xi^2, s_t^2\eta^2)$. We define

  $$s_t = t + (1-t)\sqrt{\frac{\xi^2 - \varepsilon}{\eta^2}}.$$

  It is clear that r_1 is the identity. On the other hand, notice that

  $$f\left(r_0(q)\right) = f\left(\xi^2, s_0^2\eta^2\right) c - \xi^2 + s_0^2\eta^2 = c - \xi^2 + \xi^2 - \varepsilon = c - \varepsilon,$$

  so r_0 maps all of C_2 onto the boundary of $M_{c-\varepsilon}$.

- On C_3, we let $r_t = \mathrm{Id}$ for all t. When $\xi^2 - \varepsilon = \eta^2$, it coincides with the last definition.

We need to check that r_t is continuous. In particular, we need to check it when $\xi^{\rightarrow}\varepsilon$ and $\eta^2 \rightarrow 0$. First of all, notice that

- when $\xi^2 = \varepsilon, s_t = t$,

- when $\xi^2 - \varepsilon = \eta^2$, $s_t = 1$.

Thus, the only points where it is unclear if r_t is continuous are those such that $\xi^2 = \varepsilon$ and $\eta^2 = 0$. In particular, we are to check the continuity in the region C_2. In this case, however, we have that.

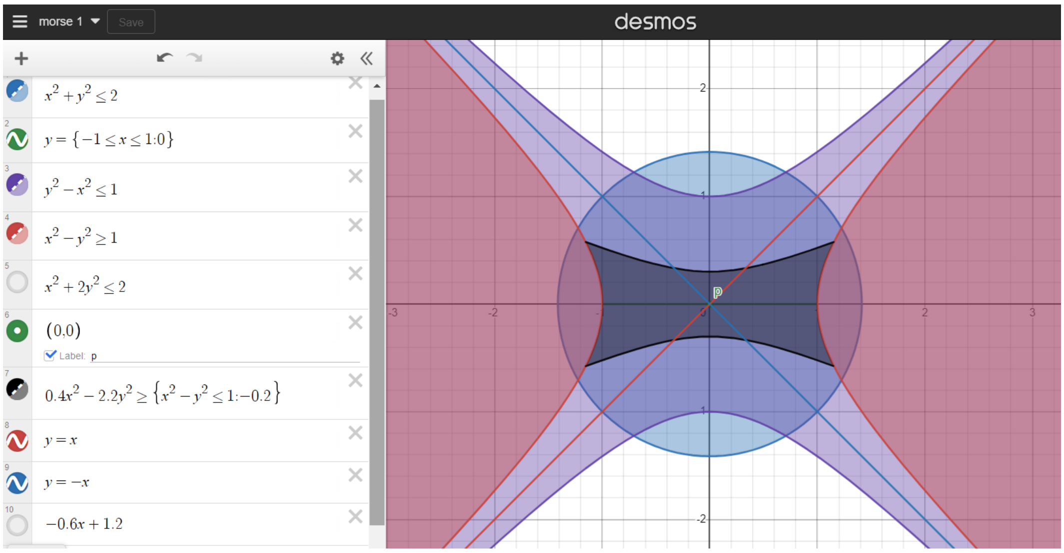
$$\xi^2 - \varepsilon \leq \eta^2 \Rightarrow 0 \leq \frac{\xi^2 - \varepsilon}{\eta^2} \leq 1,$$

so s_t stays bounded in the whole C_2. Moreover, for each $i > k$, the coordinate u_i is mapped as $u_i \mapsto s_t u_i$. In addition, $|u_i| \leq \eta^2$. Taking all of this into account, we deduce that

$$0 \leq |s_t u_i| \leq s_t \eta^2 \xrightarrow[\eta^2 \to 0, \xi^2 \to \varepsilon]{\longrightarrow} 0,$$

so, in particular, $s_t u_i \xrightarrow[\eta^2 \to 0, \xi^2 \to \varepsilon]{\longrightarrow} 0$, as we wanted to see. Thus, r_t is continuous, so it is a retraction from $M_{c-\varepsilon} \cup H$ onto $M_{c-\varepsilon} \cup e^k$. This concludes the proof $\square$

***We have taken an example of μ, which nearly follows the conditions to check how the proof works using desmos.

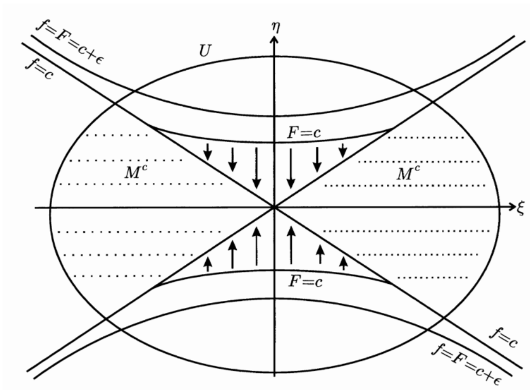


Remark 4.30.

If there are j critical points, $p_1, ..., p_j$ in the level set $f^{-1}(c)$ with indices $\lambda_1, ..., \lambda_j$ respectively, then a similar proof like the proof of **Second Fundamental Theorem of Classical Morse Theory** shows that for some collection of attaching maps $f_1, ..., f_j$.
$M^{c+\epsilon} \simeq M^{c-\epsilon} \cup_{f_1} D^{\lambda_1} \cup_{f_2} ... \cup_{f_j} D^{\lambda_j}$

Remark 4.31.

An easy modification of the above proof shows that M^c is a deformation retract of $M^{c+\varepsilon}$. Indeed, $F^{-1}((-\infty, c])$ is a deformation retract of $F^{-1}((-\infty, c+\varepsilon]) = M^{c+\varepsilon}$ by **First Fundamental Theorem of Morse Theory** , and M^c is a deformation retract of $F^{-1}((-\infty, c])$. The following diagram illustrates a deformation retraction of $F^{-1}((-\infty, c])$ to M^c.



Now, we will see the proof of two results, which will help us to reach our final and most important outcome of this chapter.

Lemma 4.32 (J.H.C. Whitehead).

Let X be a topological space, and suppose that $f_0 : S^{k-1} \to X$ and $f_1 : S^{k-1} \to X$ are homotopic. Then, the identity map of X extends to a homotopy equivalence

$$h : X \cup_{f_0} D^k \to X \cup_{f_1} D^k.$$

Proof. Denote the characteristic maps by $f_0 : D^k \to X \cup_{f_0} D^k$ and $f_1 : D^k \to X \cup_{f_1} D^k$, and let $f_t : [0,1] \times S^{k-1} \to X$ be a homotopy from $f_0 : S^{k-1} \to X$ to $f_1 : S^{k-1} \to X$. Define $h_0 : X \cup_{f_0} D^k \to X \cup_{f_1} D^k$ by $h_0(x) = x$ if $x \in X$ and for all $u \in S^{k-1}$

$$h_0\left(f_0(ru)\right) = \begin{cases} f_1(2ru) & \text{if } 0 \leq 2r \leq 1 \\ f_{2-2r}(u) & \text{if } 1 \leq 2r \leq 2, \end{cases}$$

and define $h_1 : X \cup_{f_1} D^k \to X \cup_{f_0} D^k$ by $h_1(x) = x$ if $x \in X$ and for all $u \in S^{k\sim1}$

$$h_1\left(f_1(ru)\right) = \begin{cases} f_0(2ru) & \text{if } 0 \leq 2r \leq 1 \\ f_{2r-1}(u) & \text{if } 1 \leq 2r \leq 2. \end{cases}$$

It is easy to verify that h_0 and h_1 are single-valued and hence continuous. We have for all $u \in S^{k-1}$

$$(h_0 \circ h_1)(f_0(ru)) = \begin{cases} h_1(f_1(2ru)) & \text{if } 0 \le 2r \le 1 \\ h_1(f_{2-2r}(u)) & \text{if } 1 \le 2r \le 2. \end{cases}$$

Since $h_1(x) = x$ for all $x \in X$ it follows that for all $u \in S^{k-1}$

$$(h_0 \circ h_1)(f_0(ru)) = \begin{cases} f_0(4ru) & \text{if } 0 \le 4r \le 1 \\ f_{4r-1}(u) & \text{if } 1 \le 4r \le 2 \\ f_{2-2r}(u) & \text{if } 1 \le 2r \le 2. \end{cases}$$

Let $\xi_t : X \cup_{f_0} D^k \to X \cup_{f_1} D^k$ be the homotopy which is defined by $\xi_t(x) = x$ for all $x \in X$ and for all $u \in S^{k-1}$

$$\xi_t(f_0(ru)) = \begin{cases} f_0((4-3t)ru) & \text{if } 0 \le r \le \frac{1}{4-3t} \\ f_{(4-3t)r-1}(u) & \text{if } \frac{1}{4-3t} \le r \le \frac{2-t}{4-3t} \\ f_{\frac{1}{2}(4-3t)(1-r)}(u) & \text{if } \frac{2-t}{4-3t} \le r \le 1. \end{cases}$$

It is easy to verify that ξ_t is single-valued and hence continuous, $\xi_0 = h_1 \circ h_0$, and $\xi_1 = 1$. A homotopy $\eta_t : X \cup_{f_1} D^k \to X \cup_{f_0} D^k$ such that $\eta_0 = h_0 \circ h_1$ and $\eta_1 = 1$ is defined by replacing f_0 with f_1 and g_λ with $g_{1-\lambda}$ in the above expression for ξ_t where $\lambda = (4-3t)r-1$ or $(4-3t)(1-r)/2$. □

Lemma 4.33 (P. Hilton).
Let X be a topological space, and let

$$f : S^{k-1} \to X$$

be an attached map. Any homotopy equivalence $h : X \to Y$ extends to a homotopy equivalence

$$H : X \cup_f D^k \to Y \cup_{h \circ f} D^k.$$

Proof. Define $H : X \cup_f D^k \to Y \cup_{h \circ f} D^k$ by

$$H(x) = \begin{cases} h(x) & \text{if } x \in X \\ x & \text{if } x \in D^k. \end{cases}$$

Let $g : Y \to X$ be a homotopy inverse of h and define

$$G : Y \cup_{h \circ f} D^k \to X \cup_{g \circ h \circ f} D^k$$

by

$$G(y) = \begin{cases} g(y) & \text{if } x \in Y \\ y & \text{if } x \in D^k. \end{cases}$$

Since $g \circ h \circ f$ is homotopic to f, it follows from Lemma 4.32 that there is a homotopy equivalence.

$$F : X \cup_{gohof} D^k \to X \cup_f D^k.$$

We will first prove that the composition.

$$F \circ G \circ H : X \cup_f D^k \to X \cup_f D^k$$

is homotopic to the identity. Let h_t be a homotopy between $g \circ h$ and the identity. Using the specific definitions of F, G, and H, we see that

$$\begin{aligned} (F \circ G \circ H)(x) &= (g \circ h)(x) && \text{for } x \in X, \\ (F \circ G \circ H)(tu) &= 2tu && \text{for } 0 \leq t \leq \frac{1}{2}, \quad u \in \partial D^k \\ (F \circ G \circ H)(tu) &= \left(h_{2-2t} \circ f\right)(u) && \text{for } \frac{1}{2} \leq t \leq 1, \quad u \in \partial D^k. \end{aligned}$$

The required homotopy $q_\tau : X \cup_f D^k \to X \cup_f D^k$ is now defined by the formula

$$\begin{aligned} q_\tau(x) &= h_\tau(x) && \text{for } x \in X \\ q_\tau(tu) &= \frac{2}{1+\tau} tu && \text{for } 0 \leq t \leq \frac{1+\tau}{2}, \quad u \in \partial D^k \\ q_\tau(tu) &= \left(h_{2-2t+\tau} \circ f\right)(u) && \text{for } \frac{1+\tau}{2} \leq t \leq 1, \quad u \in \partial D^k. \end{aligned}$$

Therefore, H has a left homotopy inverse, namely $F \circ G : Y \cup_{h \circ f} D^k \to X \cup_f D^k$, and a similar proof shows that G also has a left homotopy inverse.

We can now complete the proof of the lemma as follows. Since

$$F \circ (G \circ H) \simeq \text{ identity },$$

And F is known to have a left homotopy inverse (by Lemma 4.32), it follows that.

$$(G \circ H) \circ F \simeq \text{ identity.}$$

Similarly,

$$G \circ (H \circ F) \simeq \text{ identity}$$

The fact that G has a left homotopy inverse implies that.

$$(H \circ F) \circ G \simeq \text{ identity.}$$

Thus, $H \circ (F \circ G) \simeq$ identity and $F \circ G$ is also a right homotopy inverse for H. Therefore, H is a homotopy equivalence. □

Theorem 4.34 (Handle Decomposition Theorem)**.**
Let $f : M \longrightarrow \mathbb{R}$ be a Morse function on a smooth manifold M. Suppose that M^t is compact for all $t \in \mathbb{R}$ Then M has the homotopy type of a CW-complex X with one cell of dimension k for each critical point of index k.

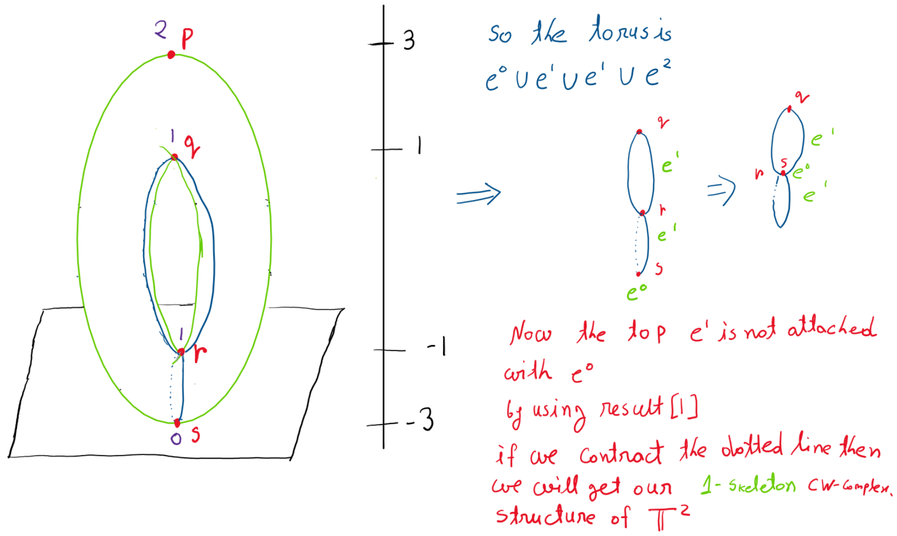
Proof. We will prove this by induction.
Let $c_0 < c_1 < c_2 < \ldots$ be the critical values of $f : M \longrightarrow \mathbb{R}$. Sequence $\{c_i\}$ has no accumulation point since M^t is compact for all t.For the base case M^t is vacuous for all $t < c_0$, and M^{t_0} is homotopic to a discrete set of points for all $c_0 < t_0 < c_1$, i.e $X_0 = \{p \in Cr(f) | f(p) = c_o\}$.
Let's assume that $c_{i-l} < t_{i-l} < c_i$ for some $i \in \mathbb{N}$ and there is a homotopy equivalence $h_{i-1} : M^{t_{i-1}} \longrightarrow X_{i-1}$ where X_{i-1} is some CW-complex.
Now by Remark 4.30 and 4.31 $\exists \epsilon > 0$ such that

$$M^{c_i+\epsilon} \simeq M^{c_i-\epsilon} \cup_{f_1} D^{\lambda_1} \cup_{f_2} ... \cup_{f_j} D^{\lambda_j}$$

where $p_1, ..., p_j$ are the critical points with indices $\lambda_1, ..., \lambda_j$ in $f^{-1}(c_i)$ and $f_1, ..., f_j$ are attaching maps.
By **First Fundamental Theorem of Classical Morse Theory** we get a homotopy equivalence $g_i : M^{c_i-\epsilon} \longrightarrow M^{t_{i-1}}$.Now by Theorem 2.68 the map $h_{i-1} \circ g_{i-1} \circ f_k : S^{\lambda_k - 1} \longrightarrow X_{i-1}$ homotopic to a map $\Psi : S^{\lambda_k-1} \longrightarrow X_{i-1}^{\lambda_K-1}$ Where $X_{i-1}^{\lambda_K-1}$ is a $\lambda_K - 1$ skeleton of X_{i-1}.
Using Lemma 4.33 we can say that $M^{c+\epsilon} \simeq X_i \overset{\text{def}}{=} X_{i-1} \cup_{\Psi_1} D^{\lambda_1} \cup_{\Psi_2} ... \cup_{\Psi_j} D^{\lambda_j}$ By Remark 4.30, it follows that M^t has the homotopy type of a CW-complex for all $t \in \mathbf{R}$.
If M is compact, this completes the proof. If M is not compact, but all the critical points

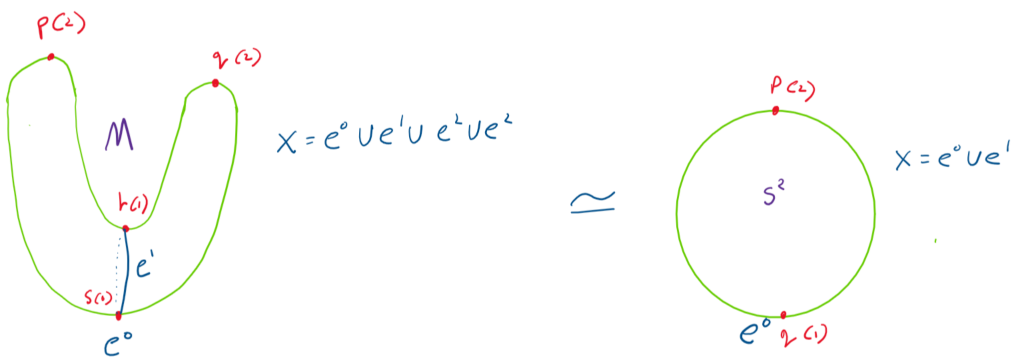
lie in M_t for some $t \in \mathbb{R}$ then a proof similar to that of **First Fundamental Theorem of Classical Morse Theory** shows that M_t is a deformation retract of M, so the proof is again complete. □

Example 4.35.



Example 4.36.

Different Morse functions give rise to different CW-complex structures of the same ManifoldManifoldManifold, but the following example gives us the intuition that they are always homeomorphic[$M \simeq S^2$]. This says about an aspect of topological invariant property of **Morse Theory**.



Chapter 5

Morse Inequality

Let's delve into the interplay between Morse theory and algebraic topology. Consider a compact smooth manifold M and a field F. The homology group $H_k(M;F)$ over F is finite-dimensional, defining the k-th Betti number of M, denoted as $b_k(F)$, which is simply the dimension of $H_k(M;F)$. When $F = \mathbb{Z}$, $H_k(M;\mathbb{Z})$ modulo its torsion subgroup forms a finitely generated free $\mathbb{Z}$-module. In this case, $b_k(\mathbb{Z})$ represents the rank of this module. For simplicity, we often denote b_k instead of $b_k(F)$ when the context implies F.

Now, let's introduce a Morse function $f : M \to \mathbb{R}$ on M. The quantity ν_k denotes the count of critical points of f with index k, where $k = 0, \ldots, m$, with m being the dimension of M. Thanks to the CW-Homology Theorem and Theorem 4.34, we obtain the "weak Morse inequalities":

$$\nu_k \geq b_k(F)$$

These inequalities arise from the fact that ν_k equals the rank of the chain module $\underline{C}_k(M;F)$, and $H_k(M;F)$ is a quotient of this module. Notably, these relations hold regardless of the choice of field F.

In particular, the weak Morse inequalities imply a compelling observation: the total count of critical points of f is at least as large as the sum of the Betti numbers:

$$\sum_{k=0}^{m} \nu_k \geq \sum_{k=0}^{m} b_k(F)$$

This observation serves as the foundational motivation behind the Arnold Conjecture.

It's worth noting that while ν_k depends on the specific Morse function f (without reliance on F), $b_k(F)$ is contingent upon the topology of M and the field F (with no reliance on the choice of Morse function). Now, let's explore the "strong Morse inequalities", which provide further insights into the relationships between ν_k and $b_k(F)$.

Theorem 5.1 (**Euler-Poincare Theorem**).

Let $(\underline{C}_*, \underline{\partial}_*)$ be a finitely generated chain complex, and assume that $\underline{C}_k = 0$ for all $k > m$. Let $c_k = \operatorname{rank} \underline{C}_k$ and $b_k = \operatorname{rank} H_k(\underline{C}_*)$ for all $k = 0, \ldots, m$. Then,

$$\sum_{k=0}^{m} (-1)^k c_k = \sum_{k=0}^{m} (-1)^k b_k$$

Proof. The exact sequence

$$0 \to \ker \underline{\partial}_k \to \underline{C}_k(X; F) \xrightarrow{\partial_k} \operatorname{im} \underline{\partial}_k \to 0$$

shows that $c_k = \operatorname{rank} \ker \underline{\partial}_k + \operatorname{rank} \operatorname{im} \underline{\partial}_k$ for all $k = 0, \ldots, m$. Similarly,

$$0 \to \operatorname{im} \underline{\partial}_{k+1} \to \ker \underline{\partial}_k \to H_k(X; F) \to 0$$

shows that rank ker $\underline{\partial}_k = \operatorname{rank} \operatorname{im} \underline{\partial}_{k+1} + b_k$, and hence

$$\operatorname{rank} \ker \underline{\partial}_k = c_k - \operatorname{rank} \operatorname{im} \underline{\partial}_k = \operatorname{rank} \operatorname{im} \underline{\partial}_{k+1} + b_k$$

for all $k = 0, \ldots, m$. Thus,

$$\sum_{k=0}^{m} (-1)^k \left(c_k - \operatorname{rank} \operatorname{im} \underline{\partial}_k\right) = \sum_{k=0}^{m} (-1)^k \left(\operatorname{rank} \operatorname{im} \underline{\partial}_{k+1} + b_k\right)$$

which implies that

$$\sum_{k=0}^{m} (-1)^k c_k = \sum_{k=0}^{m} (-1)^k b_k$$

□

Theorem 5.2 (Morse Inequalities).

For any Morse function $f : M \to \mathbb{R}$ on a compact smooth manifold M of dimension m we have the following.

(a) $\sum_{k=0}^{n} (-1)^{k+n} \nu_k \geq \sum_{k=0}^{n} (-1)^{k+n} b_k(F)$ for every $n = 0, \ldots, m$.

(b) $\sum_{k=0}^{m} (-1)^k \nu_k = \sum_{k=0}^{m} (-1)^k b_k(F)$.

Proof. Part (b) of Theorem is The Euler-Poincaré Theorem (Theorem 5.1) applied to the chain complex $(\underline{C}_*(X;F), \underline{\partial}_*)$. To prove part (a) let $n \leq m$ and consider the truncated chain complex $\left(\underline{C}_*^{(n)}(X;F), \underline{\partial}_*\right)$ given by

$$\underline{C}_k^{(n)}(X;F) = \begin{cases} \underline{C}_k(X;F) & \text{if } k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

By The Euler-Poincaré Theorem we have

$$(-1)^n \sum_{k=0}^{n} (-1)^k \operatorname{rank} \underline{C}_k^{(n)}(X;F) = (-1)^n \sum_{k=0}^{n} (-1)^k \operatorname{rank} H_k \left(\underline{C}_*^{(n)}(X;F)\right)$$

Since $\nu_k = \operatorname{rank} \underline{C}_k^{(n)}(X;F)$ for all $k = 0, \ldots, n$, $b_k = \operatorname{rank} H_k \left(\underline{C}_*^{(n)}(X;F)\right)$ for all $k = 0, \ldots, n-1$ and $H_n(\underline{C}_*(X;F))$ is a quotient of $H_n\left(\underline{C}^{(n)}(X;F)\right)$, we have

$$\nu_n - \nu_{n-1} + \cdots + (-1)^n \nu_0 \geq b_n - b_{n-1} + \cdots + (-1)^n b_0$$

□

Remark 5.3.

Part (b) of the preceeding theorem shows that the Euler characteristic, $\mathcal{X}(M) = \sum_{k=0}^{m} (-1)^k b_k(F)$, is independent of the field F. If F is a field of characteristic zero or $F = \mathbb{Z}$, then the same is true for the Betti numbers. This can be seen as a corollary to the Universal Coefficient Theorem for Homology.

Definition 5.4.

The Poincaré polynomial of M is defined to be

$$P_t(M) = \sum_{k=0}^{m} b_k(F) t^k$$

and the **Morse polynomial** of f is defined to be

$$M_t(f) = \sum_{k=0}^{m} \nu_k t^k$$

Theorem 5.5 (Polynomial Morse Inequalities)**.**

For any Morse function $f : M \to \mathbb{R}$ on a smooth manifold M we have

$$M_t(f) = P_t(M) + (1+t)R(t)$$

where $R(t)$ is a polynomial with non-negative integer coefficients. That is, $R(t) = \sum_{k=0}^{m-1} r_k t^k$ where $r_k \in \mathbb{Z}$ satisfies $r_k \geq 0$ for all $k = 0, \ldots, m-1$.

Proof. Let $z_k = \operatorname{rank}\ker \underline{\partial}_k$ for all $k = 0, \ldots, m$. As in the proof of Theorem 5.1 , the exact sequence

$$0 \to \ker \underline{\partial}_k \to \underline{C}_k(X;F) \xrightarrow{\underline{\partial}_k} \operatorname{im}\underline{\partial}_k \to 0$$

implies that $\nu_k = z_k + \operatorname{rank}\operatorname{im}\underline{\partial}_k$ for all $k = 0, \ldots, m$, and

$$0 \to \operatorname{im}\underline{\partial}_{k+1} \to \ker \underline{\partial}_k \to H_k(X;F) \to 0$$

implies that $b_k = z_k - \operatorname{rank}\operatorname{im}\underline{\partial}_{k+1}$ for all $k = 0, \ldots, m$. Hence,

$$\begin{aligned}
M_t(f) - P_t(M) &= \sum_{k=0}^{m} \nu_k t^k - \sum_{k=0}^{m} b_k t^k \\
&= \sum_{k=0}^{m} (z_k + \operatorname{rank}\operatorname{im}\underline{\partial}_k)\, t^k - \sum_{k=0}^{m} \left(z_k - \operatorname{rank}\operatorname{im}\underline{\partial}_{k+1}\right) t^k \\
&= \sum_{k=0}^{m} \left(\operatorname{rank}\operatorname{im}\underline{\partial}_k + \operatorname{rank}\operatorname{im}\underline{\partial}_{k+1}\right) t^k \\
&= \sum_{k=0}^{m} (\nu_k - z_k + \nu_{k+1} - z_{k+1})\, t^k \\
&= \sum_{k=0}^{m} (\nu_k - z_k)\, t^k + \sum_{k=0}^{m} (\nu_{k+1} - z_{k+1})\, t^k \\
&= t\sum_{k=1}^{m} (\nu_k - z_k)\, t^{k-1} + \sum_{k=1}^{m} (\nu_k - z_k)\, t^{k-1} \quad (\text{ since } \nu_0 = z_0) \\
&= (t+1)\sum_{k=1}^{m} (\nu_k - z_k)\, t^{k-1}.
\end{aligned}$$

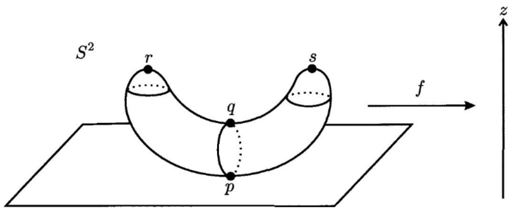
Therefore, $M_t(f) = P_t(M) + (1+t)R(t)$ where $R(t) = \sum_{k=0}^{m-1} (\nu_{k+1} - z_{k+1})\, t^k$. Note that $\nu_{k+1} - z_{k+1} \geq 0$ for all $k = 0, \ldots, m-1$ because z_{k+1} is the rank of a subgroup of $\underline{C}_{k+1}(X;F)$ and $\nu_{k+1} = \operatorname{rank}\underline{C}_{k+1}(X;F)$. □

Remark 5.6.

Theorem 5.5 occasionally suffers from misinterpretations in the literature. Two common errors include asserting that the polynomial $R(t)$ is invariably positive and claiming that the Morse polynomial consistently surpasses the Poincaré polynomial for all $t \in \mathbb{R}$. While these assertions hold true for $t \geq 0$, they do not universally apply when $t < 0$. Below, we present an illustrative example that serves as a counterexample to both of these misconceptions.

Example 5.7 (**Wiener-Dog Counterexample**).

Consider the function f defined by projection onto the z axis in the following picture.



The critical points p, q, r, and s have indices $0, 1, 2$, and 2 respectively. Hence, $M_t(f) = 2t^2 + t + 1$, and since $P_t(S^2) = t^2 + 1$, we see that

$$M_t(f) = 2t^2 + t + 1 = t^2 + 1 + (1+t)t = P_t(M) + (1+t)R(t)$$

Thus, $R(t) = t$, and we see that $R(t) < 0$ for all $t < 0$. Moreover, it is easy to check that $M_t(f) < P_t(f)$ for $-1 < t < 0$.

Lemma 5.8.

Theorem 5.2 is equivalent to Theorem 5.5

Proof. By utilizing part (b) of Theorem 5.2, we can express $M_{-1}(f)$ as follows:

$$M_{-1}(f) = \sum_{k=0}^{m} (-1)^k \nu_k = \sum_{k=0}^{m} (-1)^k b_k(F) = P_{-1}(M)$$

Consequently, $M_t(f) - P_t(M)$ is divisible by $1+t$, and we can represent $M_t(f)$ as $P_t(M) + (1+t)R(t)$, where $R(t) = \sum_{n=0}^{m-1} r_n t^n$. It's evident that $r_n \in \mathbb{Z}$ for all $n = 0, \ldots, m-1$, as both $M_t(f)$ and $P_t(M)$ possess integer coefficients. Our task is to demonstrate that $r_n \geq 0$ for all $n = 0, \ldots, m-1$. We will establish the equivalence of these inequalities with those stated in part (a) of Theorem 5.2.

To illustrate, let's start with $M_t(f) = P_t(M) + (1+t)R(t)$, which implies:

$$\nu_0 = b_0(F) + r_0$$

Additionally, $\nu_1 = b_1(F) + r_1 + r_0$, yielding $\nu_1 = b_1(F) + r_1 + \nu_0 - b_0(F)$, thus:

$$\nu_1 - \nu_0 = b_1(F) - b_0(F) + r_1$$

Continuing in this manner, we observe that:

$$\nu_n - \nu_{n-1} + \cdots + (-1)^n \nu_0 = b_n(F) - b_{n-1}(F) + \cdots + (-1)^n b_0(F) + r_n$$

for all $n = 0, \ldots, m-1$. Consequently, the inequalities articulated in part (a) of Theorem 5.2 imply $r_n \geq 0$ for all $n = 0, \ldots, m-1$. Furthermore, by setting $t = -1$ in the equation $M_t(f) = P_t(M) + (1+t)R(t)$, we readily recover part (b) of Theorem 5.2. Hence, Theorem 5.5 effectively implies Theorem 5.2. □

Definition 5.9.

If $f : M \to \mathbb{R}$ is a Morse function such that $M_t(f) = P_t(M)$, then f is called a perfect Morse function.

Note that if a manifold admits a perfect Morse function, then its homology doesn't have any torsion.

Theorem 5.10 (**Morse's Lacunary Principle**)**.**

If $M_t(f)$ has no consecutive powers of t, then

$$M_t(f) = P_t(M)$$

In fact, $\nu_k = b_k$ and $R(t)$ is identically zero.

Proof. This is a direct consequence of Theorem 5.5 □

Example 5.11 (Bott's perfect Morse function)**.**

Using the result from Example 4.17 and considering $f : \mathbb{C}P^n \to \mathbb{R}$ as the same Morse function defined in Exaple 4.17 we have

$$M_t(f) = 1 + t^2 + \cdots + t^{2n}$$

and the preceeding theorem implies that

$$P_t\left(\mathbb{C}P^n\right) = 1 + t^2 + \cdots + t^{2n}.$$

Now we will try to see an easy approach to an important result by previous results.

Theorem 5.12.

Let M be a compact manifold of odd dimension, then the Euler characteristic is zero, i.e. $\mathcal{X}(M) = 0$.

Proof. Let $f : M \to \mathbb{R}$ be a Morse function, and assume that the dimension m of the manifold M is odd. Since $\nu_k(f) = \nu_{m-k}(-f)$ we have the following.

$$\begin{aligned}
\mathcal{X}(M) &= \sum_{k=0}^{m}(-1)^k \nu_k(f) \\
&= \sum_{k=0}^{m}(-1)^k \nu_{m-k}(-f) \\
&= (-1)^m \sum_{k=0}^{m}(-1)^{m-k} \nu_{m-k}(-f) \\
&= (-1)^m \sum_{k=0}^{m} \nu_k(-f) \\
&= (-1)^m \mathcal{X}(M)
\end{aligned}$$

Hence, $\mathcal{X}(M) = 0$ if m is odd. □

Chapter 6

Stable/Unstable Manifold Theorem

Let $f : M \to \mathbb{R}$ be a smooth function on a finite dimensional compact smooth Riemannian manifold (M, g). Recall from Definition 4.22 that the gradient vector field of f with respect to the metric g is the unique smooth vector field ∇f such that

$$g(\nabla f, V) = df(V) = V \cdot f$$

for all smooth vector fields V on M. The gradient vector field determines a smooth flow $\varphi : \mathbb{R} \times M \to M$ by $\varphi_t(x) = \gamma_x(t)$ where $\frac{d}{dt}\gamma_x(t) = -\left.\nabla f\right|_{\gamma_x(t)}$ and $\gamma_x(0) = x$. Since M is compact, the flow φ_t satisfies the following.

1. $\varphi_t : M \to M$ is a diffeomorphism for all $t \in \mathbb{R}$.

2. $\varphi_{t_1} \circ \varphi_{t_2} = \varphi_{t_1+t_2}$ for any $t_1, t_2 \in \mathbb{R}$.

That is, φ_t is a 1-parameter group of diffeomorphisms defined on $\mathbb{R} \times M$

Definition 6.1.

Let $p \in M$ be a non-degenerate critical point of f.

1. The **stable manifold** of p is defined to be

$$W^s(p) = \left\{ x \in M \mid \lim_{t\to\infty} \varphi_t(x) = p \right\}.$$

2. The **unstable manifold** of p is defined to be

$$W^u(p) = \left\{ x \in M \mid \lim_{t\to-\infty} \varphi_t(x) = p \right\}$$

Now we will try to understand an important theorem as we can refer the proof details from Morse Homology by Audin.

Theorem 6.2 (**Stable/Unstable Manifold Theorem for a Morse Function**).

Let $f : M \to \mathbb{R}$ be a Morse function on a compact smooth Riemannian manifold (M, g)

of dimension $m < \infty$. If $p \in M$ is a critical point of f, then the tangent space at p splits as

$$T_pM = T_p^sM \oplus T_p^uM$$

where the Hessian is positive definite on T_p^sM and negative definite on T_p^uM. Moreover, the stable and unstable manifolds are surjective images of smooth embeddings

$$E^s : T_p^sM \to W^s(p) \subseteq M$$

$$E^u : T_p^uM \to W^u(p) \subseteq M$$

Hence, $W^s(p)$ is a smoothly embedded open disk of dimension $m - \lambda_p$, and $W^u(p)$ is a smoothly embedded open disk of dimension λ_p, where λ_p is the index of the critical point p.

Now we will try to calculate the Stable/Unstable Manifold for some Morse function.

Example 6.3.

Let's consider a function on $D \subseteq \mathbb{R}^2$ where D is a closed disc around origin

$$f : D \to \mathbb{R}$$

$$f(x, y) = x^2 - y^2$$

Now $\nabla f = (2x, -2y)$ So critical points is only origin $(0, 0)$ & $M_{(0,0)}f$ is non-degenerate so it is a Morse function

$$M_{(0,0)}f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now index at $(0, 0)$ is 1.

for $(a, b) \in D \quad \varphi : (-\infty, \infty) \to D, \varphi(t) = (\varphi_1(t), \varphi_2(t))$ where $\varphi(0) = (a, b)$[D is compact]

Now the negative gradient flow is

$$-\nabla f|_{\varphi(t)} = \frac{d}{dt}\varphi(t)$$

$$\Rightarrow [-2\varphi_1(t), 2\varphi_2(t)] = [\varphi_1'(t), \varphi_2'(t)]$$

so $\varphi_1'(t) = -2\varphi_1(t)$

$$\Rightarrow \int_0^t \frac{\varphi_1'(t)}{\varphi_1(t)} dt = \int_0^t -2$$
$$\Rightarrow \ln|\varphi_1(t)|]_0^t = -2t$$
$$\Rightarrow |\varphi_1(t)| = |a|e^{-2t}$$
$$\Rightarrow \varphi_1(t) = ae^{-2t} \text{ or } \Rightarrow \varphi_1(t) = -ae^{-2t}$$

Putting $'0'$ in place of t we get second one is not valid

so $\varphi_1(t) = ae^{-2t}$.

Similarly $\varphi_2(t) = be^{2t}$

So flow lines are (ae^{-2t}, be^{2t})

$$W^s((0,0)) = \left\{x \in D \mid \lim_{t\to\infty} \varphi_t(x) = (0,0)\right\}$$
$$\text{for } \lim_{t\to\infty} be^{2t} \neq 0$$

So if we take points from x-axis then

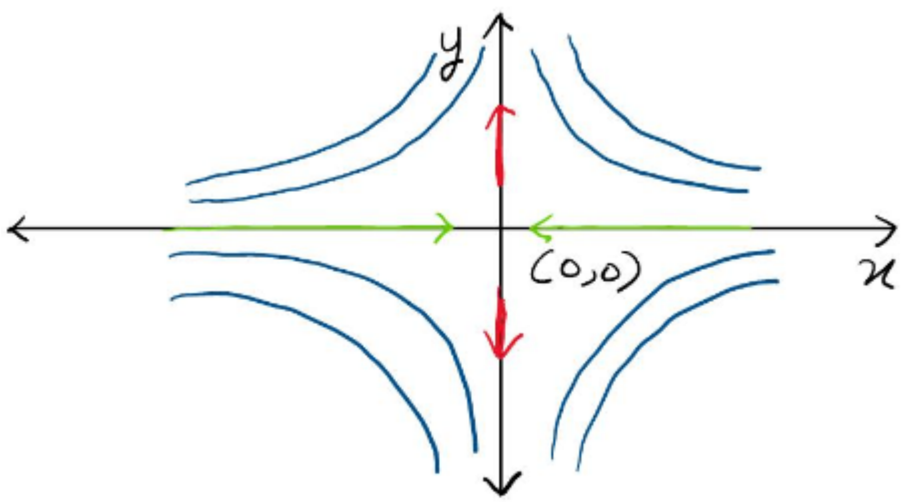
$$\lim_{t\to\infty} \left(ae^{-2t}, 0\right) = (0,0)$$

So

$$W^s((0,0)) = \{(x,0) \in D \mid x \in \mathbb{R}\}$$

Similarly we can say that

$$W^u((0,0)) = \{(0,y) \in D \mid y \in \mathbb{R}\}$$



Example 6.4 (Sphere)**.**

Let's define $\hat{f} : \mathbb{R}^3 \to \mathbb{R} \quad \hat{f}(x, y, z) = z$

$$\nabla \hat{f} = (0, 0, 1)$$

Now let's Consider $f : S^2 \to \mathbb{R} \quad f(x, y, z) = z$

From the calculation in Example 4.14 we can see that it is a Morse function with critical points $(0, 0, 1)\&(0, 0, -1)$ with index 2,0 respectively

Now $S^2 \subseteq \mathbb{R}^3$ is an embedded submanifold. so $\nabla \hat{f}(x)$ projects orthogonally onto $\nabla f(x)$ as we are looking the orthogonal projection of $T_m\mathbb{R}^n$ onto T_mS^2 Where $m \in S^2$

So now

$$\begin{aligned} \nabla f &= \nabla \hat{f} - \frac{\nabla \hat{f} \cdot (x, y, z)}{\|(x, y, z)\|}(x, y, z) \\ &= (0, 0, 1) - \left(zx, zy, z^2\right) \\ &= \left(-zx, -zy, 1 - z^2\right) \end{aligned}$$

$(x, y, z) \in S^2 \Rightarrow \|(x, y, z)\| = 1$

For $(a, b, c) \in S^2$ define flow $\varphi : (-\infty, \infty) \to S^2[S^2 \text{ is Compact}]$

$$\varphi, (t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$$

$$\varphi(0) = (a, b, c)$$

So now

$$\begin{aligned} -\left.(\nabla f)\right|_{\varphi(t)} &= \frac{d}{dt}\varphi(t) \\ \Rightarrow \left[\varphi_1(t)\varphi_3(t), \varphi_2(t)\varphi_3(t), \varphi_3^2(t) - 1\right] &= [\varphi_1'(t), \varphi_2'(t), \varphi_3'(t)] \end{aligned}$$

So

$$\begin{aligned} \varphi_3'(t) &= \varphi_3^2(t) - 1 \\ \Rightarrow \int_0^t \frac{\varphi_3'(t)}{\varphi_3^2(t) - 1} dt &= \int_0^t dt \\ \Rightarrow \frac{1}{2} \ln \left|\frac{\varphi_3(t) - 1}{\varphi_3(t) + 1}\right|\Bigg|_0^t &= t \\ \Rightarrow \left|\frac{\varphi_3(t) - 1}{\varphi_3(t) + 1}\right| &= \left|\frac{c - 1}{c + 1}\right| e^{2t} \end{aligned}$$

putting $t = 0$ and checking the sign we get

$$\begin{aligned}
\Rightarrow \quad & \frac{1-\varphi_3(t)}{\varphi_3(t)+1} = \left(\frac{1-c}{c+1}\right) e^{2t} \\
\Rightarrow \quad & \frac{2}{\varphi(t)+1} - 1 = \left(\frac{1-c}{c+1}\right) e^{2t} \\
\Rightarrow \quad & \varphi_3(t) + 1 = \frac{2}{\left(\frac{1-c}{c+1}\right) e^{2t} + 1} \\
\Rightarrow \quad & \varphi_3(t) = -\frac{\left(\frac{1-c}{c+1}\right) e^{2t} - 1}{\left(\frac{1-c}{c+1}\right) e^{2t} + 1}
\end{aligned}$$

Now $\varphi_1(t)\varphi_3(t) = \varphi_1'(t)$

$$\begin{aligned}
& \varphi_1(t)\varphi_3(t) = \varphi_1'(t) \\
\Rightarrow & \int_0^t -\frac{\left(\frac{1-c}{c+1}\right) e^{2t} - 1}{\left(\frac{1-c}{c+1}\right) e^{2t} + 1} dt = \int_0^t \frac{\varphi_1'(t)}{\varphi_1(t)} dt \\
\Rightarrow & -\left[\ln\left|\left(\frac{1-c}{c+1}\right) e^{2t} + 1\right| + t\right]\Bigg|_0^t = \ln|\varphi_1(t)| - \ln|a|
\end{aligned}$$

Using similar argument we get

$$\begin{aligned}
& \Rightarrow -\ln\left[\left(\frac{1-c}{c+1}\right) e^{2t} + 1\right] + t + \ln\left(\frac{2}{c+1}\right) = \ln\varphi_1(t) - \ln a \\
& \Rightarrow \ln\left[\frac{\left(\frac{1-c}{c+1} e^{2t} + 1\right)\varphi_1(t)}{a\left(\frac{2}{c+1}\right)}\right] = t \\
& \Rightarrow \quad \varphi_1(t) = \frac{\frac{2a}{c+1} e^t}{\left(\frac{1-c}{c+1}\right) e^{2t} + 1}
\end{aligned}$$

Similarly we get

$$\varphi_2(t) = \frac{\frac{2b}{c+1} e^t}{\left(\frac{1-c}{c+1}\right) e^{2t} + 1}$$

So flow lines are $\left(\frac{\left(\frac{2a}{c+1}\right)e^t}{\left(\frac{1-c}{c+1}\right)e^{2t}+1}, \frac{\left(\frac{2b}{c+1}\right)e^t}{\left(\frac{1-c}{c+1}\right)e^{2t}+1}, -\frac{\left(\frac{1-c}{c+1}\right)e^t-1}{\left(\frac{1-c}{c+1}\right)e^t+1}\right)$ for any $(a, b, c) \in S^2$

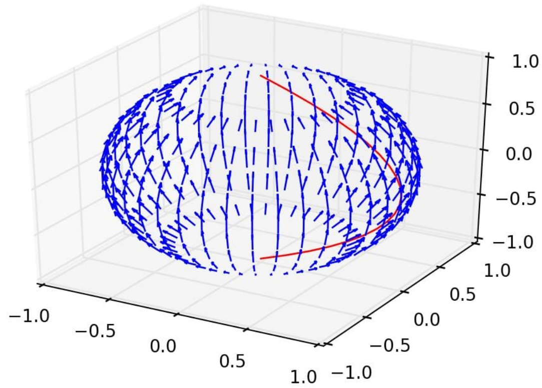
$$\lim_{t\to\infty} \varphi(t) = (0, 0, -1) \quad \& \quad \lim_{t\to-\infty} \varphi(t) = (0, 0, 1)$$

So

$$W^u((0, 0, 1)) = S^2 \backslash \{(0, 0, -1)\} \quad W^s((0, 0, 1)) = \{(0, 0, 1)\}$$

$$W^u((0, 0, -1)) = \{(0, 0, -1)\} \quad W^s((0, 0, -1)) = S^2 \backslash \{(0, 0, 1)\}$$

We have tried to plot it using Matlab.



Example 6.5.

$$\varphi : \mathbb{R}^2 \to \mathbb{T}^2$$

$$\varphi(t,s) = (\sin t, (2+\cos t)\sin s, (2+\cos t)\cos s)$$

$$f : \mathbb{T}^2 \to \mathbb{R}$$

$$f(x,y,z) = z$$

From Example 4.16 we can see that $f \circ \varphi$ is a Morse function

So $-\nabla(f \circ \varphi) = (\sin t \cos s, (2+\cos t)\sin s)$ [using flat metric]

$$\text{for } (a,b) \in \mathbb{R}^2 \quad \gamma : (-\infty,\infty) \to \mathbb{T}^2 \quad [\mathbb{T}^2 \text{is compact.}]$$

$$\gamma(c) = (t(c), s(c))$$

$$\gamma(0) = (a,b)$$

So

$$-\nabla f \cdot \varphi|_{\gamma(c)} = \frac{d}{dt}\gamma(c)$$

$$\Rightarrow [\sin(t(c))\cos(s(c)), (2+\cos(t(c)))\sin(s(c))] = [t'(c), s'(c)]$$

So $s'(c) = (2+\cos t(c))\sin(s(c))$

Solving this ODE we get $\tan\left(\frac{s(c)}{2}\right) = e^{c(2+\cos(t(c)))}\tan\left(\frac{a}{2}\right)$

$$\left.\begin{matrix} c \to \infty & s(c) \to \pi \\ c \to -\infty & s(c) \to 0 \end{matrix}\right] \cdots (1)$$

So $t'(c) = \sin(t(c))\cos(s(c))$

Solving this ODE we get $\tan\left(\frac{t(c)}{2}\right) = e^{c\cos(s(c))}\tan\left(\frac{b}{2}\right)$ and using (1) we get the following result.

$$\left.\begin{array}{ll} c \to \infty & t(c) \to 0 \\ c \to -\infty & t(c) \to 0 \end{array}\right] \cdots (2)$$

All of these happens when $t \neq 0, \pi \quad \& \quad s \neq 0, \pi$

We know critical points are from Example 4.16 $(0,0,3), (0,0,-3), (0,0,2), (0,0,-2)$ for $(s,t) = (0,0); (0,\pi), (\pi,0), (\pi,\pi)$

for $t = 0$ our flow line will be $\left(0, 2\tan^{-1}\left(e^{2c}\tan\left(\frac{a}{2}\right)\right) = k\right)$

$$c \to \infty \text{ this will go to } \quad (0,\pi)$$

$$c \to -\infty \text{ this will go to } \quad (0,0)$$

for $s = 0$ our flow line will be $\left(2\tan^{-1}\left(e^{-c}\tan\left(\frac{b}{2}\right)\right) = k, 0\right)$

$$c \to \infty \text{ this will go to } \quad (\pi,0)$$

$$c \to -\infty \text{ this will go to } \quad (0,0)$$

$$W^s((0,0,2)) = \{(\sin t(k), 0, (2 + \cos t(k))\} \backslash \{(0,0,1)\}$$

for $t = \pi$ our flow line will be $\left(\pi, 2\tan^{-1}\left(e^{2c}\tan\left(\frac{a}{2}\right)\right) = k\right)$

$$c \to \infty \text{ this will go to } \quad (\pi,\pi)$$

$$c \to -\infty \text{ this will go to } \quad (\pi,0)$$

$$W^u((0,0,2)) = \{(0, \sin s(k), \cos s(k))\} \backslash \{(0,0,-2)\}$$

$$W^s((0,0,-2)) = \{(0, \sin s(k), \cos s(k))\} \backslash \{(0,0,2)\}$$

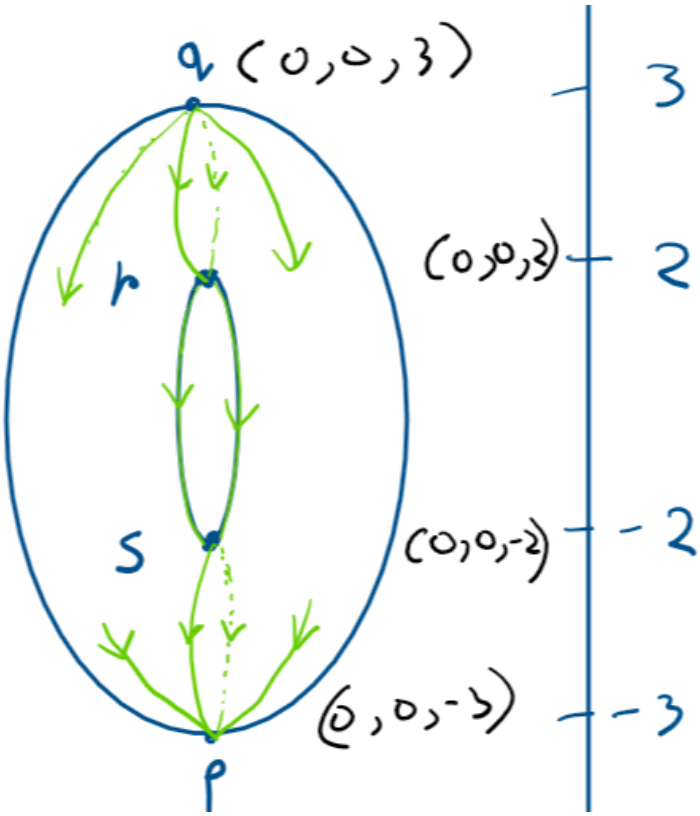
for $s = \pi$ our flow line will be $\left(2\tan^{-1}\left(e^{-c}\tan\left(\frac{b}{2}\right)\right) = k, \pi\right)$

$$c \to \infty \text{ this will go to } \quad (0,\pi)$$

$$c \to -\infty \text{ this will go to } \quad (\pi,\pi)$$

$$W^u((0,0,-2)) = \{(\sin t(k), 0, -(2 + \cos t(k))\} \backslash \{(0,0,-3)\}$$

By taking cases we get stable and unstable manifold for $r \& s$



From (1) & (2)

$$W^s((0,0,-3)) = \mathbb{T}^2 \setminus (W^s((0,0,-2)) \cup W^s((0,0,2)) \cup \{(0,0,3)\})$$

$$W^s((0,0,3)) = \{(0,0,3)\}$$

$$W^u((0,0,-3)) = \{(0,0,-3)\}$$

$$W^u((0,0,3)) = T^2 \setminus (W^u((0,0,-2)) \cup W^u((0,0,2)) \cup \{(0,0,-3)\})$$

So we have calculated all the stable unstable manifold of T^2

Chapter 7

Intersection Number

First We will discuss some results and definition to define the intersection number properly.

Definition 7.1 (Transversality).

Let $f : M \to N$ and $g : Z \to N$ be smooth maps where M, N, and Z are smooth manifolds. We say that f is transverse to g, $f \pitchfork g$, if and only if whenever $f(x) = g(z) = y$ we have

$$df_x\left(T_xM\right) + dg_z\left(T_zZ\right) = T_yN$$

If $Z \subseteq N$ and $g : Z \to N$ is the inclusion map, then we will denote $f \pitchfork g$ by $f \pitchfork Z$.

We will try to realize this via some example.

- Firstly we will give an example where $\dim T_xX + \dim T_xZ = \dim T_xY$ but X and Z do not meet transversally. Let $X = Z$ be the x-axis, $Y = \mathbb{R}^2$. Both X and Z are 1-dimensional, but the span of T_xX and T_xZ is also 1-dimensional.

- Secondly we will give an example where X and Z do not meet transversally and $X \cap Z$ is not a submanifold of Y. Let $Y = \mathbb{R}^2, Z$ be the x-axis, and let X be a curve that intersects Z over both an interval and a point outside of the interval. The (disjoint) union of the interval and the point is not a manifold.

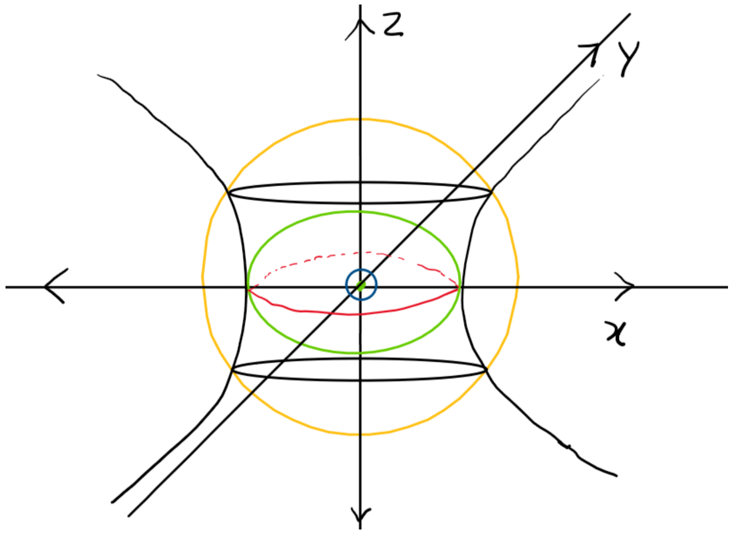
- Thirdly we will try to give an example of X, Z, Y, Y' where X is transverse to Z as submanifolds in Y but not as submanifolds in Y'. Let $Y = \mathbb{R}^2, Y' = \mathbb{R}^3, X$ be the x-axis, Z be the y-axis. The span of T_xX and T_xZ is equal to T_xY, a 2-dimensional subspace of T_xY'.

- Consider a hyperboloid X and a sphere of radius $a \leqslant 1$ given by

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$$

$$Z_a = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = a\}$$

Now we will try to find out for what values of $a \leqslant 1$ do X and Z_a meet transversally in $Y = \mathbb{R}^3$.



Let $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x^2 + y^2 - z^2$, and let $g : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x^2 + y^2 + z^2$. Then $X = f^{-1}(1), Z_a = g^{-1}(a)$. We need to check whether 1 and a are regular values of f and g respectively. By calculating the Jacobian, $df_{(x,y,z)} = \begin{pmatrix} 2x & 2y & -2z \end{pmatrix}$ and $dg_{(x,y,z)} = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix}$. Both of these are surjective as long as $(x, y, z) \neq (0, 0, 0)$.

Since $T_{(x,y,z)}X = \ker df_{(x,y,z)}$ and $T_{(x,y,z)}Z_a = \ker dg_{(x,y,z)}$, the intersection is transverse if $\ker df_{(x,y,z)} + \ker dg_{(x,y,z)} = \mathbb{R}^3$ for every $(x, y, z) \in X \cap Z_a$. If $a < 1$, then $X \cap Z_a$ is empty. Hence X meets Z_a transversally by definition. If $a = 1$, then $X \cap Z_a$ is all points where $x^2 + y^2 - z^2 = x^2 + y^2 + z^2$, that is, when $z = 0$. Therefore $df_{(x,y,0)} = \begin{pmatrix} 2x & 2y & 0 \end{pmatrix} = dg_{(x,y,0)}$. Both kernels are the z-axis, hence the intersection is not transverse.

Theorem 7.2 (Inverse Image Theorem)**.**

Let $Z \subseteq N$ be an immersed submanifold and $f : M \to N$ a smooth map. If $f \pitchfork Z$, then $f^{-1}(Z)$ is a submanifold of M whose codimension in M is the same as the codimension of Z in N, i.e.

$$\dim M - \dim f^{-1}(Z) = \dim N - \dim Z$$

Moreover, the normal bundle of Z in N pulls back to the normal bundle of $f^{-1}(Z)$ in M, i.e. $\nu f^{-1}(Z) = f^*(\nu Z)$.

Proof. See Introduction to Smooth Manifold by Lee. □

Corollary 7.3.

If M and Z are immersed submanifolds of N of dimension m, z, and n respectively and $M \pitchfork Z$, then $M \cap Z$ is an immersed submanifold of N of dimension $m + z - n$

Proof. Applying previous theorem to the inclusion $i : M \to N$ we get $m - \dim(M \cap Z) = n - z$. □

Definition 7.4 (smooth homotopy)**.**

Let $f_0, f_1 : M \to N$ be smooth maps between smooth manifolds M and N. The maps f_0 and f_1 are said to be smoothly homotopic if and only if there exists a smooth map $H : M \times [0,1] \to N$ such that

$$H(x, 0) = f_0(x)$$

$$H(x, 1) = f_1(x)$$

for all $x \in X$. The map H is called a smooth homotopy from f_0 to f_1.

Definition 7.5.

We will call a property of a class of smooth maps $f : M \to N$ locally stable provided that for every $x \in M$ there is a neighborhood $U \subseteq M$ of x such that whenever $f|_U : U \to N$ possesses the property and $H : U \times [0,1] \to N$ is a smooth homotopy of $f|_U$, then for some $\varepsilon > 0$ each $f_t = H(\cdot, t) : U \to N$ with $t < \varepsilon$ also possesses the property. We will call the property globally stable if the above condition holds for $U = M$.

Theorem 7.6 (stability theorem)**.**

If M and N are smooth manifolds, then following classes of smooth maps $f : M \to N$ are locally stable:

1. immersions
2. submersions
3. local diffeomorphisms
4. maps transverse to a specified closed submanifold $Z \subseteq N$. If M is compact, then the preceeding classes are globally stable.

Proof. See Lectures on Morse Homology by Banyaga and Hurtubise. □

Theorem 7.7 (Homotopy Transversality Theorem for Smooth Maps)**.**
Let $f : M \to N$ and $g : Z \to N$ be smooth maps where M, N, and Z are smooth manifolds. Then there is an arbitrarily small smooth homotopy g_t of g such that $g_0 = g$ and $g_1 \pitchfork f$.

Theorem 7.8.
(Homotopy Transversality Theorem for Embeddings) Let M, N, and Z be smooth manifolds, and assume that Z is compact. Let $f : M \to N$ be smooth and let $g : Z \to N$ be a smooth embedding. Then there is an arbitrarily small smooth homotopy g_t of g to a smooth embedding $g_1 : Z \to N$ such that $g_0 = g$ and $g_1 \pitchfork f$. Moreover, the smooth homotopy can be chosen such that $g_t : Z \to N$ an embedding for all t, i.e. g_0 is isotopic to g_1.

As the proof require some extra tools the proof can be referred from Topology and Geometry by Bredon.

7.1 Orientation

Let V be a real vector space of finite dimension $m > 0$. On the set of ordered bases of V define a relation $\mathcal{R}$ by

$$v\mathcal{R}w$$

if and only if the change of basis matrix $C = (c_{ij})$ from $v = (v_1, \ldots, v_m)$ to $w = (w_1, \ldots, w_m)$, i.e. $w_i = \sum_{j=1}^{m} c_{ij} v_j$, has positive determinant. The relation $\mathcal{R}$ is an equivalence relation, and there are exactly two equivalence classes.

Definition 7.9.

An orientation of a real vector space V is a choice of one of the equivalence classes θ of the relation $\mathcal{R}$, which we call the positive orientation. The couple (V, θ) is called an oriented vector space. If dim $V = 0$, then an orientation is an assignment of +1 or -1 to the point $V = \{0\}$. If (V, θ) and (V', θ') are two oriented vector spaces of the same positive dimension and $L : V \to V'$ a linear isomorphism, then L is said to be orientation preserving if and only iffor all $v = (v_1, \ldots, v_m) \in \theta$ we have $(L(v_1), \ldots, L(v_m)) \in \theta'.A$ linear isomorphism that is not orientation preserving is said to be orientation reversing.

Definition 7.10.

An orientation of a differentiable manifold with boundary M of dimension m is a choice of orientation θ_x for each tangent space T_xM that satisfies the following compatability requirement: Around every point in M there is a coordinate chart $\phi : U \to \mathbb{R}^m$ (or $\phi : U \to \mathbb{R}^m_+$) which is orientation preserving, i.e. for every point $x \in U$ the linear isomorphism

$$d\phi_x : T_xM \to \mathbb{R}^m$$

is orientation preserving where $\mathbb{R}^m$ is given its standard orientation. A differentiable manifold M that possesses an orientation is called orientable.

Theorem 7.11.

Let M be a differentiable manifold with boundary of dimension m. Then the following are equivalent.

1. M is orientable.

2. There is a collection $\Phi = \{(U, \phi)\}$ of coordinate systems on M such that

$$M = \bigcup_{(U,\phi)\in\Phi} U \text{ and } \det\left(d_y\left(\phi_j \circ \phi_i^{-1}\right)\right) > 0 \text{ on } \phi_i\left(U_i\right) \cap \phi_j\left(U_j\right)$$

whenever (U_i, ϕ_i) and (U_j, ϕ_j) belong to Φ.

3. There is a no-where vanishing m-form on M.

Proof. Refer to Introduction to Smooth Manifold by Lee. □

Remark 7.12.

If $V = W \oplus W'$, where W and W' are oriented vector spaces with orientations θ_W and θ'_W respectively, then a unique orientation θ can be determined on V as follows: If $(w_1, \ldots, w_l) \in \theta_W$ and $w' = \left(w'_1, \ldots, w'_{m-l}\right) \in \theta'_W$, then the vector $v = \left(w_1, \ldots, w_l, w'_1, \ldots, w'_{m-l}\right)$ belongs to the orientation θ.

Similarly, if (V, θ) is an oriented vector space and (W, θ_W) is an oriented subspace, then any complementary subspace W', satisfying $V = W \oplus W'$, can be endowed with an orientation θ'_W such that for $w = (w_1, \ldots, w_l) \in \theta_W$ and $w' = \left(w'_1, \ldots, w'_{m-l}\right) \in \theta'_W$, the vector $v = \left(w_1, \ldots, w_l, w'_1, \ldots, w'_{m-l}\right)$ lies in the orientation θ.

Moreover, if M is an oriented differentiable manifold with boundary and N is an oriented differentiable manifold, then an induced orientation exists on the manifold with boundary $M \times N$.

In the case where M is a finite-dimensional smooth manifold with boundary, the Collaring Theorem asserts the existence of an embedding $f : [0, 1) \times \partial M \to M$ onto a neighborhood of ∂M in M, satisfying $f(x) = x$ for all $x \in \partial M$. This embedding is termed a "collar" on ∂M, and ∂M is said to be "collared" in M".

7.2 Intersection Number

When considering two immersed submanifolds M and Z within a smooth manifold N that intersect transversally, denoted as $M \pitchfork Z$, their intersection $M \cap Z$ forms another immersed submanifold of N. The dimension of this intersection is given by the formula:

$$\dim(M \cap Z) = \dim(M) + \dim(Z) - \dim(N)$$

by Corollary 7.3 this intersection is non-empty only if its dimension is non-negative, implying:

$$\dim(M) + \dim(Z) \geq \dim(N)$$

If $\dim(M) + \dim(Z) < \dim(N)$ and $M \cap Z$ is still non-empty, it means M and Z do not intersect transversally. However, by the Homotopy Transversality Theorem, it's possible to smoothly perturb M such that it becomes transverse to Z. This means there exists a smooth homotopy $g_t : M \to N$ from the inclusion $i : M \to N$ to a map $g_1 : M \to N$ that intersects Z transversally. But, if $\dim(M) + \dim(Z) < \dim(N)$, the dimension of $g_1^{-1}(Z)$ becomes negative, implying it's empty. Consequently, $g_1(M) \cap Z$ is also empty, indicating that by perturbing M, it can be made disjoint from Z. Thus, there's no meaningful "intersection theory" for such M and Z.

Now, when $\dim(M) + \dim(Z) = \dim(N)$, the intersection $M \cap Z$ consists of points. If both M and Z are closed submanifolds of N, and at least one of them is compact, then $M \cap Z$ is a finite collection of points. Similarly, if $f : M \to N$ is a smooth map transverse to Z, and $\dim(M) + \dim(Z) = \dim(N)$, then $f^{-1}(Z)$ is a submanifold of M of dimension zero, implying it's also a finite collection of points provided Z is closed and M is compact.

Assuming M, N, and Z are oriented, with Z being closed and M compact, and $\dim(M) + \dim(Z) = \dim(N)$, and considering a smooth map $f : M \to N$ transverse to Z, we can assign signs to the points $x \in f^{-1}(Z)$ based on the orientation of the bases of the involved vector spaces. If the transition matrix between the bases has a positive determinant, we assign a positive sign to the point, denoted as $\mathrm{sign}(x) = +1$. Otherwise, if the determinant is negative, we assign a negative sign, denoted as $\text{sign}(x) = -1$. This sign assignment helps capture the orientation information of the intersection points.

Definition 7.13 (Intersection Number)**.**

The oriented intersection number, $I(f, Z) \in \mathbb{Z}$, is defined to be

$$I(f, Z) = \sum_{x \in f^{-1}(Z)} \text{sign}(x)$$

Remark 7.14.

The summation in question is guaranteed to be finite due to the compactness of $f^{-1}(Z)$, which is a consequence of it being a closed subset of a compact space, as per Theorem 3.5.2 from the book Topology by Munkres. It's worth noting that when Z reduces to a point, the condition $f \pitchfork Z$ indicates that this point serves as a regular value of the function f.

If the point Z possesses an orientation of $+1$, then upon transformation to the new basis $v'_y = df_x(v_x)$, the sign assigned to x is determined by whether the linear transformation df_x preserves orientation or not. Consequently, the sign of x is either $+1$ or -1.

Furthermore, it's important to observe that if the preimage of Z, denoted as $f^{-1}(Z)$, is empty, the contribution of Z to the intersection index, denoted as $I(f, Z)$, is zero.

Theorem 7.15.

Let $f, g : M \to N$ be two smooth maps which are both transverse to $Z \subset N$. If f is homotopic to g, then, then $I(f, Z) = I(g, Z)$.

Proof. Given that f and g are smooth maps homotopic as continuous maps, they are also smoothly homotopic, as per Corollary III.2.6 from Differential Manifold by Kosinski. Let $F : [0, 1] \times M \to N$ be a smooth homotopy from f to g. Since both $f \pitchfork Z$ and $g \pitchfork Z$, we can select F such that $F \pitchfork Z$, a consequence of a slightly stronger version of Homotopy Transversality Theorem for Smooth maps (This is in details in the book by

Differential Topology by Guillemin, Pollack). Considering $F \pitchfork Z$ and $\partial F \pitchfork Z$, it follows that $F^{-1}(Z)$ is an immersed submanifold of $[0,1] \times M$. By a version of Inverse Image theorem applicable to manifolds with boundary, we have:

$$1 + \dim(M) - \dim(F^{-1}(Z)) = \dim(N) - \dim(Z)$$

This results in $\dim(F^{-1}(Z)) = 1$, as $\dim(M) + \dim(Z) = \dim(N)$. Furthermore, any compact 1-dimensional manifold is orientable, with the sum of orientation numbers at the boundary points equating to zero, as established in in details in the book by Differential Topology by Guillemin, Pollack. We have:

$$\sum_{x \in \partial F^{-1}(Z)} \operatorname{sign}(x) = 0$$

Given that $\partial F^{-1}(Z) = f^{-1}(Z) \cup -g^{-1}(Z)$, with the orientation on $\partial F^{-1}(Z)$ chosen appropriately and the negative sign indicating the reversal of orientation on $g^{-1}(Z)$, we can deduce:

$$\sum_{x \in f^{-1}(Z)} \operatorname{sign}(x) = \sum_{x \in g^{-1}(Z)} \operatorname{sign}(x)$$

□

Remark 7.16.

Remark 7.12 suggests that if Z is an oriented immersed submanifold within an oriented smooth manifold N, it induces an orientation on the normal bundle of Z in N. Consequently, when $f : M \to N$ is transverse to Z, Inverse Image Theorem ensures an induced orientation on the normal bundle of $f^{-1}(Z)$. Moreover, if M is oriented, Remark 7.12 implies an induced orientation on $f^{-1}(Z)$. These principles extend to cases where M is an oriented manifold with boundary. In the context of these induced orientations, $\partial F^{-1}(Z) = f^{-1}(Z) \cup -g^{-1}(Z)$, as established in the proof of the preceding theorem.

With these considerations, we can define an intersection number $I(f, Z)$ for any smooth map $f : M \to N$ and any closed submanifold Z. This definition does not assume Z to be transverse to f, but it still requires all the other previously stated assumptions.

Definition 7.17.

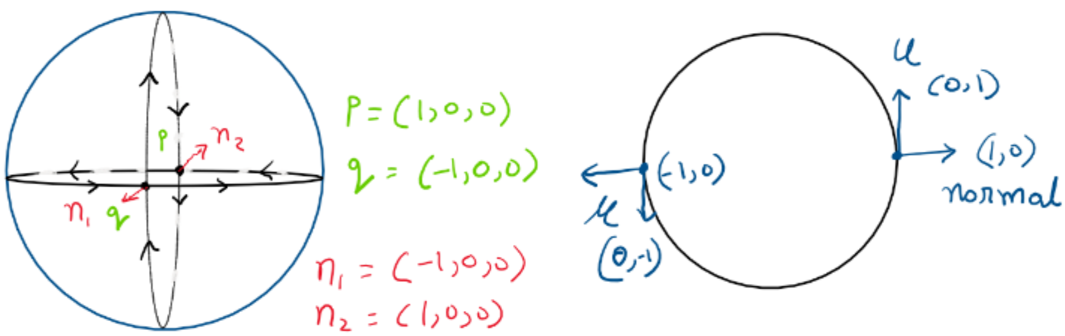
Assume that M, N, and Z are oriented smooth manifolds, Z is a closed submanifold of N, M is compact, and $\dim(M) + \dim(Z) = \dim(N)$. For any smooth map $f : M \to N$, the Homotopy Transversality Theorem (Theorem 7.7) implies that there is a smooth map f_1 transverse to Z and homotopic to f, and we define the oriented intersection number $I(f, Z)$ to be $I(f_1, Z)$. This number is well define because $I(f_1, Z)$ is independent of the choice of f_1 by Theorem 7.15.

Remark 7.18.

Every continuous map $f : M \to N$ between smooth manifolds M and N is homotopic to a smooth map. Hence, the preceeding definition of the oriented intersection number $I(f, Z)$ also applies to a continuous map $f : M \to N$.

Example 7.19.

$$\begin{array}{ll} f : S^1 \longrightarrow S^2 & f(x,y) = (x,y,0) \\ g : S^1 \longrightarrow S^2 & g(x,y) = (x,0,y) \\ & f(1,0) = (1,0,0) = g(1,0) \\ & f(-1,0) = (-1,0,0) = g(-1,0) \end{array}$$



So it has only 2 intersection point $(1,0,0),(-1,0,0)$

$$T_{(1,0)}S^1 = \langle(0,1)\rangle$$

$$df_{(1,0)}\left(T_{(1,0)}S^1\right) = \langle(0,1,0)\rangle = \langle v\rangle$$

$$dg_{(1,0)}\left(T_{(1,0)}S^1\right) = \langle(0,0,1)\rangle = \langle w\rangle$$

$$\det(v,w,n_2) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1$$

$$\text{So } \operatorname{sign}((1,0)) = 1$$

$$T_{(-1,0)}S^1 = \langle(0,-1)\rangle$$

$$df_{(-1,0)}\left(T_{(-1,0)}S^1\right) = \langle(0,-1,0)\rangle$$

$$dg_{(1,0)}\left(T_{(-1,0)}S^1\right) = \langle(0,0,-1)\rangle$$

$$\det(v,\omega,n_1) = \begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{vmatrix} = -1$$

$$\text{So } \operatorname{sign}((-1,0)) = -1$$

$$\text{If } g\left(S^1\right) = Z \text{ then } I(f,Z) = \sum_{x\in f^{-1}(Z)} \operatorname{sign}(x) = \operatorname{sign}((1,0)) + \operatorname{sing}((-1,0))$$

$$= 1 - 1$$

$$= 0$$

Chapter 8

Morse-Smale function

Definition 8.1.

A Morse function $f : M \to \mathbb{R}$ on a finite dimensional smooth Riemannian manifold (M, g) is said to satisfy the Morse-Smale transversality condition if and only if the stable and unstable manifolds of f intersect transversally, i.e.

$$W^u(q) \pitchfork W^s(p)$$

for all $p, q \in \mathrm{Cr}(f)$. A Morse function that satisfies the Morse-Smale transversality condition is called a Morse-Smale function.

The Morse-Smale transversality condition yields the following immediate consequence.

Proposition 8.2.

Let $f : M \to \mathbb{R}$ be a Morse-Smale function on a finite dimensional compact smooth Riemannian manifold (M, g). If p and q are critical points of f such that $W^u(q) \cap W^s(p) \neq \emptyset$, then $W^u(q) \cap W^s(p)$ is an embedded submanifold of M of dimension $\lambda_q - \lambda_p$.

Proof. According to Theorem 6.2, $W^u(q)$ and $W^s(p)$ emerge as smooth embedded submanifolds of M, with dimensions λ_q and $m - \lambda_p$ respectively. Utilizing Theorem 6.2 and Corollary 7.3, we determine that $W^u(q) \cap W^s(p)$ constitutes a smooth embedded submanifold whose dimension is calculated as:

$$\dim W^u(q) + \dim W^s(p) - m = \lambda_q + (m - \lambda_p) - m = \lambda_q - \lambda_p$$

□

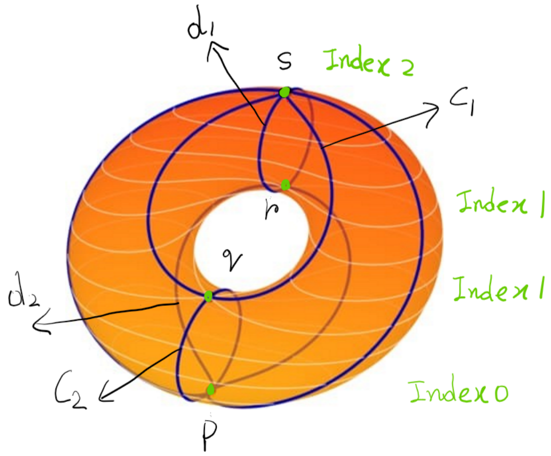
Corollary 8.3.

If $f : M \to \mathbb{R}$ is a Morse-Smale function on a finite dimensional compact smooth Riemannian manifold (M, g), then the index of the critical points is strictly decreasing along gradient flow lines. That is, if p and q are critical points of f with $W(q, p) \neq \emptyset$, then $\lambda_q > \lambda_p$.

Proof. When $W(q, p) \neq \emptyset$, it implies that there is at least one flow line from q to p contained within $W(q, p)$. Since a gradient flow line is one-dimensional, we conclude that $\dim W(q, p) \geq 1$. □

Example 8.4 (Tilted Torus)**.**

The torus T^2 positioned vertically on the plane $z = 0$ within $\mathbb{R}^3$, with the standard height function $f : T^2 \to \mathbb{R}$ as discussed in Example 4.16 and Example 6.5, does not qualify as a Morse-Smale function. This conclusion arises from Corollary 8.3, as the gradient flow lines of the standard height function $f : T^2 \to \mathbb{R}$ originate at the critical point r of index 1 and terminate at the critical point q of the same index. Nevertheless, Kupka-Smale Theorem (Theorem 8.5) suggests that, there exists an infinitesimally small perturbation of the standard height function on T^2 that does satisfy the Morse-Smale condition. One plausible approach to conceptualize such a perturbation involves tilting the torus slightly and observing the resultant gradient flow lines. So if we take the function $f : \mathbb{T}^2 \to \mathbb{R}$ as $f(x, y, z) = x + z$ instead of $f(x, y, z) = z$ then we can see that this is an example of a Morse-Smale function. Here d_1, c_2 are flow lines from index 2 critical points to index 1 and d_2, c_1 are flow lines from index 1 critical points to index 0



Theorem 8.5 (Kupka-Smale Theorem)**.**

If (M, g) is a finite dimensional compact smooth Riemannian manifold, then the set of Morse-Smale gradient vector fields of class C^r is a generic subset of the set of all gradient vector fields on M of class C^r for all $1 \leq r \leq \infty$.

Proof. Refer to Lectures on Morse Homology by Banyaga & Hurtubise. □

Now we will discuss about some corollaries of λ-Lemma. To get more details on the proof of the λ-Lemma and the corollaries refer to Lectures on Morse Homology by Banyaga & Hurtubise.

Corollary 8.6 (Transitivity for Gradient Flows)**.**

Let p, q, and r be critical points of a Morse-Smale function $f : M \to \mathbb{R}$. If $W(r, q) \neq \emptyset$ and $W(q, p) \neq \emptyset$, then $W(r, p) \neq \emptyset$. Moreover,

$$\overline{W(r,p)} \supseteq W(r,q) \cup W(q,p) \cup \{p, q, r\}.$$

The preceeding corollary allows us to define a partial ordering on the critical points of a Morse-Smale function $f : M \to \mathbb{R}$ on a finite dimensional compact smooth Riemannian manifold (M, g) as follows.

Definition 8.7.

Let p and q be critical points of $f : M \to \mathbb{R}$. We say that q is succeeded by $p, q \succeq p$, if and only if $W(q, p) = W^u(q) \cap W^s(p) \neq \emptyset$,i.e. there exists a gradient flow line from q to p. The set of critical points of f, $\mathrm{Cr}(f)$, together with the partial ordering $\succeq$ is called the phase diagram of f.

Corollary 8.8.

If p and q are critical points of relative index one, i.e. if $\lambda_q - \lambda_p = 1$, then

$$\overline{W(q,p)} = W(q,p) \cup \{p, q\}$$

Moreover, $W(q,p)$ has finitely many components, i.e. the number of gradient flow lines from q to p is finite.

Proof. $W(q,p) \cup \{p,q\}$ is closed due to Corollary 8.3, which indicates the absence of any intermediate critical points between q and p in the phase diagram of f. Consequently, $W(q,p) \cup \{p,q\} \subseteq M$ is compact, being a closed subset of a compact space.

The gradient flow lines originating from q and terminating at p constitute an open cover of $W(q,p)$. This cover can be expanded to cover $W(q,p) \cup \{p,q\}$ by incorporating small open sets in $W(q,p) \cup \{p,q\}$ surrounding p and q along with each gradient flow line.

As every open cover of a compact space possesses a finite subcover, it follows that the number of gradient flow lines from q to p must be finite. $\square$

Chapter 9

Morse Homology Theorem

Let (M, g) denote a finite-dimensional compact smooth oriented Riemannian manifold, and let $f : M \to \mathbb{R}$ be a Morse-Smale function. In this context, "Morse-Smale" implies that for all critical points p and q of f, the unstable manifold $W^u(q)$ and the stable manifold $W^s(p)$ intersect transversally. We denote the intersection as $W(q,p) = W^u(q) \cap W^s(p)$. This intersection is either empty or a smooth manifold of dimension $\lambda_q - \lambda_p$ by Proposition 8.2, where λ_q represents the index of q and λ_p denotes the index of p. In cases where $W(q,p)$ is not empty, we write $q \succeq p$. We define $\mathrm{Cr}(f)$ as the set of all critical points of f, and $\mathrm{Cr}_k(f)$ represents those critical points q with $\lambda_q = k$.

9.1 Orientation Conventions

For each $p \in \mathrm{Cr}(f)$, we select a basis B_p^u of $T_p^u M = T_p W^u(p)$ to establish the orientation of $T_p W^u(p)$. This orientation of $T_p^u M$ subsequently determines an orientation of $T_p^s M = T_p W^s(p)$, as $T_p M = T_p^s M \oplus T_p^u M$. Consequently, the embedded submanifolds $W^u(p)$ and $W^s(p)$ possess orientations that align with the orientation of M at p. These orientations extend to determine orientations of $T_v T_p^u M \approx T_p^u M$ for all $v \in T_p^u M$, and $T_v T_p^s M \approx T_p^s M$ for all $v \in T_p^s M$. Thus, an orientation is established on $T_x W^u(p)$ for all $x \in W^u(p)$ through the embedding $E^u : T_p^u M \to W^u(p)$ as defined by the Stable/Unstable Manifold Theorem (Theorem 6.2). Similarly, an orientation is determined on $T_x W^s(p)$ for all $x \in W^s(p)$ through the embedding $E^s : T_p^s M \to W^s(p)$.

9.2 Counting Flow Lines with Sign - The two Definitions of $n(q,p)$

Consider two critical points p, q of indices $\lambda_p = k-1$ and $\lambda_q = k$ respectively, with $q \succeq p$. Let $\gamma : \mathbb{R} \to M$ denote a gradient flow line from q to p:

$$\frac{d}{dt}\gamma(t) = -(\nabla f)(\gamma(t)), \quad \lim_{t\to-\infty}\gamma(t) = q, \quad \lim_{t\to\infty}\gamma(t) = p$$

At any point $x \in \gamma(\mathbb{R}) \subset W(q,p)$, we can complete $-(\nabla f)(x)$ to form a positive basis $\left(-(\nabla f)(x), \hat{B}^u_x\right)$ of $T_xW^u(q)$, thereby establishing the orientation of $W^u(q)$ at x. If we select any positive basis B^s_x of $T_xW^s(p)$, it determines the orientation of $W^s(p)$ at x. Consequently, $\left(B^s_x, \hat{B}^u_x\right)$ forms a basis for T_xM. Assigning +1 or -1 to the flow γ depends on whether $\left(B^s_x, \hat{B}^u_x\right)$ forms a positive orientation for T_xM. As the orientations on $W^u(q)$ and $W^s(p)$ are defined to ensure that E^s and E^u are orientation preserving, this assignment remains consistent across $x \in \gamma(\mathbb{R})$.

If $\lambda_q - \lambda_p = 1$, then $W(q,p) \cup \{q,p\}$ constitutes a compact 1-dimensional manifold (Corollary 8.8), where the flow is directed for time $t \in \mathbb{R}$. Consequently, $\mathcal{M}(q,p) = W(q,p)/\mathbb{R}$ forms a compact zero-dimensional manifold, i.e., it comprises a finite number of elements, and the count of elements in $\mathcal{M}(q,p)$ equals the number of flows γ from q to p. To each flow γ from q to p, we assign a number +1 or -1 using the orientations. The integer $n(q,p) \in \mathbb{Z}$ is defined as the sum of these numbers.

Remark 9.1.

Assigning +1 or -1 to a gradient flow line involves several choices. For example, we could complete $-(\nabla f)(x)$ to form a positive basis $\left(\hat{B}^u_x, -(\nabla f)(x)\right)$ instead of $\left(-(\nabla f)(x), \hat{B}^u_x\right)$. Similarly, we could use $\left(B^u_q, \hat{B}^s_p\right)$ instead of $\left(B^s_p, \hat{B}^u_q\right)$ as the basis for T_xM. It is evident that altering one of these conventions merely changes the sign of $n(q,p)$, thereby affecting only the sign of the Morse-Smale-Witten boundary operator.

An Alternate Definition of $n(q,p)$ Using Intersection Numbers:-

Let c be a regular value in the open interval (a,b) where $f(p) = a$ and $f(q) = b$. We consider the unstable sphere of q:

$$S^u(q) = W^u(q) \cap f^{-1}(c)$$

and the stable sphere of p:

$$S^s(p) = W^s(p) \cap f^{-1}(c)$$

inside the level set $f^{-1}(c)$. By the preimage theorem, $f^{-1}(c)$ is an $(m-1)$-dimensional manifold oriented such that for any $x \in f^{-1}(c)$, a basis $v_1, \ldots, v_{m-1}$ of $T_x f^{-1}(c)$ is positive if and only if $-(\nabla f)(x), v_1, \ldots, v_{m-1}$ forms a positive basis for $T_x M$. Additionally, Corollary 7.3 asserts that $S^u(q)$ is a $(k-1)$-dimensional manifold and $S^s(p)$ is an $(m-k)$-dimensional manifold, with orientations following the same convention as $f^{-1}(c)$.

Since the manifolds $S^u(q)$ and $S^s(p)$ intersect transversally in the submanifold $f^{-1}(c)$, Corollary 7.13 implies that $S^u(q) \cap S^s(p)$ is a 0-dimensional manifold, where each point corresponds to a connecting orbit in $W(q,p)$, i.e., $S^u(q) \cap S^s(p) \approx \mathcal{M}(q,p)$. As this set is finite (Corollary 8.8), the integer $n(q,p) \in \mathbb{Z}$ can also be defined as the intersection number of the oriented manifolds $S^u(q)$ and $S^s(p)$ inside the oriented manifold $f^{-1}(c)$.

Definition 9.2 (Morse-Smale-Witten Chain Complex)**.**
Let $f : M \to \mathbb{R}$ be a Morse-Smale function on a compact smooth oriented Riemannian manifold M of dimension $m < \infty$, and assume that orientations for the unstable manifolds of f have been chosen. Let $C_k(f)$ be the free abelian group generated by the critical points of index k, and let

$$C_*(f) = \bigoplus_{k=0}^{m} C_k(f)$$

The homomorphism $\partial_k : C_k(f) \to C_{k-1}(f)$ defined by

$$\partial_k(q) = \sum_{p \in Cr_{k-1}(f)} n(q,p)p$$

is called the Morse-Smale-Witten boundary operator, and the pair $(C_*(f), \partial_*)$ is called the Morse-Smale-Witten chain complex of f.

Remark 9.3.

The integer $n(q,p) \in \mathbb{Z}$ in the preceeding definition is well defined by Corollary 8.8 of the λ-Lemma.

9.3 Morse Homology Calculation

Theorem 9.4 (Morse Homology Theorem)**.**

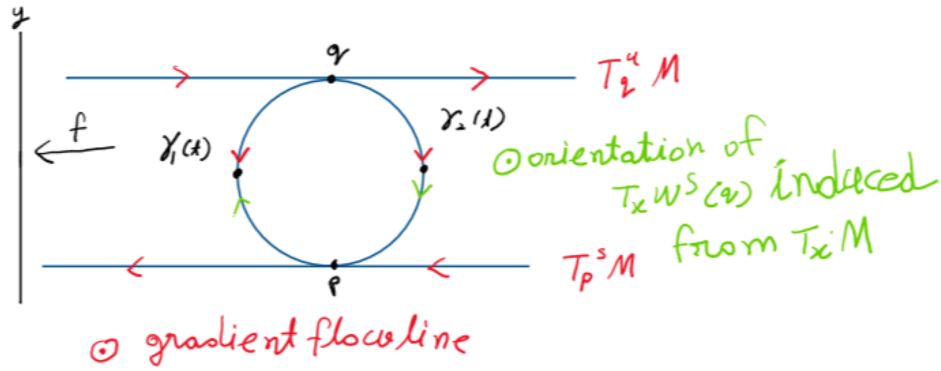
The pair $(C_*(f), \partial_*)$ is a chain complex, and its homology is isomorphic to the singular homology $H_*(M;\mathbb{Z})$.

Proof. Refer to Lectures on Morse Homology by Banyaga & Hurtubise. [Also we can say that this theorem proves the well-definedness of the Morse Homology] □

This can be also shown using the de-Rham Cohomology. But here they have used the concept of the filtered Conley index pair; then we can use the similar proof technique as CW-Homology theorem to get the desired result. Now we will try to calculate some examples using the concepts of $n(p, q)$.

Example 9.5 (S^1)**.**

f is the height function on this $M = S^1$.



Here we have choosen the orientation from left to right for the manifold and for all the unstable manifold. So T_qM, $T_q(W^u(q))$, has the same orientation. Now from the orientation of the manifold the orientation of stable manifolds induced as $T_p(W^s(p))$ from right to left. Now if x is in the left flow line from q to p $-\nabla(f)(x)$ can't be completed as a positive basis of T_xM so $n(q, p) = -1$.

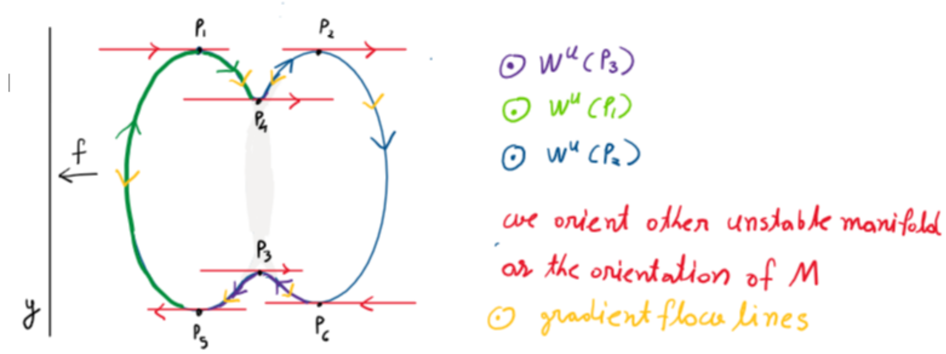
Now if x is in the right flow line from q to p $-\nabla(f)(x)$ can be completed as a positive basis of T_xM and $\hat{B}_u(x)$ is null and $B^s(x) = -\nabla(f)(x)$ so $(B^s(x), \hat{B}_u(x)) = (-\nabla(f)(x))$ and it matches with the orientation of T_xM So $n(q, p) = 1$.

$C_1(f) =< q >$, $C_0(f) =< p >$, $\partial_1(q) = p - p = 0$. So all the boundary operators will be 0. So we have

$$H_k\left((C_*(f), \partial_*)\right) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 9.6 (Deformed S^1)**.**

f is the height function on this manifold.



Here we have choosen the orientation from left to right for the manifold and for all the unstable manifold. So $T_{p_1}M$, $T_{p_2}M$, $T_{p_3}M$, $T_{p_1}(W^u(p_1))$, $T_{p_2}(W^u(p_2))$, $T_{p_3}(W^u(p_3))$ has the same orientation. Now from the orientation of the manifold the orientation of stable manifolds induced as $T_{p_4}(W^s(p_4))$ from left to right, $T_{p_5}(W^s(p_5))$, right to left, $T_{p_6}(W^s(p_6))$ right to left. Now if x is in the flow line from p_1 to p_5 $-\nabla(f)(x)$ can't be completed as a positive basis of T_xM so $n(p_1, p_5) = -1$ similarly we can say $n(p_2, p_4) = -1$,$n(p_3, p_6) = -1$.

Now if x is in the flow line from p_2 to p_6 $-\nabla(f)(x)$ can be completed as a positive basis of T_xM and $\hat{B}_u(x)$ is null and $B^s(x) = -\nabla(f)(x)$ so $(B^s(x), \hat{B}_u(x)) = (-\nabla(f)(x))$ and it matches with the orientation of T_xM So $n(p_2, p_6) = 1$ similarly we can say $n(p_1, p_4) = 1$,$n(p_3, p_5) = 1$.

$C_1(f) =< p_1, p_2, p_3 >$, $C_0(f) =< p_4, p_5, p_6 >$, $\partial_1(p_1) = p_4 - p_5$, $\partial_1(p_2) = p_6 - p_4$, $\partial_1(p_3) = p_5 - p_6$. It is easy to see that $H_1\left((C_*(f), \partial_*)\right) = \ker \partial_1 =< p_1 + p_2 + p_3 >\approx \mathbb{Z}$ since

$$\partial_1(p_1 + p_2 + p_3) = (p_4 - p_5) + (p_6 - p_4) + (p_5 - p_6) = 0$$

and $\ker \partial_0 =< p_4, p_5, p_6 >\approx \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. The image of ∂_1 is the free abelian group generated by $\partial_1(p_1) = p_4 - p_5, \partial_1(p_2) = p_6 - p_4$, and $\partial_1(p_3) = p_5 - p_6$. Hence,

$$\begin{aligned} H_0\left((C_*(f), \partial_*)\right) &= \ker \partial_0 / \operatorname{im} \partial_1 \\ &\approx < p_4, p_5, p_6 > / < p_4 - p_5, p_6 - p_4, p_5 - p_6 > \\ &\approx < p_4, p_5, p_6; p_4 = p_5 = p_6 > \\ &\approx \mathbb{Z} \end{aligned}$$

and we have

$$H_k\left((C_*(f), \partial_*)\right) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

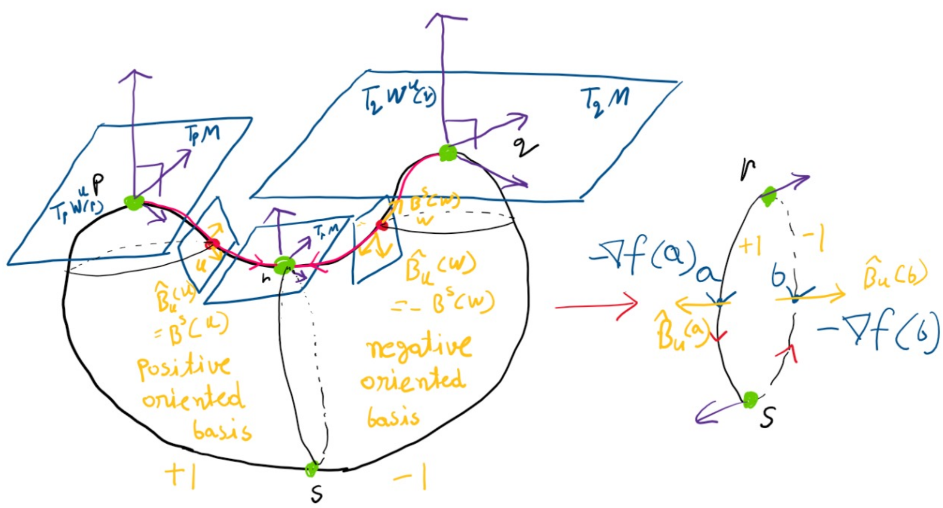
as expected.

Example 9.7 ($S^n \quad n > 1$)**.**

Here we will get two critical point of index [Refer Example 4.14] $0, n$ so all the boundary operators are all 0 so $H_j(S^n) = \mathbb{Z}$ if $j = 0, n$ otherwise it will be 0.

Example 9.8 (Deformed S^2)**.**

f is the height function on this manifold.



Here we have taken the same orientation fixing outward normal for the manifold. So T_qM, T_pM, T_rM and $T_qW^u(q)$, $T_pW^u(p)$, $T_r(W^u(r)$ have the same orientation. This implies $T_wW^u(q)$, $T_uW^u(p)$ has also the same orientation. We get the orientation of stable manifold around r from the orientation of the manifold that is from right to left.

So $B^s(w) = \nabla(f)(w)$,$B^s(u) = -\nabla(f)(u)$

Now to complete $-\nabla(f)(u)$ to a positively oriented basis of $T_pW^u(p)$ we need to take $\hat{B}_u(u)$ which is inward facing similarly we need to take $\hat{B}_u(w)$ which is outward facing. Now we can see that $(B^s(u), \hat{B}_u(u))$ is a positively oriented basis of T_uM So $n(p,r) = 1$ and $(B^s(w), \hat{B}_u(w))$ is a negatively oriented basis of T_wM So $n(q,r) = -1$. Now If we get orientation of $T_s(W^s(s))$ from front to back induced by orientation on T_sM. Then $-\nabla(f)(b)$ can be completed to a positive basis of $T_b(W^u(r))$ where $\hat{B}_u(b))$ is from left to right. $(B^s(b), \hat{B}_u(b))$ is a negatively oriented basis of T_bM as $B^s(b) = \nabla(f)(b)$. $n(r,s) = -1$. Now$-\nabla(f)(b)$ can be completed to a positive basis of $T_a(W^u(r))$ where $\hat{B}_u(b))$ is from right to left. $(B^s(a), \hat{B}_u(a))$ is a positively oriented basis of T_aM as $B^s(a) = -\nabla(f)(a)$. So the sign is negative. So $n(r,s) = +1$

So we get $C_1(f) = \langle r\rangle; C_2(f) = \langle p,q\rangle, C_0(f) = \langle s\rangle$ now $\partial_1(r) = 0; \partial_2(p) = r, \partial_2(q) = -r$.

$$\xrightarrow{0} C_2(f) \xrightarrow{\partial_2} C_1(f) \xrightarrow{\partial_1} C_0(f) \xrightarrow{\partial_0} 0$$

$$H_0\left((C_*, f)\right) = \frac{\text{Ker } \partial_0}{Im\partial_1} = \frac{\langle p\rangle}{\langle 0\rangle} \approx \mathbb{Z}$$

$$H_1\left((C_*, f)\right) = \frac{\text{Ker } \partial_1}{\text{Im } \partial_2} = \frac{\langle q\rangle}{\langle q\rangle} \approx 0$$

$$H_2\left((C_*, f)\right) = \frac{Ker\partial_2}{Im\partial_3} = \frac{\langle r+s\rangle}{\langle 0\rangle} \approx \mathbb{Z}$$

Example 9.9 ($S^m \times S^n$)**.**

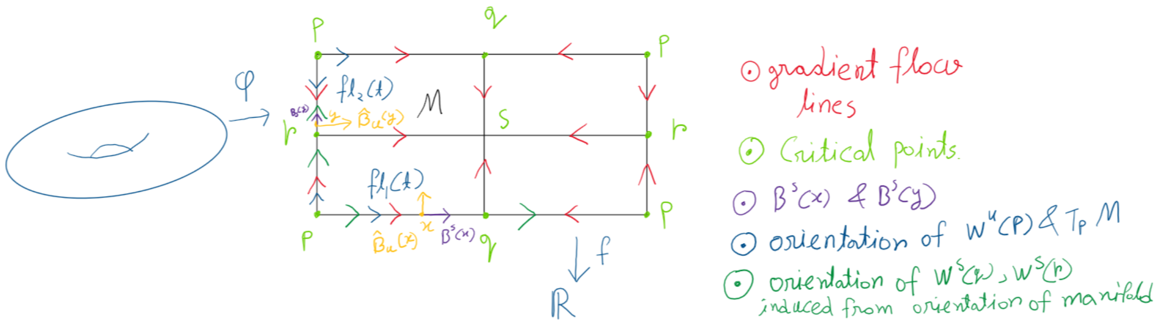
When $m \neq n \neq 1 \quad |m-n| \neq 1$ then Here will get four critical point of index $0, n, m, m+n$ so all the boundary operators are all 0 so $H_j(S^m \times S^n) = \mathbb{Z}$ if $j = 0, m, n, m+n$ otherwise it will be 0.

Now when either $m = 1$ or $n = 1 \quad |m-n| \neq 1$ then we will get the gradient calculation will be similar for index 1 to index 0 as Example 6.14 [Refer to example 4.21] so the boundary operators will be 0 as calculated in Example 9.5.

If $|m-n| = 1$ then if one of them is 1 we can get the boundary operator from previous example now after that we can see that we can deal with it similarly as the half part will be constant through out the calculation. [Refer to 6.14,4.21]. So the boundary operators will be 0 as calculated in Example 9.5.

Example 9.10 ($\mathbb{T}^2$).

As the previous height function is not Morse-Smale we are defining new function which is Morse-Smale and easy to calculate.



Torus is diffeomorphic to the quotient manifold $[0,1]\times[0,1]/\sim[(x,1)\sim(x,0)]$ and $[(0,y)\sim(1,y)]$. Now we define Morse function from M to $\mathbb{R}.f(x,y)=\cos(2\pi x)+\cos(2\pi y)$. So $\nabla f(x,y)=-2\pi(\sin(2\pi x),\sin 2\pi y)$. and critical points are $P=(0,0)=(0,1)=(1,0)=(1,1)$ $q=\left(0,\frac{1}{2}\right)=\left(1,\frac{1}{2}\right)$ $r=\left(\frac{1}{2},1\right)=\left(\frac{1}{2},0\right)$ $s=\left(\frac{1}{2},\frac{1}{2}\right)$

Index calculated as $\lambda_p=2,\lambda_r=1,\lambda_h=1,\lambda_s=0$ Now we will calculate the gradient flow lines. [M is compact] $\gamma:\mathbb{R}\to M$ is a flow line through (x,y)

$\gamma(t)=(\gamma_1(t),\gamma_2(t))$ and $\gamma_1(0)=x,\gamma_2(0)=y$

$$-\nabla f(\gamma(t))=(\gamma_1'(t),\gamma_2'(t))$$

After calculating this we get $(\gamma_1(t),\gamma_2(x))=\left(\frac{\tan^{-1}e^{c(x)t}}{\pi},\frac{\tan^{-1}e^{c(y)t}}{\pi}\right)$ $c(x)\&c(y)$ is a constant depending on x and y. Now

$$t\to\infty\quad\gamma(t)\to\left(\frac{1}{2},\frac{1}{2}\right)=s$$

[when $x\neq 0,1$ or $y\neq 0,1$]

$$t\to-\infty\quad\gamma(t)\longrightarrow p=(0,0),(1,1),(1,0),(0,1)$$

for point $(P,K),(k,P)$ when $P\in\left\{0,1,\frac{1}{2}\right\}$ we get f low line $\left(P,\frac{\tan^{-1}e^{c(k)t}}{\pi}\right),\left(\frac{\tan^{-1}e^{c(k)t}}{\pi},P\right)$ $c(k)>0\quad\forall 0<k<\frac{1}{2}$ and $c(k)<0\quad\forall\frac{1}{2}<k<1$.

Now we choose orientation by choosing normal outward direction. Our goal is now to calculate sign for $fl_1(t),fl_2(t)$ others will follow similarly we give orientation to $T_p\left(W^u(p)\right)$

same as manifold. Now we get induced orientation for $T_q\left(W^s(q)\right)$ and $T_r\left(W^s(r)\right)$ now $-\nabla(f)(x), -\nabla(f)(y)$ completes the positive oriented basis using $\hat{B}_u(x)$ & $\hat{\beta}_u(y)$. Now from the stable manifold we get $B^s(x) \& B^s(y)$. After getting this we can check that $\left(B^s(x), \hat{B}_u(x)\right)$ is positively oriented but $\left(B^s(y), \hat{B}_u(y)\right)$ is negatively oriented basis of T_xM and T_yM so $n(p,r) = -1$ and $n(p,q) = 1$ for this $x \in fl_1(t), y \in fl_2(t)$.

This way we get. $\partial_2(p) = q - q + r - r = 0, \partial_1(q) = s - s = 0$, $\partial_1(r) = s - s = 0$; so every boundary operator is ' 0 '.

$$\text{so } H_j\left(\mathbb{T}^2\right) = H_j(M) = \begin{cases} 0 & j > 2 \\ \langle p \rangle \simeq \mathbb{Z} & j = 2 \\ \langle q, r \rangle \simeq \mathbb{Z} \oplus \mathbb{Z} & j = 1 \\ \langle s \rangle \simeq \mathbb{Z} & j = 0 \end{cases}$$

Chapter 10

Morse Homology of Grassmannian

10.1 Morse Theory on the adjoint orbit of a Lie group

Definition 10.1.

A Lie group is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$, given by

$$m(g,h) = gh, \quad i(g) = g^{-1},$$

are both smooth.

1. A Lie group is, in particular, a topological group.

2. It is traditional to denote the identity element of an arbitrary Lie group by the symbol e.

A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ consists of a real vector space $\mathfrak{g}$ equipped with a bilinear operator $[] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket. It satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:

1. **Antisymmetry**: $[X,Y] = -[Y,X]$
2. **Jacobi Identity**: $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

If G is a compact Lie group, then its tangent space at the identity T_eG can be identified with the set of left-invariant vector fields on G. Under this identification, $T_eG = \mathfrak{g}$ forms a Lie algebra with the Lie bracket operation on vector fields.

The smooth representation of a Lie group G on its Lie algebra $\mathfrak{g}$ is called the adjoint representation:

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$$

The differential at the identity of G gives a smooth map:

$$\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g})$$

This makes the following diagram commute:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{ad} & \text{End}(\mathfrak{g}) \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\ G & \xrightarrow{Ad} & \text{Aut}(\mathfrak{g}) \end{array}$$

Here, $\exp(X)$ is defined as the value at 1 of the unique 1-parameter subgroup $\alpha : \mathbb{R} \to G$ whose tangent vector at zero is $X \in \mathfrak{g}$. Furthermore, for any $X, Y \in \mathfrak{g}$, we have $\text{ad}(X)(Y) = [X, Y]$.

Given $x_0 \in \mathfrak{g}$, let G_{x_0} denote the isotropy group of the adjoint representation at x_0, defined as:

$$G_{x_0} = \{g \in G \mid \text{Ad}(g)(x_0) = x_0\} \subseteq G$$

and let $G \cdot x_0$ denote the orbit of the adjoint representation at x_0, defined as:

$$G \cdot x_0 = \{\text{Ad}(g)(x_0) \mid g \in G\} \subseteq \mathfrak{g}$$

The homogeneous space G/G_{x_0} and the orbit $G \cdot x_0$ inherit smooth structures, and the map $h : G/G_{x_0} \to G \cdot x_0$ given by $h([g]) = \text{Ad}(g)(x_0)$ is a G-equivariant diffeomorphism.[Refer to Foundations of Differentiable Manifolds and Lie Groups by Warner.]

Definition 10.2.

Let $x_0 \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of a compact Lie group G. For any $A \in \mathfrak{g}$ we define the function $f_A : G \cdot x_0 \to \mathbb{R}$ by

$$f_A(x) =< x, A >$$

for all $x \in G \cdot x_0$, where $<,>: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is an inner product on $\mathfrak{g}$.

The function f_A depends on the choice of the inner product on the real vector space $\mathfrak{g}$.

Definition 10.3.

A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on a Lie algebra $\mathfrak{g}$ is said to be associative if and only if it satisfies

$$B([X,Y],Z) = B(X,[Y,Z])$$

for all $X, Y, Z \in \mathfrak{g}$, where [,] denotes the Lie bracket on $\mathfrak{g}$.

Now we for any matrix group we will see some definition and results.

$$\begin{aligned}\mathrm{Ad}(g)(X) &= gXg^{-1} \\ \mathrm{ad}(X)(Y) &= [X,Y] = XY - YX \\ \exp(X) &= e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!} \\ \det e^X &= e^{\mathrm{trace}(X)}\end{aligned}$$

for all $g \in G$ and for all $X, Y \in \mathfrak{g}$. Moreover, using the properties of the trace it is easy to show that the trace form

$$B(X,Y) \stackrel{\mathrm{def}}{=} \operatorname{Re}\operatorname{trace}(XY)$$

is an associative bilinear form, i.e.

$$\begin{aligned}\operatorname{Re}\operatorname{trace}([X,Y]Z) = \operatorname{Re}\operatorname{trace}((XY-YX)Z) &= \operatorname{Re}\operatorname{trace}(XYZ - YXZ) \\ &= \operatorname{Re}\operatorname{trace}(XYZ) - \operatorname{Re}\operatorname{trace}(YXZ) \\ &= \operatorname{Re}\operatorname{trace}(XYZ) - \operatorname{Re}\operatorname{trace}(XZY) \\ &= \operatorname{Re}\operatorname{trace}(X(YZ - ZY)) \\ &= \operatorname{Re}\operatorname{trace}(X[Y,Z])\end{aligned}$$

for all $X, Y, Z \in \mathfrak{g}$, that is symmetric and invariant under the adjoint representation.

Lemma 10.4.

Suppose that G is a matrix group with Lie algebra $\mathfrak{g}$ that satisfies the following condition: if $X \in \mathfrak{g}$, then ${}^t\bar{X} \in \mathfrak{g}$. Then the trace form is nondegenerate on $\mathfrak{g}$. Moreover, if $G = U(n), SU(n)$, or $SO(n)$, then the trace form is negative definite on the Lie algebra $\mathfrak{g}$.

Proof. Suppose that $X \in \mathfrak{g}$ satisfies $B(X, Y) = 0$ for all $Y \in \mathfrak{g}$. Then taking $Y = {}^t\bar{X}$ we have

$$\begin{aligned} 0 &= B\left(X, {}^t\bar{X}\right) \\ &= \operatorname{Re}\operatorname{trace}\left(X^t\bar{X}\right) \\ &= \operatorname{Re}\sum_{i,j} X_{ij}\bar{X}_{ij} \\ &= \sum_{ij} |X_{ij}|^2 \end{aligned}$$

and we see that $X = 0$. For $G = U(n), SU(n)$, or $SO(n)$, we have $X = -{}^t\bar{X}$ for any $X \in \mathfrak{g}$, and hence,

$$B(X, X) = -\sum_{ij} |X_{ij}|^2$$

Hence, for $G = U(n), SU(n)$, and $SO(n)$, the negative of the trace form is an associative inner product on the Lie algebra $\mathfrak{g}$. $\square$

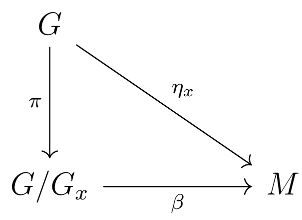
From now on we will assume that the Lie algebra $\mathfrak{g}$ has an associative inner product. Our next step is to describe the tangent and normal spaces of the orbit $G \cdot x_0$ in terms of the Lie algebra structure.

Lemma 10.5.

Let $\eta : G \times M \to M$ be a transitive smooth action of a compact Lie group G on a smooth manifold M, and let $x \in M$. For all $X \in T_xM$ there exists a $Y \in \mathfrak{g}$ such that

$$\left.\frac{d}{dt}\eta(\exp(tY), x)\right|_{t=0} = X$$

Proof. Let's define $\eta_x : G \to M$ as $\eta_x(g) = \eta(g, x)$ for all $g \in G$. Then, the following diagram commutes:



Here, β is a diffeomorphism and π is a submersion. Hence, η_x is a submersion [Refer to Foundations of Differentiable Manifolds and Lie Groups by Warner]. In particular, $d\eta_x|_e : \mathfrak{g} \to T_xM$ is surjective, where e is the identity of G.

Given $X \in T_xM$, we can choose a $Y \in \mathfrak{g}$ such that $d\eta_x(Y) = X$. Then,

$$\begin{aligned} \frac{d}{dt}\eta(\exp(tY), x)\Big|_{t=0} &= \frac{d}{dt}\eta_x(\exp(tY))\Big|_{t=0} \\ &= d\eta_x(Y) \\ &= X \end{aligned}$$

The second equality follows, for instance, from Remark 3.36 of Foundations of Differentiable Manifolds and Lie Groups by Warner. □

Lemma 10.6.

The tangent space at $x \in G \cdot x_0$ is given by

$$T_x(G \cdot x_0) = [\mathfrak{g}, x] = [x, \mathfrak{g}]$$

Proof. We will prove this for only matrix group. In the preceding lemma, if we consider $X \in T_x(G \cdot x_0)$, it implies the existence of a $Y \in \mathfrak{g}$ such that

$$\frac{d}{dt}(\mathrm{Ad}(\exp(tY))(x))\Big|_{t=0} = X$$

$$\begin{aligned} \frac{d}{dt}(\mathrm{Ad}(\exp(tY))(x))\Big|_{t=0} &= \frac{d}{dt}((\exp(tY))x(\exp(tY))^{-1})\Big|_{t=0} \\ &= \frac{d}{dt}((\exp(tY))x(\exp(-tY)))\Big|_{t=0} \\ &= \{((Y\exp(tY))x(\exp(tY))^{-1}) - ((\exp(tY))x(Y\exp(tY))^{-1})\}\big|_{t=0} \\ &= \{YIxI - IxYI\} = Yx - xY = [Y, x] \end{aligned}$$

Therefore, we have $T_x(G \cdot x_0) \subseteq [\mathfrak{g}, x]$. Conversely, for any $Y \in \mathfrak{g}$, the equality

$$[Y, x] = \frac{d}{dt}(\mathrm{Ad}(\exp(tY))(x))\Big|_{t=0}$$

shows that $[Y, x]$ represents the derivative of a path in $G \cdot x_0$ passing through x at $t = 0$. Consequently, $[\mathfrak{g}, x] \subseteq T_x(G \cdot x_0)$. Thus, we have

$$T_x(G \cdot x_0) = [\mathfrak{g}, x] = [x, \mathfrak{g}]$$

where the second equality arises from the skew-symmetry property of a Lie bracket. □

Lemma 10.7.

The normal space at $x \in G \cdot x_0 \subseteq \mathfrak{g}$ with respect to an associative inner product on $\mathfrak{g}$ is given by

$$N_x (G \cdot x_0) = \{Z \in \mathfrak{g} \mid [Z, x] = 0\}.$$

Proof. The vector $Z \in \mathfrak{g}$ is orthogonal to the tangent space at $x \in G \cdot x_0$ if and only if one (and hence all) of the following equivalent conditions hold:

$$\begin{aligned} &< Z, X >= 0 \quad \text{for all} \quad X \in T_x(G \cdot x_0) \\ \Leftrightarrow &< Z, [x, Y] >= 0 \quad \text{for all} \quad Y \in \mathfrak{g} \\ \Leftrightarrow &< [Z, x], Y >= 0 \quad \text{for all} \quad Y \in \mathfrak{g} \\ \Leftrightarrow &[Z, x] = 0 \end{aligned}$$

The second equivalence follows from associativity, and the last equivalence follows from the fact that an inner product is nondegenerate. □

Lemma 10.8.

For any $x \in G \cdot x_0$ and for any $X \in T_x (G \cdot x_0)$ the directional derivative of $f_A : G \cdot x_0 \to \mathbb{R}$ in the direction of X is

$$D_X f_A =< X, A > .$$

Proof. First, observe that the function $f_A : G \cdot x_0 \to \mathbb{R}$ naturally extends to $\tilde{f}_A : \mathfrak{g} \to \mathbb{R}$. Furthermore, along any tangent direction to $G \cdot x_0$, the directional derivatives of f_A and $\tilde{f}_A$ coincide. Therefore, we can compute the partial derivative of f_A in the direction $X \in T_x (G \cdot x_0)$ as follows.

$$\begin{aligned} D_X f_A &= \lim_{t \to 0} \frac{1}{t} \left[\tilde{f}_A(x + tX) - \tilde{f}_A(x) \right] \\ &= \lim_{t \to 0} \frac{1}{t} [< x + tX, A > - < x, A >] \\ &= \lim_{t \to 0} \frac{1}{t} [t < X, A >] \\ &=< X, A > \end{aligned}$$

□

Lemma 10.9.

A point $p \in G \cdot x_0$ is a critical point of $f_A : G \cdot x_0 \to \mathbb{R}$ if and only if

$$[p, A] = 0.$$

Proof. The point $p \in G \cdot x_0$ is a critical point of f_A if and only if one (and hence all) of the following equivalent conditions hold:

$$\begin{array}{rll} & D_X f_A = 0 & \text{for all } X \in T_p(G \cdot x_0) \\ \Leftrightarrow & \langle X, A \rangle = 0 & \text{for all } X \in T_p(G \cdot x_0) \\ \Leftrightarrow & \langle [Z, p], A \rangle = 0 & \text{for all } Z \in \mathfrak{g} \\ \Leftrightarrow & \langle Z, [p, A] \rangle = 0 & \text{for all } Z \in \mathfrak{g} \\ \Leftrightarrow & [p, A] = 0 & \end{array}$$

Here, the third equivalence follows from associativity, and the last equivalence follows from the fact that an inner product is nondegenerate. □

10.2 A Morse Function on an adjoint orbit of the unitary group

Let's now specialize to the scenario where $G = U(n+k) = \left\{ A \in M_{(n+k)\times(n+k)}(\mathbb{C}) \middle|^t \bar{A}A = I_{n\times n} \right\}$ and the associated Lie algebra $\mathfrak{g} = \mathfrak{u}(n+k) = \left\{ A \in M_{(n+k)\times(n+k)}(\mathbb{C}) \middle|^t \bar{A} = -A \right\}$, where the inner product $\langle \cdot, \cdot \rangle$ is defined as the negative of the trace form. In other words,

$$\langle A, B \rangle = -\operatorname{trace}(AB)$$

for all $A, B \in \mathfrak{u}(n+k)$. Additionally, we designate a specific point and $x_0 = \begin{pmatrix} iI_{n\times n} & 0_{n\times k} \\ 0_{k\times n} & 0_{k\times k} \end{pmatrix} \in \mathfrak{u}(n+k)$ We denote the adjoint action of $U(n+k)$ on $\mathfrak{u}(n+k)$ by $g \cdot x = \operatorname{Ad}(g)(x) = gxg^{-1}$, where $g \in U(n+k)$ and $x \in \mathfrak{u}(n+k)$. Later, we'll demonstrate that with these definitions, the orbit $U(n+k) \cdot x_0$ is diffeomorphic to $G_{n,n+k}(\mathbb{C})$, the complex Grassmann manifold comprising n-dimensional complex planes in $\mathbb{C}^{n+k}$. This will be achieved by establishing the diffeomorphism of both $G_{n,n+k}(\mathbb{C})$ and $U(n+k) \cdot x_0$ to $U(n+k)/(U(n) \times U(k))$, a smooth manifold of real dimension $2nk$. $dim(U(n+k)/(U(n) \times U(k))) = 2 \times (\frac{(n+k)(n+k+1)}{2} - \frac{n(n+1)}{2} - \frac{k(k+1)}{2} = 2nk$

Lemma 10.10.

The function $f_A : U(n+k)\cdot x_0 \to \mathbb{R}$ given by $f_A(x) = \langle x, A >$ satisfies

$$f_A(x) = -\frac{1}{2}\left(g_A(x) - C\right)$$

for some constant $C \in \mathbb{R}$ where $g_A(x) \stackrel{\text{def}}{=} \|x - A\|^2$ for all $x \in U(n+k)\cdot x_0$. Hence, the functions f_A and g_A have the same critical points, and a critical point p is degenerate for f_A if and only if it is degenerate for g_A. Moreover, a non-degenerate critical p of index λ_p for the function f_A is a non-degenerate critical point of index $2nk - \lambda_p$ for the function g_A.

Proof. Given that the trace form is invariant under the adjoint representation, we have $< g\cdot x_0, g\cdot x_0 >=< gx_0g^{-1}, gx_0g^{-1} >= \operatorname{Re}\operatorname{trace}(gx_0g^{-1}gx_0g^{-1}) = \operatorname{Re}\operatorname{trace}(gx_0x_0g^{-1}) = \operatorname{Re}\operatorname{trace}(x_0x_0g^{-1}g) = \operatorname{Re}\operatorname{trace}(x_0x_0) =< x_0, x_0 >$ for all $g \in U(n+k)$. Thus, $\|x\|^2 = \|x_0\|^2$ holds for every $x \in U(n+k)\cdot x_0$, meaning the orbit $U(n+k)\cdot x_0$ resides within the sphere of radius $\|x_0\|$ in $\mathfrak{u}(n+k)$. Consequently,

$$\begin{aligned} g_A(x) &= \|x - A\|^2 \\ &=< x - A, x - A > \\ &=< x, x > -2 < x, A > + < A, A > \\ &= -2 < x, A > +C \end{aligned}$$

for all $x \in U(n+k)\cdot x_0$, where $C = \|x_0\|^2 + \|A\|^2$. □

Let M be a manifold embedded in some Euclidean space $\mathbb{R}^r$. Define a function $E : N \to \mathbb{R}^r$ by $E(x, \vec{v}) = x + \vec{v}$ where N is the total space of the normal bundle of M in $\mathbb{R}^r$, i.e.

$$N = \{(x, \vec{v}) \in \mathbb{R}^r \times \mathbb{R}^r \mid x \in M \text{ and } \vec{v} \in N_x(M)\}$$

Definition 10.11 (Focal point)**.**

A point $e \in \mathbb{R}^r$ is called a focal point of $x \in M$ with multiplicity μ if and only if $E(x, \vec{v}) = e$ for some $\vec{v}$ with $(x, \vec{v}) \in N$ and the Jacobian of $E : N \to \mathbb{R}^r$ at $(x, \vec{v})$ has nullity $\mu > 0$.

Now we will discuss an Example of focal point of S^1.

Center as a focal point of S^1

Tangent at $(\cos t, \sin t)$ will be $(-\sin t, \cos t)$ so Normal at $(\cos t, \sin t)$ will be $(-\cos t, -\sin t)$

$$N_{S^1} = \{(\cos t, \sin t, x\cos t, x\sin t) \mid t \in (0, 2\pi), x \in \mathbb{R}\}$$

$$E : N_{S^1} \to \mathbb{R}^2 \quad E(\cos t, \sin t, x\cos t, x\sin t) = (x+1)(\cos t, \sin t)$$

$$\varphi : \mathbb{R}^2 \to N_{S^1}\varphi(t, \theta) = (\cos t, \sin t, x\cos t, x\sin t)$$

$$\text{Now } |J_{E\circ\varphi}| = \begin{vmatrix} -(x+1)\sin t & \cos t \\ (x+1)\cos t & \sin t \end{vmatrix} = -(x+1)$$

so $|J_{E\cdot\varphi}| = 0$ iff $x = -1$ so $(0,0)$ is the one and only focal point.

Definition 10.12.

Let $u_1, \ldots, u_m$ be local coordinates on M. The inclusion of M into $\mathbb{R}^r$ determines r smooth functions

$$x_1(u_1, \ldots, u_m), \ldots, x_r(u_1, \ldots, u_m)$$

given by projecting onto the axes in $\mathbb{R}^r$. We will denote $\vec{x}(u_1, \ldots, u_m) = (x_1, \ldots, x_r)$. The first fundamental form associated to this coordinate system is the following symmetric $m \times m$ matrix of real valued functions:

$$(g_{ij}) = \left(\frac{\partial \vec{x}}{\partial u_i} \cdot \frac{\partial \vec{x}}{\partial u_j}\right)$$

The second fundamental form is the symmetric $m \times m$ matrix of vector valued functions $\left(\vec{\ell}_{ij}\right)$ where

$$\vec{\ell}_{ij} \overset{\text{def}}{=} \text{ normal component of } \frac{\partial^2 \vec{x}}{\partial u_i \partial u_j}.$$

Lemma 10.13.

The nullity of the Jacobian of E at $(p, t\vec{v}) \in N$ equals the nullity of an $r \times r$ matrix of the form

$$\begin{pmatrix} \left.\frac{\partial \vec{x}}{\partial u_i}\right|_p \cdot \left.\frac{\partial \vec{x}}{\partial u_j}\right|_p - t\vec{v} \cdot \vec{\ell}_{ij} & 0 \\ * & I_{(r\sim m)\times(r-m)} \end{pmatrix}$$

where $I_{(r-m)\times(r-m)}$ denotes the $(r-m)\times(r-m)$ identity matrix. Hence, $p + t\vec{v}$ is a focal point of $p \in M$ with multiplicity μ if and only if the upper left $m \times m$ minor of the above matrix is singular with nullity μ.

Proof. Locally, we select $r-m$ orthonormal vector fields

$$\vec{w}_1\left(u_1,\ldots,u_m\right),\ldots,\vec{w}_{r-m}\left(u_1,\ldots,u_m\right)$$

spanning the normal bundle of $M\subseteq\mathbb{R}^r$. We introduce local coordinates $(u_1,\ldots,u_m,t_1,\ldots,t_{r-m})$ on the total space of the normal bundle N as follows: Let $(u_1,\ldots,u_m,t_1,\ldots,t_{r-m})$ correspond to the point

$$\left(\vec{x}\left(u_1,\ldots,u_k\right),\sum_{\alpha=1}^{r-m}t_\alpha\vec{w}_\alpha\left(u_1,\ldots,u_m\right)\right)\in N$$

In these coordinates, the function $E:N\to\mathbb{R}^r$ becomes

$$\vec{e}\left(u_1,\ldots,u_m,t_1,\ldots,t_{r-m}\right)=\vec{x}\left(u_1,\ldots,u_k\right)+\sum_{\alpha=1}^{r-m}t_\alpha\vec{w}_\alpha\left(u_1,\ldots,u_m\right)$$

and its partial derivatives are

$$\begin{aligned}\frac{\partial\vec{e}}{\partial u_i}&=\frac{\partial\vec{x}}{\partial u_i}+\sum_{\alpha=1}^{r-m}t_\alpha\frac{\partial\vec{w}_\alpha}{\partial u_i}\\ \frac{\partial\vec{e}}{\partial t_\alpha}&=\vec{w}_\alpha\end{aligned}$$

By multiplying the Jacobian of $\vec{e}$ on the left by the $r\times r$ nonsingular matrix whose rows consist of the linearly independent vectors $\frac{\partial\vec{x}}{\partial u_1},\ldots,\frac{\partial\vec{x}}{\partial u_m},\vec{w}_1,\ldots,\vec{w}_{r-m}$, we obtain an $r\times r$ matrix whose nullity equals the nullity of the Jacobian of E. This matrix has the following form:

$$\begin{pmatrix}\left(\frac{\partial\vec{x}}{\partial u_i}\cdot\frac{\partial\vec{x}}{\partial u_j}+\sum_\alpha t_\alpha\frac{\vec{w}_\alpha}{\partial u_i}\cdot\frac{\partial\vec{x}}{\partial u_j}\right) & 0\\ \left(\sum_\alpha t_\alpha\frac{\partial\vec{w}_\alpha}{\partial u_i}\cdot\vec{w}_\beta\right) & I_{(r-m)\times(r-m)}\end{pmatrix}.$$

Using the identity

$$0=\frac{\partial}{\partial u_i}\left(\vec{w}_\alpha\cdot\frac{\partial\vec{x}}{\partial u_j}\right)=\frac{\partial\vec{w}_\alpha}{\partial u_i}\cdot\frac{\partial\vec{x}}{\partial u_j}+\vec{w}_\alpha\cdot\frac{\partial^2\vec{x}}{\partial u_i\partial u_j}$$

we deduce that the upper left $m\times m$ minor of the matrix is

$$\left(\frac{\partial\vec{x}}{\partial u_i}\cdot\frac{\partial\vec{x}}{\partial u_j}-\sum_{\alpha=1}^{r-m}t_\alpha\vec{w}_\alpha\cdot\vec{\ell}_{ij}\right)$$

where $\sum_{\alpha=1}^{r-m}t_\alpha\vec{w}_\alpha$ is some vector $t\vec{v}$ that is normal to M. □

Lemma 10.14.

The point $p\in M\subseteq\mathbb{R}^r$ is a critical point of $g_A(x)=\|x-A\|^2$ if and only if $A-p$ is normal to M at p. Moreover, if $\vec{v} = A - p$ is normal to M at p, then the Hessian of g_A at the critical point p satisfies

$$H_p\left(g_A\right) = 2\left(\left.\frac{\partial \vec{x}}{\partial u_i}\right|_p \cdot \left.\frac{\partial \vec{x}}{\partial u_j}\right|_p - \vec{v} \cdot \vec{\ell}_{ij}\right)$$

Proof. In line with Definition 10.12, let's denote

$$g_A\left(\vec{x}\left(u_1, \ldots, u_m\right)\right) = \|\vec{x} - A\|^2 = \vec{x} \cdot \vec{x} - 2\vec{x} \cdot A + A \cdot A$$

This gives us

$$\frac{\partial g_A}{\partial u_i} = 2\frac{\partial \vec{x}}{\partial u_i} \cdot (\vec{x} - A)$$

Observing this, we note that g_A reaches a critical point at p if and only if $p - A$ (or equivalently $A - p$) is perpendicular to M at p. The second partial derivatives of g_A are

$$\frac{\partial^2 g_A}{\partial u_i \partial u_j} = 2\left(\frac{\partial \vec{x}}{\partial u_i} \cdot \frac{\partial \vec{x}}{\partial u_j} + \frac{\partial^2 \vec{x}}{\partial u_i \partial u_j} \cdot (\vec{x} - A)\right)$$

Thus, if $\vec{v} = A - p$ is perpendicular to M at p, then p is a critical point of g_A, and the Hessian of g_A at p is represented by

$$H_p\left(g_A\right) = 2\left(\left.\frac{\partial \vec{x}}{\partial u_i}\right|_p \cdot \left.\frac{\partial \vec{x}}{\partial u_j}\right|_p - \vec{v} \cdot \vec{\ell}_{ij}\right)$$

□

Lemma 10.15.

A point $p \in M$ is a degenerate critical point of the function $g_A : M \to \mathbb{R}$ given by $g_A(x) = \|x - A\|^2$ if and only if A is a focal point of $p \in M$. The nullity of p as a critical point of g_A of is equal to the multiplicity of A as a focal point of $p \in M$.

Proof. If p represents a critical point of $g_A(x) = \|x - A\|^2$, then according to Lemma 10.14, $\vec{v} = A - p$ is perpendicular to M at p. Hence, $E(p, \vec{v}) = A$ where $(p, \vec{v}) \in N$. Furthermore, Lemmas 10.14 and 10.13 affirm that the nullity of the Hessian $H_p\left(g_A\right)$ and the nullity of the Jacobian of E at $(p, \vec{v})$ are equivalent. □

Lemma 10.16.

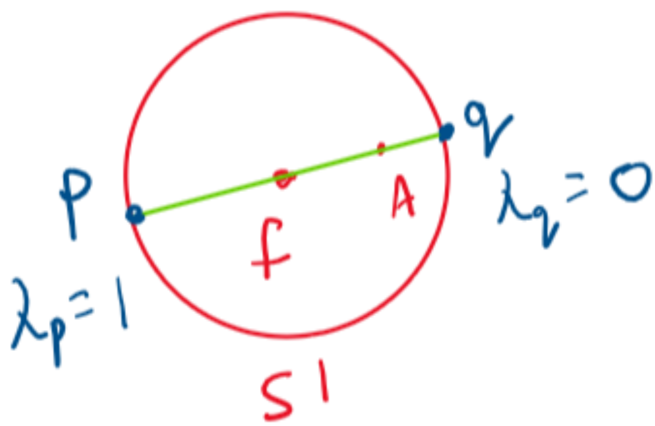
The index of $g_A = \|x - A\|^2$ at a non-degenerate critical point $p \in M$ is equal to the number of focal points of $p \in M$ that lie on the line segment from p to A; each focal point being counted with multiplicity.

Proof. The index of the matrix

$$H_p(g_A) = 2\left(\left.\frac{\partial \vec{x}}{\partial u_i}\right|_p \cdot \left.\frac{\partial \vec{x}}{\partial u_j}\right|_p - \vec{v} \cdot \vec{\ell}_{ij} \right)$$

equals the count of its negative eigenvalues. By choosing local coordinates such that $\left(\left.\frac{\partial \vec{x}}{\partial u_i}\right|_p \cdot \left.\frac{\partial \vec{x}}{\partial u_j}\right|_p \right)$ becomes the identity matrix, this count equates to the number of eigenvalues of $\left(\vec{v} \cdot \vec{\ell}_{ij}\right)$ that exceed 1.

For each eigenvalue $\lambda > 1$, according to Lemma 10.13, the point $p + \frac{1}{\lambda}\vec{v}$ stands as a focal point of $p \in M$. Furthermore, the multiplicity of the focal point $p + \frac{1}{\lambda}\vec{v}$ matches the multiplicity of λ as an eigenvalue. Similarly, if $p + t\vec{v}$ represents a focal point of $p \in M$ with multiplicity μ where $0 < t < 1$, Lemma 10.13 suggests that $\frac{1}{t} > 1$ emerges as an eigenvalue of $\left(\vec{v} \cdot \vec{\ell}_{ij}\right)$ with multiplicity μ. □



As the focal point of S^1 is the center of the S^1 denoted as f here.and if we choose any point inside the circle as A then we can see that the previous theorem is true as only critical point is p, q the diagonally opposite point of the circle where the diagonal passes through A. As line joining between A, p passes through f with nullity of J_E as 1 so

$\lambda_p = 1$. As line joining between A, q doesn't passes through f(only focal point of S^1) so $\lambda_q = 0$.

Remark 10.17.

When $\vec{v}$ serves as a unit vector normal to M at p, the matrix $\left(\vec{v} \cdot \vec{\ell}_{ij}\right)$ earns the title of the second fundamental form of M at p in the direction of $\vec{v}$. The eigenvalues of this matrix are recognized as the principal curvatures of M at p in the normal direction of $\vec{v}$, while the reciprocals of these eigenvalues are termed the principal radii of curvature.

We will now utilize the preceding results for the manifold $M = U(n+k) \cdot x_0 \subset \mathfrak{u}(n+k)$. It's worth noting that according to Proposition 10.7,

$$N = \{(x, Z) \in \mathfrak{u}(n+k) \times \mathfrak{u}(n+k) \mid x \in U(n+k) \cdot x_0 \text{ and } [Z, x] = 0\}$$

To determine the Jacobian of $E : N \to \mathfrak{u}(n+k)$, we'll require a basis for the tangent space of N.

Lemma 10.18.

Let $N \subset \mathfrak{g} \times \mathfrak{g}$ be the total space of the normal bundle of $G \cdot x_0$, let $x \in G \cdot x_0$, and let $Z \in N_x (G \cdot x_0)$. If $[Y_1, x], \ldots, [Y_{2nk}, x]$ is a basis for $T_x (G \cdot x_0)$ and $Z_{2nk+1}, \ldots, Z_{2\binom{n+k}{2}}$ is a basis for $N_x (U(n+k) \cdot x_0)$, then

$$\begin{array}{ccc} X_1 & = & ([Y_1, x], [Y_1, Z]) \\ \vdots & \vdots & \vdots \\ X_{2nk} & = & ([Y_{2nk}, x], [Y_{2nk}, Z]) \\ X_{2nk+1} & = & \left(\overrightarrow{0}, Z_{2nk+1}\right) \\ \vdots & \vdots & \vdots \\ X_{2\binom{n+k}{2}} & = & \left(\overrightarrow{0}, Z_{2\binom{n+k}{2}}\right). \end{array}$$

is a basis for $T_{(x,Z)}N$.

Proof. For $j = 1, \ldots, 2nk$, let's define paths $\gamma_j : \mathbb{R} \to U(n+k) \cdot x_0$ by $\gamma_j(t) = (\exp tY_j) \cdot x$. These paths through x satisfy $\gamma_j'(0) = [Y_j, x]$ for all $j = 1, \ldots, 2nk$. We assert that

$\gamma_j^{\perp}(t) = (\exp tY_j) \cdot Z \in N_{\gamma_j(t)}(U(n+k) \cdot x)$ for all $j = 1, \ldots, 2nk$. To comprehend this, observe that:

$$\begin{aligned}\left[\gamma_j(t), \gamma_j^{\perp}(t)\right] &= [(\exp tY_j) \cdot x, (\exp tY_j) \cdot Z] \\ &= (\exp tY_j) \cdot [x, Z] \\ &= (\exp tY_j) \cdot 0 \\ &= 0\end{aligned}$$

Thus, $p_j(t) = \left(\gamma_j(t), \gamma_j^{\perp}(t)\right) \in N$ for all $t \in \mathbb{R}$ and $p_j(0) = (x, Z)$. Consequently, $p_j'(0) = ([Y_j, x], [Y_j, Z]) \in T_{(x,Z)}N$ for all $j = 1, \ldots, 2nk$.

Now, let's define $p_j(t) = (x, Z + tZ_j)$ for all $j = 2nk+1, \ldots, 2\binom{n+k}{2}$. Clearly, $p_j(t) \in N$ for all $t \in \mathbb{R}$ and $p_j(0) = (x, Z)$. Therefore, the derivative $p_j'(0) = \left(\overrightarrow{0}, Z_j\right) \in T_{(x,Z)}N$ for all $j = 2nk+1, \ldots, 2\binom{n+k}{2}$. The linear independence stems evidently from the selection of Y_j's and Z_j's. □

Lemma 10.19.

Let $(x, Z) \in N$. For all $j = 1, \ldots, 2nk$ we have,

$$D_{X_j}E\big|_{(x,Z)} = [Y_j, x] + [Y_j, Z],$$

and for all $j = 2nk+1, \ldots, 2\binom{n+k}{2}$ we have,

$$D_{X_j}E\big|_{(x,Z)} = Z_j.$$

Proof. First we need to extend E to whole $\mathfrak{u}(n+k) \times \mathfrak{u}(n+k)$.

For any $j = 1, \ldots, 2nk$ we have,

$$\begin{aligned}D_{X_j}E\big|_{(x,Z)} &= \lim_{t \to 0} \frac{1}{t}\left(E\left((x, Z) + tX_j\right) - E(x, Z)\right) \\ &= \lim_{t \to 0} \frac{1}{t}\left(E\left(x + t[Y_j, x], Z + t[Y_j, Z]\right) - (x + Z)\right) \\ &= \lim_{t \to 0} \frac{1}{t}\left(x + t[Y_j, x] + Z + t[Y_j, Z] - x - Z\right) \\ &= [Y_j, x] + [Y_j, Z].\end{aligned}$$

For any $j = 2nk+1, \ldots, 2\binom{n+k}{2}$ we have,

$$\begin{aligned} D_{X_j}E\big|_{(x,Z)} &= \lim_{t\to 0}\frac{1}{t}\left(E\left((x,Z)+tX_j\right)-E(x,Z)\right) \\ &= \lim_{t\to 0}\frac{1}{t}\left(E\left(x,Z+tZ_j\right)-(x+Z)\right) \\ &= \lim_{t\to 0}\frac{1}{t}\left(x+B+tZ_j-x-B\right) \\ &= Z_j. \end{aligned}$$

□

Theorem 10.20.

If $A \in \mathfrak{u}(n+k)$ has distinct eigenvalues, then the function $f_A : U(n+k)\cdot x_0 \to \mathbb{R}$ given by $f_A(x) =< x, A >$ is Morse function.

Proof. Let $A \in \mathfrak{u}(n+k)$ be a matrix with distinct eigenvalues, and $p \in U(n+k)\cdot x_0$ be a critical point of f_A. Given that $p \in U(n+k)\cdot x_0$, there exists $g \in U(n+k)$ such that $g\cdot p = x_0$. Since the inner product $<,>$ is invariant under the adjoint representation, we can express $f_A(x) =< x, A >$ as $f_A(x) =< g\cdot x, g\cdot A >= f_{g\cdot A}(g\cdot x)$ for all $x \in U(n+k)\cdot x_0$. Consequently, p is a non-degenerate critical point of f_A if and only if x_0 is a non-degenerate critical point of the function $f_{g\cdot A}$. As A and $g\cdot A = gAg^{-1}$ share the same eigenvalues, it suffices to prove the theorem in the case where the critical point $p = x_0$.

According to Proposition 10.9, x_0 is a critical point of f_A if and only if $[x_0, A] = 0$, or equivalently, if and only if x_0 commutes with A. Since x_0 and A commute, there exists $g \in U(n+k)$ such that both gx_0g^{-1} and gAg^{-1} are diagonal. Successive conjugations by permutation matrices can bring gx_0g^{-1} back to x_0 while maintaining gAg^{-1} diagonal. Hence, there exists a $g \in U(n+k)$ such that $g\cdot x_0 = x_0$ and $g\cdot A$ is diagonal. By the same reasoning, we can assume that A is diagonal.

As per Lemma 10.10, x_0 is a non-degenerate critical point for the function $f_A(x) =< x, A >$ if and only if it is a non-degenerate critical point for the function $g_A(x) = \|x-A\|^2$. Therefore, by Lemma 10.15, x_0 is a non-degenerate critical point of f_A if and only if the Jacobian of $E : N \to \mathfrak{u}(n+k)$ is non-singular at $(x_0, A-x_0) \in N$.

The computation of the partial derivatives of E was accomplished in Lemma 10.19. To conclude the proof, we need to select specific matrices $Y_1, \ldots, Y_{2nk}$ such that $[Y_1, x_0], \ldots, [Y_{2nk}, x_0]$ forms a basis for $T_{x_0}(U(n+k) \cdot x_0)$ and verify that $[Y_1, A], \ldots, [Y_{2nk}, A], Z_{2nk+1}, \ldots, Z_{(n+k)^2}$ are linearly independent. (Note that $[Y_j, x_0] + [Y_j, A - x_0] = [Y_j, A]$.)

The following selections demonstrate linear independence clearly. For $j = 1, \ldots, 2nk$, we choose $Y_j = \begin{pmatrix} A_{n\times n} & B_{n\times k} \\ C_{k\times n} & D_{k\times k} \end{pmatrix}$ and $C = {}^t\,\overline{B}$ by assigning a 1 or i in some entry of B with zeros elsewhere. For $j = 2nk+1, \ldots, (n+k)^2$, we select $Z_j = \begin{pmatrix} A_{n\times n} & B_{n\times k} \\ C_{k\times n} & D_{k\times k} \end{pmatrix}$ by either placing a basis element of $\mathfrak{u}(n)$ in A zeros elsewhere or by placing a basis element of $u(k)$ in D with zeros elsewhere. □

10.3 Calculating Gradient

Lemma 10.21.

Define $J(X) = [X, x]$ for all $X \in T_x(U(n+k) \cdot x_0)$. Then J is an almost complex structure on $U(n+k) \cdot x_0$, i.e. $J^2 = -1$.

Proof. According to Lemma 10.6, $X \in T_x(U(n+k) \cdot x_0)$ if and only if $X = [Y, x]$ for some $Y \in \mathfrak{u}(n+k)$. Since $g \cdot [Y, x] = [g \cdot Y, g \cdot x]$ for all $g \in U(n+k)$, we only need to verify that $[[[Y, x_0], x_0], x_0] = -[Y, x_0]$ for all $Y \in \mathfrak{u}(n+k)$.

The matrix $[Y, x_0]$ takes the form:

$$[Y, x_0] = \begin{pmatrix} 0 & -iB \\ iC & 0 \end{pmatrix}$$

where the upper left block of zeroes is $n \times n$ and the lower right block of zeroes is $k \times k$. Since

$$[[Y, x_0], x_0] = \begin{pmatrix} 0 & iB \\ -iC & 0 \end{pmatrix}$$

we observe that $[[[Y, x_0], x_0], x_0] = -[Y, x_0]$. □

Lemma 10.22.

For all $A \in \mathfrak{u}(n+k)$ and $x \in U(n+k) \cdot x_0$ the projection of A onto $T_x(U(n+k) \cdot x_0)$ is $-[[A, x], x]$.

Proof. Consider $A \in \mathfrak{u}(n+k)$ and $x \in U(n+k) \cdot x_0 \subseteq \mathfrak{u}(n+k)$. The projection of A onto $T_x(U(n+k) \cdot x_0)$ is defined as the unique vector $X \in T_x(U(n+k) \cdot x_0)$ such that $A - X \in N_x(U(n+k) \cdot x_0)$. As per Lemma 10.7, this condition is equivalent to $[A - X, x] = 0$, or $[A, x] = [X, x]$. Thus, the lemma asserts that $[A, x] = -[[[A, x], x], x]$, implying that $[-, x]$ serves as an almost complex structure on $U(n+k) \cdot x_0$. This fact was demonstrated in Lemma 10.21. □

Theorem 10.23.

The gradient vector field of f_A is $(\nabla f_A)(x) = -[[A, x], x]$.

Proof. The vector $(\nabla f_A)(x)$ is the unique element of $T_x(U(n+k) \cdot x_0)$ which satisfies

$$< (\nabla f_A)(x), X >= D_X f_A$$

for all $X \in T_x(U(n+k) \cdot x_0)$. So by Lemma 10.8,

$$< (\nabla f_A)(x), X >=< A, X >$$

for all $X \in T_x(U(n+k) \cdot x_0)$. That is, $(\nabla f_A)(x)$ is the projection of A onto the tangent space $T_x(U(n+k) \cdot x_0)$. The result now follows from the previous lemma. □

Remark 10.24.

For any $x \in U(n+k) \cdot x_0$, the path $\sigma_x(t) = \exp(t[A, x]) \cdot x$ satisfies $\sigma_x'(0) = -\nabla(f_A)(x)$. However, σ_x is not a gradient flow line. To see this we compute $\sigma_x'(t_0)$ as follows. Define $\tilde{\sigma}_x(t) = \sigma_x(t + t_0)$.

$$\begin{aligned}
\sigma_x'(t_0) &= \tilde{\sigma}_x'(0) \\
&= \frac{d}{dt} \exp((t + t_0)[A, x]) \cdot x \bigg|_{t=0} \\
&= \frac{d}{dt} \exp(t[A, x]) \cdot (\exp(t_0[A, x]) \cdot x) \bigg|_{t=0} \\
&= [[A, x], \exp(t_0[A, x]) \cdot x] \\
&\neq [[A, \sigma_x(t_0)], \sigma_x(t_0)]
\end{aligned}$$

The correct formula for the gradient flow lines of f_A are given in later; We will describe the gradient flow lines in terms of the action of $GL_{n+k}(\mathbb{C})$ on the complex Grassmann manifold $G_{n,n+k}(\mathbb{C})$.

10.4 The critical point of $f_A : U(n+k) \cdot x_0 \to \mathbb{R}$

Now, we'll select a particular Morse function defined on the orbit $U(n+k) \cdot x_0$ and determine its critical points and indices. Consider the matrix:

$$A = \begin{pmatrix} i & & & 0 \\ & 2i & & \\ & & \ddots & \\ 0 & & & (n+k)i \end{pmatrix}$$

Proposition 10.25.

The function $f_A : U(n+k) \cdot x_0 \to \mathbb{R}$ is a Morse function whose critical points are the diagonal matrices in $\mathfrak{u}(n+k)$ which have exactly n entries equal to i and k entries equal to 0 along the diagonal.

Proof. According to Lemma 10.9, a point $p \in U(n+k) \cdot x_0$ becomes a critical point of the function f_A if and only if it commutes with the matrix A. Given that A is diagonal with distinct eigenvalues, this criterion implies that p must also be diagonal. Since conjugation by an element of $U(n+k)$ preserves the eigenvalues of a matrix, p must have exactly n entries equal to i and k entries equal to 0 along its diagonal. Furthermore, by conjugating x_0 with permutation matrices, it's evident that all such diagonal matrices fall within the orbit $U(n+k) \cdot x_0$. □

To compute the indices of the critical points of f_A, we will employ Lemmas 10.10 and 10.16. Before delving into the computation, let's introduce some additional notation. Consider an n-tuple $\sigma = (r_1, \ldots, r_n)$ comprising integers from 1 to $n+k$, where $r_1 < r_2 < \cdots < r_n$. This tuple, termed a Schubert symbol, encapsulates vital information for our analysis. For any Schubert symbol σ, we denote x_σ as the diagonal matrix in $\mathfrak{u}(n+k)$ with an i in rows $r_1, \ldots, r_n$ and zeros elsewhere. Thus, σ effectively specifies the critical point x_σ under consideration.

To determine the index of x_σ, we need to identify specific matrices $Y_1, \ldots, Y_{2nk}$ such that $[Y_1, x_\sigma], \ldots, [Y_{2nk}, x_\sigma]$ form a basis for $T_{x_\sigma}(U(n+k) \cdot x_0)$. To this end, we define the matrix

$Y_{r,s}(z)$ as follows: it takes the value z in the (r, s) entry, $-\bar{z}$ in the (s, r) entry, and zeros elsewhere. This leads us to the following proposition, which is self-evident.

Proposition 10.26.

If $D \in \mathfrak{u}(n+k)$ is diagonal with entries $(d_1, \ldots, d_{n+k})$ along the diagonal then,

$$[Y_{r,s}(z), D] = Y_{r,s}\left(z\left(d_s - d_r\right)\right)$$

for all $z \in \mathbb{C}$.

Theorem 10.27.

The index of x_σ is twice the number of rows above each i which consist entirely of zeros. That is,

$$\text{index of } x_\sigma = 2\sum_{j=1}^{n}(r_j - j) = 2\left(\sum_{j=1}^{n} r_j\right) - n(n+1).$$

Proof. According to Lemma 10.16, to determine the index of x_σ as a critical point of the function $g_A(x) = \|x - A\|^2$, we count the points B (taking multiplicities into account) along the line segment between x_σ and A where the Jacobian of the function $E : N \to \mathfrak{u}(n+k)$ given by $E(x, \vec{v}) = x + \vec{v}$ is singular at $(x_\sigma, B - x_\sigma)$ in N. The multiplicity of B corresponds to the dimension of the kernel of the Jacobian of E at $(x_\sigma, B - x_\sigma)$. Since x_σ also serves as a critical point of f_A, its index as a critical point of f_A is $2nk$ minus its index as a critical point of g_A, as per Lemma 10.10.

Now, let's delve into the computation by introducing matrices $Y_{r,s}(1)$ and $Y_{r,s}(i)$ in $\mathfrak{u}(n+k)$, where $1 \leq r < s \leq n+k$, with either r or s belonging to $\{r_1, \ldots, r_n\}$, but not both. As $[Y_{r,s}(1), x_\sigma] = \pm Y_{r,s}(i)$ and $[Y_{r,s}(i), x_\sigma] = \pm Y_{r,s}(1)$, it's evident that $\{[Y_{r,s}(1), x_\sigma], [Y_{r,s}(i), x_\sigma]\}$ forms a basis for $T_{x_\sigma}(U(n+k) \cdot x_0)$. We select the obvious matrices $Z_{2nk+1}, \ldots, Z_{(n+k)^2}$ as a basis for $N_{x_\sigma}(U(n+k) \cdot x_0)$, which possess only two non-zero entries either 1, -1, or i in the position (r, s) where r, s either they both belong to σ or don't.

Lemma 10.16 provides insight into the Jacobian of E at $(x_\sigma, t(A - x_\sigma))$ in N, where $0 < t < 1$. We obtain the columns of this Jacobian by computing the commutators, leveraging Proposition 8.31:

$$[Y_{r,s}(1), x_\sigma + t(A - x_\sigma)] = \begin{cases} Y_{r,s}(it(s+1-r) - i) & \text{if } r \in \{r_1, \dots, r_n\} \\ Y_{r,s}(i - it(r+1-s)) & \text{if } s \in \{r_1, \dots, r_n\} \end{cases}$$

$$[Y_{r,s}(i), x_\sigma + t(A - x_\sigma)] = \begin{cases} Y_{r,s}(1 - t(s+1-r)) & \text{if } r \in \{r_1, \dots, r_n\} \\ Y_{r,s}(t(r+1-s) - 1) & \text{if } s \in \{r_1, \dots, r_n\} \end{cases}$$

Analyzing these commutators, it's evident that the Jacobian has a non-trivial kernel if and only if one or more of them are identically zero. The dimension of the kernel at such a point is determined by counting the number of zero commutators. Thus, to compute the index of x_σ, we count the number of commutators for which there exists a t such that $0 < t < 1$ making the commutator zero. Since $r < s$, such a t exists if and only if r belongs to $\{r_1, \dots, r_n\}$. For a fixed r in $\{r_1, \dots, r_n\}$, the number of allowed s values equals the number of rows below row r in x_σ that consist entirely of zeros. Since both $[Y_{r,s}(1), x_\sigma + t(A - x_\sigma)]$ and $[Y_{r,s}(i), x_\sigma + t(A - x_\sigma)]$ become zero for $t = 1/(s+1-r)$, the index of x_σ as a critical point of g_A is twice the sum of these numbers. Considering there are $r_j - j$ rows of zeros above row r_j, and since there are k rows of zeros in the matrix, there are $k - (r_j - j)$ rows of zeros below row r_j for all $j = 1, \dots, n$. Consequently, the index of x_σ as a critical point of g_A is given by:

$$2\sum_{j=1}^{n} (k - (r_j - j)) = 2nk - 2\sum_{j=1}^{n}(r_j - j)$$

and the index of x_σ as a critical point of f_A is $2\sum_{j=1}^{n} (r_j - j)$. □

Example 10.28.

Now to understand the technique of the proof we will calculate the index of a critical point of $\mathrm{Gr}_{2,4}$.

Now we know that the critical point of f_A on $G.x_0$. is a diagonal matrix of dimension 4×4 with two ' i ' & two ' 0 '. $A = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 3i & 0 \\ 0 & 0 & 0 & 4i \end{pmatrix}$

So the critical points are $x_{(1,2)}, x_{(1,3)}, x_{(1,4)}, x_{(2,3)}, x_{(2,4)}, x_{(3,2)}$ The indices are $0, 2, 4, 4, 6, 8$ consecutively.

Now we will try to calculate index of one of the critical point then others will follow similarly.

Now first we need to calculate the index of this critical point as a critical point of the function $g_A(x) = \|x - A\|^2$.

Now if we join $x_{(2,4)}$ with A by a line segment then. index will be sum of nullity of each focal point on the line segment.

Now let's consider the 4×4 matrices from $\mathfrak{u}(n+k)$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_5 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, A_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, A_7 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, A_8 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

We can see that they are linearly independent and $T_{x_{(2,4)}}\left(U(4) \cdot x_0\right) = \langle A_i \rangle_{i \in [8]}$

$$B_9 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_{10} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, B_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Similarly they are also linearly independent and $N_{x_{(2,4)}}\left(U(4), x_0\right) = \langle B_i \rangle_{i \in [9,12]}$ Now for $0 < t < 1 \quad \left(x_{(2,4)}, t\left(A - x_{(2,4)}\right)\right) \in N_x(2,4)$

J_E has columns $\left[A_i, x_{(2,4)} + t\left(A - x_{(2,4)}\right)\right] \quad \forall i \in [8]$ and $B_i \quad \forall i \in [0,12]$. So the Columns are

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & -i(2t-1) \\ 0 & 0 & -i(2t-1) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & i(2t-1) \\ 0 & 0 & i(2t-1) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & i(2t+1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i(2t+1) & 0 & 0 & 0 \end{pmatrix}$$

As $0 < t < 1$ so only for $t = \frac{1}{2}$ we get a focal point with nullity 2 so $x_{(2,4)}$ has index 2 in g_A and this implies $8 - 2 = 6$ index in f_A.

10.5 A Morse function on the complex Grassmannian

We've established the existence of a Morse function on a specific orbit of the adjoint representation of $U(n+k)$ on its Lie algebra $\mathfrak{u}(n+k)$. Now, let's connect this function to a Morse function on the complex Grassmann manifold $G_{n,n+k}(\mathbb{C})$.

Consider $G_{n,n+k}(\mathbb{C})$, the complex Grassmann manifold comprising n-dimensional complex planes in $\mathbb{C}^{n+k}$, and let $\mathbb{C}^n \subseteq \mathbb{C}^{n+k}$ be the subspace spanned by the first n standard basis vectors. There's a transitive action

$$U(n+k) \times G_{n,n+k}(\mathbb{C}) \to G_{n,n+k}(\mathbb{C})$$

defined by mapping $\mathbb{C}^n \in G_{n,n+k}(\mathbb{C})$ to its image under the linear transformation determined by a matrix in $U(n+k)$. Clearly, the stabilizer of $\mathbb{C}^n$ consists of all matrices of the form

$$\begin{pmatrix} U_n & 0 \\ 0 & U_k \end{pmatrix}$$

where $U_n \in U(n)$ and $U_k \in U(k)$. Thus, we obtain a diffeomorphism

$$\psi_1 : G_{n,n+k}(\mathbb{C}) \to U(n+k)/(U(n) \times U(k))$$

Leveraging this diffeomorphism, we can embed $G_{n,n+k}(\mathbb{C})$ into the Lie algebra $\mathfrak{u}(n+k)$ as follows: Define a map $\psi_2 : U(n+k)/(U(n) \times U(k)) \to \mathfrak{u}(n+k)$ by $\psi_2([U]) = U \cdot x_0 = Ux_0U^{-1}$, where $[U]$ denotes the coset represented by $U \in U(n+k)$. A lemma confirms that ψ_2 is a diffeomorphism, thereby establishing

$$\psi \stackrel{\text{def}}{=} \psi_2 \circ \psi_1 : G_{n,n+k}(\mathbb{C}) \xrightarrow{\psi_1} U(n+k)/(U(n) \times U(k)) \xrightarrow{\psi_2} U(n+k) \cdot x_0$$

as a diffeomorphism. With this, we define a Morse function $f_A : G_{n,n+k}(\mathbb{C}) \to \mathbb{R}$ by

$$f_A(x) = \langle \psi(x), A \rangle = -\operatorname{trace}(\psi(x)A)$$

where $\psi(x)A$ represents matrix multiplication.

Lemma 10.29.

The map ψ_2 is a well defined diffeomorphism onto the orbit $U(n+k) \cdot x_0 \subseteq \mathfrak{u}(n+k)$.

Proof. When a compact Lie group G smoothly acts on a manifold M, the quotient space G/G_x is diffeomorphic to the orbit $G \cdot x$ for all $x \in M$ (Theorem 3.62 of Foundations of Differentiable Manifolds and Lie Groups by Warner). Hence, we only need to confirm that $U(n) \times U(k)$ acts as the stabilizer of x_0. Let's identify $U(n) \times U(k) \subseteq \mathfrak{u}(n+k)$ with matrices of the form

$$\begin{pmatrix} U_n & 0 \\ 0 & U_k \end{pmatrix}$$

where $U_n \in U(n)$ and $U_k \in U(k)$. For any $U \in U(n) \times U(k)$, we have:

$$\begin{aligned} Ux_0U^{-1} &= \begin{pmatrix} U_n & 0 \\ 0 & U_k \end{pmatrix} \begin{pmatrix} iI_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {}^t\bar{U}_n & 0 \\ 0 & {}^t\bar{U}_k \end{pmatrix} \\ &= \begin{pmatrix} iU_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {}^t\bar{U}_n & 0 \\ 0 & {}^t\bar{U}_n \end{pmatrix} \\ &= \begin{pmatrix} iU_n{}^t\bar{U}_n & 0 \\ 0 & 0 \end{pmatrix} \\ &= x_0. \end{aligned}$$

This demonstrates that $U(n) \times U(k) \subseteq U(n+k)_{x_0}$. Now, assume $U \in U(n+k)$ satisfies $Ux_0U^{-1} = x_0$. Let

$$U = \begin{pmatrix} A_n & B \\ C & D_k \end{pmatrix}$$

where A_n is an $n \times n$ complex matrix and D_k is a $k \times k$ complex matrix. Then,

$$\begin{aligned} x_0 &= Ux_0{}^t\bar{U} \\ &= i \begin{pmatrix} A_n & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} {}^t\bar{A}_n & {}^t\bar{C} \\ {}^t\bar{B} & {}^t\bar{D}_k \end{pmatrix} \\ &= i \begin{pmatrix} A_n{}^t\bar{A}_n & A_n{}^t\bar{C} \\ C^t\bar{A}_n & C^t\bar{C} \end{pmatrix}. \end{aligned}$$

In particular, this says that $A_n{}^t\bar{A}_n = I_n$, and hence, $A_n \in U(n)$. So, all the columns and rows of A_n have unit length, and hence, $B = C = 0$. Thus, $U^t\bar{U} = I_{n+k}$ implies that $D_k \in U(k)$, and hence, $U \in U(n) \times U(k)$.

□

Now we will see an explicit example of Morse function on Grassmannian.

Lemma 10.30.
If $U = \begin{pmatrix} A_n & B \\ C & D_k \end{pmatrix} \in U(n+k)$ where A_n is some $n \times n$ complex matrix and D_k is some

$k \times k$ complex matrix, then $f_A(U \cdot x_0)$ is given by,

$$-\sum_{j=1}^{n} j \left(\text{ length of } j^{\text{th}} \text{ row of } A_n \right)^2 - \sum_{j=1}^{k} (j+n) \left(\text{ length of } j^{\text{th}} \text{ row of } C \right)^2 .$$

Proof. As in the proof of the preceding lemma we have,

$$\begin{aligned} Ux_0U^{-1} &= \begin{pmatrix} A_n & B \\ C & D_k \end{pmatrix} \begin{pmatrix} iI_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {}^t\bar{A}_n & {}^t\bar{C} \\ {}^t\bar{B} & {}^t\bar{D}_k \end{pmatrix} \\ &= i \begin{pmatrix} A_n & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} {}^t\bar{A}_n & {}^t\bar{C} \\ {}^t\bar{B}^t & {}^t\bar{D}_k \end{pmatrix} \\ &= i \begin{pmatrix} A_n{}^t\bar{A}_n & A_n{}^t\bar{C} \\ C^t\bar{A}_n & C^t\bar{C} \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} f_A(U \cdot x_0) &= < Ux_0U^{-1}, A > \\ &= -\operatorname{trace}\left(i \begin{pmatrix} A_n^t\bar{A}_n & A_n^t\bar{C} \\ C^t\bar{A}_n & C^t\bar{C} \end{pmatrix} (-i) \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & n+k \end{pmatrix} \right) \\ &= -\sum_{j=1}^{n} j \sum_{i=1}^{n} \|a_{ji}\|^2 - \sum_{j=1}^{k} (j+n) \sum_{i=1}^{n} \|c_{ji}\|^2 \end{aligned}$$

□

10.6 Stable & Unstable Manifold and Morse-Smale function

Now, let's delve into describing the gradient flow lines of $f_A : G_{n,n+k}(\mathbb{C}) \to \mathbb{R}$. This description will be articulated in terms of the action of $GL_{n+k}(\mathbb{C})$ on $G_{n,n+k}(\mathbb{C})$, which represents the set of n-dimensional complex planes in $\mathbb{C}^{n+k}$. This action is defined as follows: for any $G \in GL_{n+k}(\mathbb{C})$ and any plane $P \in G_{n,n+k}(\mathbb{C})$, the product $G \cdot P$ is defined to be $G(P)$, the image of the plane P under the linear transformation determined by G.

To embark on this, we introduce the following general lemma.

Lemma 10.31.

Let $\eta : G \times M \to M$ be a smooth action of a compact Lie group G on a smooth manifold M. Let $x \in M$ and let $\eta_x : G \to M$ be induced from η. Then $T_e(G_x) \subseteq \ker d\eta_x(e)$ where

$e \in G$ is the identity and G_x is the stabilizer of x. Moreover, $T_e(G_{g\cdot x}) = Ad_g(T_e(G_x))$ for all $g \in G$.

Proof. Consider a path $\gamma(t)$ in G_x with $\gamma(0) = e$. Since $\gamma(t)$ lies entirely within G_x, the function $\eta_x(\gamma(t))$ is constant along this path. Therefore, its derivative with respect to t evaluated at $t = 0$ is zero, i.e., $\frac{d}{dt}\eta_x(\gamma(t))\big|_{t=0} = d\eta_x(e)(\gamma'(0)) = 0$. This demonstrates that $T_e(G_x) \subseteq \ker d\eta_x(e)$.

To prove the second statement, let's recall that $G_{g\cdot x} = gG_xg^{-1}$ and then consider paths of the form $g\sigma(t)g^{-1}$, where $\sigma(t) \in G_x$ and $\sigma(0) = e$. □

Lemma 10.32.

Let $P \in G_{n,n+k}(\mathbb{C})$. The gradient flow of the Morse function $f_A : G_{n,n+k}(\mathbb{C}) \to \mathbb{R}$ through P is given by $\gamma_P(t) = \exp(itA)(P)$.

Proof. Consider $\mathbb{C}^n \subseteq \mathbb{C}^{n+k}$, the subspace spanned by the first n standard basis elements of $\mathbb{C}^{n+k}$. Let $U \in U(n+k)$ be a unitary matrix such that $U(\mathbb{C}^n) = P$, and define $x = Ux_0U^{-1} \in \mathfrak{u}(n+k)$. In Remark 10.24, we observed that the path $\sigma_x(t) = \exp(t[A,x]) \cdot x$ in $\mathfrak{u}(n+k)$ satisfies $\sigma_x'(0) = [[A,x],x]$, which is minus the gradient vector of f_A at x. Under the diffeomorphism $U(n+k) \cdot x_0 \approx G_{n,n+k}(\mathbb{C})$, this path corresponds to $\tilde{\sigma}_x(t) = \exp(t[A,x])(P)$. Since $\gamma_P(t + t_0) = \gamma_{\tilde{P}}(t)$ for some $\tilde{P} \in G_{n,n+k}(\mathbb{C})$, it's sufficient to prove that $\gamma_P'(0) = \tilde{\sigma}_x'(0)$. Define $\eta : GL_{n+k}(\mathbb{C}) \times G_{n,n+k}(\mathbb{C}) \to G_{n,n+k}(\mathbb{C})$ as the smooth action of $GL_{n+k}(\mathbb{C})$ on $G_{n,n+k}(\mathbb{C})$ and let $\eta_P : GL_{n+k}(\mathbb{C}) \to G_{n,n+k}(\mathbb{C})$ be induced from η. In this notation, $\gamma_P(t) = \eta_P(\exp(itA))$ and $\tilde{\sigma}_x(t) = \eta_P(\exp(t[A,x]))$. By the chain rule, we have $\gamma_P'(0) = d\eta_P(I_{n+k})(iA)$ and $\tilde{\sigma}_x'(0) = d\eta_P(I_{n+k})([A,x])$ where I_{n+k} is the $(n+k)\times(n+k)$ identity matrix. Thus, it suffices to show $d\eta_P(I_{n+k})(iA - [A,x]) = 0$. To demonstrate that $iA - [A,x] \in \ker d\eta_P(I_{n+k})$, we apply the previous lemma and show that,

$$iA - [A,x] \in T_{I_{n+k}}(GL_{n+k}(\mathbb{C})_P) = U\left(T_{I_{n+k}}(GL_{n+k}(\mathbb{C})_{\mathbb{C}^n})\right)U^{-1}$$

The stabilizer of $\mathbb{C}^n \in G_{n,n+k}(\mathbb{C})$ consists of matrices in $GL_{n+k}(\mathbb{C})$ whose lower left $k \times n$ block is zero. Since $GL_{n+k}(\mathbb{C})$ is open in $\mathbb{C}^{(n+k)^2}$, the tangent space at I_{n+k} of the stabilizer of $\mathbb{C}^n$ consists of matrices in the tangent space whose lower left $k \times n$ block is zero. Therefore, we express $iA - [A, x]$ as such a matrix conjugated with U:

$$\begin{aligned} iA - [A, x] &= iA - \left[A, Ux_0U^{-1}\right] \\ &= iA - U\left[U^{-1}AU, x_0\right]U^{-1} \\ &= U\left(iU^{-1}AU - \left[U^{-1}AU, x_0\right]\right)U^{-1} \end{aligned}$$

Letting $Y = U^{-1}AU$, it's clear that $iY - [Y, x_0]$ has its lower left $k \times n$ block equal to zero. □

Remark 10.33.

In Remark 10.24, it was demonstrated that $\tilde{\sigma}_x'(t)$ differs from $-\nabla(f)\left(\tilde{\sigma}_x(t)\right)$ for $t \neq 0$. It's natural to question why the preceding proof applies to the path $\gamma_P(t) = \exp(itA)(P)$ but not to $\tilde{\sigma}_x(t) = \exp(t[A, x])(P)$, given that both paths satisfy

$$\gamma_P'(0) = \tilde{\sigma}_x'(0) = -\nabla\left(f_A\right)(P).$$

The fundamental distinction lies in the fact that iA remains constant with respect to the point P, whereas $[A, x]$ does not. Consequently, while the path $\gamma_P(t)$ fulfills $\gamma_P\left(t + t_0\right) = \gamma_{\tilde{P}}(t)$ for some $\tilde{P} \in G_{n,n+k}(\mathbb{C})$, the same does not hold for the path $\tilde{\sigma}_x(t)$.

It's crucial to note that the matrix $\exp(itA) \in GL_{n,n+k}(\mathbb{C})$ is not unitary. Therefore, the aforementioned description of gradient flow lines does not straightforwardly extend to adjoint orbits $U(n+k) \cdot x_0 \subseteq \mathfrak{u}(n+k)$.

Our objective now is to delineate the unstable manifolds $W^u\left(x_\sigma\right)$ of f_A. We begin by revisiting the concept of Schubert cells associated with a Schubert symbol $\sigma = (r_1, \ldots, r_n)$. The associated Schubert cell $e(\sigma) \subseteq G_{n,n+k}(\mathbb{C})$ encompasses all planes P in $G_{n,n+k}(\mathbb{C})$ such that $\dim\left(P \cap \mathbb{C}^{r_j}\right) = j$ and $\dim\left(P \cap \mathbb{C}^{r_j - 1}\right) = j - 1$ for $j = 1, \ldots, n$. Here, $\mathbb{C}^j \subseteq \mathbb{C}^{n+k}$ represents the subspace spanned by the first j standard basis elements.

Recall that $G_{n,n+k}(\mathbb{C})$ is topologized as a quotient of the Stiefel manifold $V_{n,n+k}(\mathbb{C}) \subseteq \mathbb{C}^{n(n+k)}$. This implies the existence of a map $\pi : V_{n,n+k}(\mathbb{C}) \to G_{n,n+k}(\mathbb{C})$, which sends an

n-tuple of linearly independent vectors in $\mathbb{C}^{n+k}$ to the n-plane they span. An open set $U \subseteq G_{n,n+k}(\mathbb{C})$ if and only if $\pi^{-1}(U)$ is open in $V_{n,n+k}(\mathbb{C}) \subset \mathbb{C}^{n(n+k)}$.

For any Schubert symbol σ, let P_σ be the plane spanned by the standard basis elements $e_{r_1}, \ldots, e_{r_n}$. Since P_σ corresponds to x_σ under the diffeomorphism $\psi : G_{n,n+k}(\mathbb{C}) \to U(n+k) \cdot x_0$, Theorem 10.20 asserts that P_σ serves as a critical point of $f_A : G_{n,n+k}(\mathbb{C}) \to \mathbb{R}$.

Theorem 10.34.

For any Schubert symbol $\sigma = (r_1, \ldots, r_n)$,

$$W^u\left(P_\sigma\right) = e(\sigma).$$

Proof. Let $\sigma = (r_1, \ldots, r_n)$ denote any Schubert symbol, and consider $P \in e(\sigma)$. To demonstrate that $P \in W^u\left(P_\sigma\right)$, we aim to show that for any open neighborhood $U \subset G_{n,n+k}(\mathbb{C})$ containing P_σ, there exists $T < 0$ such that for all $t < T$, we have $\gamma_P(t) = \exp(itA)(P) \in U$.

Given that $P \in e(\sigma)$, we can select a basis $v_1, \ldots, v_n$ of P such that for all $j = 1, \ldots, n$, v_j has a 1 in the r_j^{th} entry and zeros in entries $r_j + 1, \ldots, n+k$. It's worth noting that $\exp(itA)$ is essentially a diagonal matrix with entries $e^{-t}, e^{-2t}, \ldots, e^{-(n+k)t}$ along the diagonal. Moreover, the vectors:

$$e^{r_1 t} \exp(itA)\left(v_1\right), \ldots, e^{r_n t} \exp(itA)\left(v_n\right)$$

span the plane $\exp(itA)(P)$. For $j = 1, \ldots, n$, the m^{th} entry of the vector $e^{r_j t} \exp(itA)\left(v_j\right)$ is:

$$\begin{array}{ll} z e^{(r_j - m)t} & \text{if } m < r_j - 1 \\ 1 & \text{if } m = r_j \\ 0 & \text{if } m = r_j + 1, \ldots, n+k \end{array}$$

Considering $\tilde{\gamma}_P(t) = \left(e^{r_1 t} \exp(itA)\left(v_1\right), \ldots, e^{r_n t} \exp(itA)\left(v_n\right)\right)$ as a path in $V_{n,n+k}(\mathbb{C})$, we have $\pi\left(\tilde{\gamma}_P(t)\right) = \gamma_P(t)$ for all $t \in \mathbb{R}$. Since $(e_{r_1}, \ldots, e_{r_n}) \in \pi^{-1}(U)$, where $\pi^{-1}(U)$ is open in $V_{n,n+k}(\mathbb{C})$, and $\lim_{t \to -\infty} \tilde{\gamma}_P(t) = (e_{r_1}, \ldots, e_{r_n})$, we can select $T < 0$ such that for all $t < T$, we have $\tilde{\gamma}_P(t) \in \pi^{-1}(U)$. This implies that $\gamma_P(t) \in U$ for all $t < T$.

Having shown $e(\sigma) \subseteq W^u\left(P_\sigma\right)$, we now consider $P \in W^u\left(P_\sigma\right)$. Since the Schubert cells partition $G_{n,n+k}(\mathbb{C})$, there exists a Schubert symbol $\tilde{\sigma}$ such that $P \in e(\tilde{\sigma})$. Consequently,

$P \in W^u(P_{\tilde{\sigma}})$ by the argument presented in the preceding paragraph. Since the unstable manifolds are disjoint, this implies that $\tilde{\sigma} = \sigma$, and hence $P \in e(\sigma)$. □

To express the corresponding result for the stable manifolds $W^s(P_\sigma)$ of f_A, we introduce some notation. If $e_1, \ldots, e_{n+k}$ denotes the standard basis for $\mathbb{C}^{n+k}$, we define the "inverse standard basis" as $\tilde{e}_j = e_{n+k-j+1}$ for $j = 1, \ldots, n+k$. This leads to the "inverse filtration" of $\mathbb{C}^{n+k}$ as follows:

$$\tilde{\mathbb{C}}^0 \subset \tilde{\mathbb{C}}^1 \subset \tilde{\mathbb{C}}^2 \subset \cdots \subset \tilde{\mathbb{C}}^{n+k}$$

where $\tilde{\mathbb{C}}^j$ denotes the subspace spanned by the first j inverse standard basis elements. For a Schubert symbol $\sigma = (r_1, \ldots, r_n)$, we define the "inverse Schubert cell" $\tilde{e}(\sigma)$ by stipulating that $P \in \tilde{e}(p)$ if and only if for all $j = 1, \ldots, n$, the following conditions hold:

1) $\dim\left(P \cap \tilde{\mathbb{C}}^{n+k+1-r_j}\right) = n + 1 - j$

2) $\dim\left(P \cap \tilde{\mathbb{C}}^{n+k-r_j}\right) = n - j.$

An essential observation about the inverse Schubert cell $\tilde{e}(\sigma)$ is that $P \in \tilde{e}(\sigma)$ if and only if one can select a basis $v_1, \ldots, v_n$ for p where v_j has a zero in entries $1, \ldots, r_j - 1$ and a 1 in the r_j^{th} entry for all $j = 1, \ldots, n$. The proof of the forthcoming theorem parallels that of the previous one.

Theorem 10.35.

For any Schubert symbol $\sigma = (r_1, \ldots, r_n)$ we have

$$W^s(P_\sigma) = \tilde{e}(\sigma)$$

The preceeding two theorems give us the following description of the intersections of the stable and unstable manifolds of f_A.

Corollary 10.36.

If $\sigma = (r_1, \ldots, r_n)$ and $\tilde{\sigma} = (\tilde{r}_1, \ldots, \tilde{r}_n)$ are Schubert cells, then $W^u(P_\sigma) \cap W^s(P_{\tilde{\sigma}}) \neq \emptyset$ if and only if $r_j \geq \tilde{r}_j$ for all $j = 1, \ldots, n$.

Proof. A plane P belongs to $W^u(P_\sigma)$ if and only if we can select a basis $v_1, \ldots, v_n$ for P such that v_j has a 1 in the r_j^{th} entry and zeros in entries $r_j + 1, \ldots, n + k$. Conversely, a plane P belongs to $W^s(p_{\tilde{\sigma}})$ if and only if we can choose a basis $\tilde{v}_1, \ldots, \tilde{v}_n$ for P where $\tilde{v}_j$ has zeros in entries $1, \ldots, \tilde{r}_j - 1$ and a 1 in the $\tilde{r}_j^{\text{th}}$ entry for all $j = 1, \ldots, n$.

If $r_j \geq \tilde{r}_j$ for all $j = 1, \ldots, n$, then the vectors $w_1, \ldots, w_n$ where w_j has a 1 in entries r_j and $\tilde{r}_j$ and zeros elsewhere will form a basis for a plane in $W^u(P_\sigma) \cap W^s(P_{\tilde{\sigma}})$. However, if $r_j < \tilde{r}_j$ for some $1 \leq j \leq n$, then a contradiction arises, demonstrating that there can be no flow from P_σ to $P_{\tilde{\sigma}}$.

Suppose there exists an n-plane $P \in W^u(p_\sigma) \cap W^s(p_{\tilde{\sigma}})$, and let $v_1, \ldots, v_n$ and $\tilde{v}_1, \ldots, \tilde{v}_n$ be as defined earlier. By adding certain multiples of $\tilde{v}_{j+1}, \ldots, \tilde{v}_n$ to $\tilde{v}_j$, we can construct a vector $v \in P$ that has zeros in entries $1, 2, \ldots, \tilde{r}_j - 1, \tilde{r}_{j+1}, \tilde{r}_{j+2}, \ldots, \tilde{r}_n$ and a 1 in the $\tilde{r}_j^{\text{th}}$ entry. However, v cannot belong to the span of $v_1, \ldots, v_n$. Therefore, the n-plane p would need to contain the $n + 1$ linearly independent vectors $v_1, \ldots, v_n, v$. This contradiction demonstrates that $W^u(P_\sigma) \cap W^s(P_{\tilde{\sigma}}) = \emptyset$. $\square$

Following the previous corollary, we establish a partial ordering on the Schubert cells as follows. Given Schubert cells $\sigma = (r_1, \ldots, r_n)$ and $\tilde{\sigma} = (\tilde{r}_1, \ldots, \tilde{r}_n)$, we define $\sigma \geq \tilde{\sigma}$ if and only if $r_j \geq \tilde{r}_j$ for all $j = 1, \ldots, n$. Notably, under this definition, $\sigma \geq \tilde{\sigma}$ if and only if $P_\sigma \succeq P_{\tilde{\sigma}}$.

For any pair of critical points P_σ and $P_{\tilde{\sigma}}$ of $f_A : G_{n,n+k}(\mathbb{C}) \to \mathbb{R}$, we define $W(P_\sigma, P_{\tilde{\sigma}}) = W^u(P_\sigma) \cap W^s(P_{\tilde{\sigma}})$. Now, we aim to demonstrate that for all critical points P_σ and $P_{\tilde{\sigma}}$ of f_A, $W^u(P_\sigma) \pitchfork W^s(P_{\tilde{\sigma}})$, i.e., $f_A : G_{n,n+k}(\mathbb{C}) \to \mathbb{R}$ constitutes a Morse-Smale function.

Lemma 10.37.

Let $\pi : E \to B$ be a smooth fiber bundle. Let V, W be submanifolds of B and let $p \in V \cap W$. The manifolds V and W meet transversely at p if and only if there exists some $q \in \pi^{-1}(p)$ with $\pi^{-1}(V) \pitchfork \pi^{-1}(W)$ at q.

Proof. First note that since $\pi : E \to B$ is a submersion $\pi^{-1}(V)$ and $\pi^{-1}(W)$ are submanifolds of E by Theorem 5.11. Also, transversality is a local property and so it suffices to prove the lemma for a trivial bundle $E = B \times F$. In this case we have,

$$\pi^{-1}(V) = V \times F \subseteq B \times F$$

$$\pi^{-1}(W) = W \times F \subseteq B \times F.$$

For any $q = (p, x) \in \pi^{-1}(p)$ we have,

$$T_q\left(\pi^{-1}(V)\right) = T_p(V) \times T_x(F)$$

$$T_q\left(\pi^{-1}(W)\right) = T_p(W) \times T_x(F)$$

Clearly,

$$(T_p(V) \times T_x(F)) \oplus (T_p(W) \times T_x(F)) = T_p(B) \times T_x(F)$$

if and only if

$$T_p(V) \oplus T_p(W) = T_p(B)$$

□

Theorem 10.38.

The function $f_A : G_{n,n+k}(\mathbb{C}) \to \mathbb{R}$ is a Morse-Smale function.

Proof. Let $V_{n,n+k}(\mathbb{C}) \subseteq \mathbb{C}^{n(n+k)}$ represent the Stiefel manifold. Then $\pi : V_{n,n+k}(\mathbb{C}) \to G_{n,n+k}(\mathbb{C})$ forms a smooth fiber bundle. Let $\sigma = (r_1, \ldots, r_n)$ and $\tilde{\sigma} = (\tilde{r}_1, \ldots, \tilde{r}_n)$ be Schubert symbols satisfying $\sigma \geq \tilde{\sigma}$. For any $P \in W(P_\sigma, P_{\tilde{\sigma}})$, we can select a basis $v_1, \ldots, v_n$ for P such that v_j has a 1 in the r_j^{th} entry and 0 in entries $r_j + 1, \ldots, n+k$. Likewise, we can choose a basis $\tilde{v}_1, \ldots, \tilde{v}_n$ for P with zeros in entries $1, \ldots, \tilde{r}_j - 1$ and a 1 in the $\tilde{r}_j^{\text{th}}$ entry.

Let $q = (v_1, \ldots, v_n) \in V_{n,n+k}(\mathbb{C})$. Our goal is to demonstrate

$$\pi^{-1}\left(W^u\left(P_\sigma\right)\right) \pitchfork \pi^{-1}\left(W^s\left(P_{\sigma'}\right)\right)$$

at q. Recall $V_{n,n+k}(\mathbb{C}) \subseteq \mathbb{C}^{n(n+k)}$. The tangent space $T_q\left(\pi^{-1}\left(W^u\left(P_\sigma\right)\right)\right)$ consists of vectors $(v_1, \ldots, v_n) \in \mathbb{C}^{n(n+k)}$ where v_j has entries r_j+ $1, \ldots, n+k$ equal to zero ($j = 1, \ldots, n$). Similarly, $T_q\left(\pi^{-1}\left(W^s\left(P_{\sigma'}\right)\right)\right)$ comprises frames $(\tilde{v}_1, \ldots, \tilde{v}_n) \in \mathbb{C}^{n(n+k)}$ where $\tilde{v}_j$

has entries $1, \ldots, \tilde{r}_j - 1$ equal to zero ($j = 1, \ldots, n$). Given $r_j \geq \tilde{r}_j$ for all $j = 1, \ldots, n$, we obtain $T_q\left(\pi^{-1}\left(W^u\left(P_\sigma\right)\right) \oplus T_q\left(\pi^{-1}\left(W^s\left(P_{\tilde{\sigma}}\right)\right) = \mathbb{C}^{n(n+k)} = T_q\left(V_{n,n+k}(\mathbb{C})\right)\right.\right.$. □

10.7 The homology of $\mathrm{Gr}_{n,n+k}$

The insights we've gained pave the way for a straightforward computation of the homology of $G_{n,n+k}(\mathbb{C})$ utilizing the Morse-Smale function $f_A : G_{n,n+k}(\mathbb{C}) \to \mathbb{R}$.

Recall that Theorem 10.20 establishes that the Morse function f_A possesses even indices, as f_A is a Morse-Smale function. This indicates that ∂_n associated with the CW-complex defined by f_A are all null. Hence, according to the CW-Homology Theorem and the Second Fundamental Theorem of Morse Theory, the homology of $G_{n,n+k}(\mathbb{C})$ can be ascertained by enumerating the critical points according to their indices. This culminates in the following assertion.

Theorem 10.39.

The homology group $H_j\left(G_{n,n+k}(\mathbb{C}); \mathbb{Z}\right)$ is isomorphic to the free abelian group generated by the critical points of f_A of index j for all $j \in \mathbb{Z}_+$.

Definition 10.40.

A partition of $j \in \mathbb{Z}_+$is an unordered sequence of positive integers with sum j. The number of partitions of j is denoted by $p(j)$.

The following table gives the value of $p(j)$ for all $j \leq 10$.

j	0	1	2	3	4	5	6	7	8	9	10
p(j)	1	1	2	3	5	7	11	15	22	30	42

For example, the integer 5 has seven partitions, namely:

$$11111, \quad 1112, \quad 113, \quad 14, \quad 122, \quad 23, \quad 5.$$

Theorem 10.41.

For all $j \in \mathbb{Z}_+$, the homology group $H_j\left(G_{n,n+k}(\mathbb{C}); \mathbb{Z}\right)$ is zero if j is odd and a free abelian group on $\tilde{r}(j/2)$ generators if j is even, where $\tilde{r}(j/2)$ denotes the number of partitions of $j/2$ into at most n integers each of which is less than or equal to k.

Proof. For every Schubert symbol $(r_1, \ldots, r_n)$ with

$$(r_1 - 1) + (r_2 - 2) + \cdots + (r_n - n) = j/2$$

we get a partition of j :

$$r_1 - 1 \leq r_2 - 2 \leq r_n - n$$

(if we ignore any leading zeros) consisting of integers less than or equal to k. Conversely, give any partition $i_1 \leq i_2 \leq \cdots \leq i_n$ of $j/2$ (which we pad with leading zeros to make length n) with integers that are less than or equal to k we have a Schubert symbol:

$$\sigma = (i_1 + 1, i_2 + 2, \ldots, i_n + n)$$

In relation to the critical point x_σ, i_j corresponds to the number of rows of zeros above the $j/2^{\text{th}}$ i along the diagonal for all $i = 1, \ldots, j/2$. This proves the theorem. □

Now we will sum up all the homology groups of $\mathrm{Gr}_{2,4}$, $\mathrm{Gr}_{2,4}$, $\mathrm{Gr}_{2,4}$ using the previous results.

Homology	$\mathrm{Gr}_{2,4}$	$\mathrm{Generators}_{\mathrm{Gr}_{2,4}}$	$\mathrm{Gr}_{2,5}$	$\mathrm{Generators}_{\mathrm{Gr}_{2,5}}$	$\mathrm{Gr}_{3,5}$	$\mathrm{Generators}_{\mathrm{Gr}_{3,5}}$
H_0	$\mathbb{Z}$	$x_{(1,2)}$	$\mathbb{Z}$	$x_{(1,2)}$	$\mathbb{Z}$	$x_{(1,2,3)}$
H_2	$\mathbb{Z}$	$x_{(1,3)}$	$\mathbb{Z}$	$x_{(1,3)}$	$\mathbb{Z}$	$x_{(1,2,4)}$
H_4	$\mathbb{Z} \oplus \mathbb{Z}$	$x_{(1,4)}, x_{(2,3)}$	$\mathbb{Z} \oplus \mathbb{Z}$	$x_{(1,4)}, x_{(2,3)}$	$\mathbb{Z} \oplus \mathbb{Z}$	$x_{(1,2,5)}, x_{(1,3,4)}$
H_6	$\mathbb{Z}$	$x_{(2,4)}$	$\mathbb{Z} \oplus \mathbb{Z}$	$x_{(1,5)}, x_{(2,4)}$	$\mathbb{Z} \oplus \mathbb{Z}$	$x_{(1,3,5)}, x_{(2,3,4)}$
H_8	$\mathbb{Z}$	$x_{(3,4)}$	$\mathbb{Z} \oplus \mathbb{Z}$	$x_{(2,5)}, x_{(3,4)}$	$\mathbb{Z} \oplus \mathbb{Z}$	$x_{(1,4,5)}, x_{(2,3,5)}$
H_{10}	0	0	$\mathbb{Z}$	$x_{(3,5)}$	$\mathbb{Z}$	$x_{(2,4,5)}$
H_{12}	0	0	$\mathbb{Z}$	$x_{(4,5)}$	$\mathbb{Z}$	$x_{(3,4,5)}$
H_j [j is odd]	0	0	0	0	0	0
H_j [$j > 12$]	0	0	0	0	0	0

So we can see that $\mathrm{Gr}_{2,5} \approx \mathrm{Gr}_{3,5}$

Reference

Journals and Lecture Notes

1. Kevin Iga Ralph L. Cohen and Paul Norbury, Topics in morse theory: Lecture notes, Lecture Notes (2009), 232.

2. R. Bohm, Morse theory and handle decompositions, University of Chicago Mathematics Research Experience for Undergraduates program (2019), 23.

3. J. Nicolas, M. Gamboa, Morse theory for partially ordered sets in context, 25

4. Sebastian Poder Morse Homology of $\mathbb{RP}^n$ (U.U.D.M Project Report 2013)

Books

1. A. Banyaga,D. Hurtubise, Lectures on Morse Homology (Springer Science & Business Media, New York, ed. 1, 2004).

2. A. Hatcher, Algebraic Topology (Cambridge University Press, Cambridge, 2002).

3. M. Audin, M. Damian, Morse Theory and Floer Homology (Springer, London, 2014), Translated by Reine Erne.

4. J. M. Lee, Introduction to Smooth Manifold (Springer Science & Business Media, New York, ed. 3, 2003),

5. J. Milnor, Morse Theory. (AM-51), Volume 51. Princeton University Press, 1969.

6. F. Warner, Foundations of differentiable manifolds and Lie groups (Springer-Verlag, New York, 1983)

7. M. Hirsch, Differential Topology (Springer-Verlag, USA, 1994)

8. V. Guillemin and A. Pollack, Differential topology, Prentice-Hall Inc., Englewood Cliffs, N.J., 1974.

9. J. Munkres, Topology (Prentice Hall, 3 ed., 2000)

10. Glen E. Bredon, Topology and geometry , Springer-Verlag, New York, 1993, ISBN 0-387-97926-3. MR 94d:55001

11. J. Munkres, Elements of Algebraic Topology (A. Wesley Publishing Company, 1984)

12. L. I. Nicolaescu, An Invitation to Morse Theory (Springer Science & Business Media, USA, ed. 1, 2007)

Thesis

1. Joaquim Brugues Mora, Morse theory and Floer Homology, Master Thesis (2019).