MODULI OF CURVES

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Contents

1. References

Our references will be:

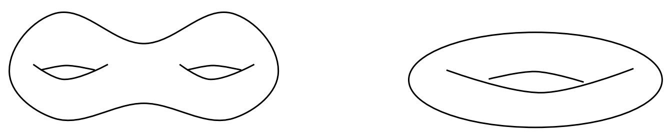
(1) *Moduli of Curves*; J. Harris and I. Morrison, GTM 187 Springer (**HM**).
(2) *Geometry of Algebraic Curves II*; E. Arbarello, M. Cornalba and P. Griffiths, A Series of Comprehensive Studies in Mathematics, Vol 268, Springer. (**ACG**).
(3) *Mirror Symmetry*, K. Hori, S. Katz, A. Klemm, R. Pandharipande, R, Thomas C. Vafa, R. Vakil, E. Zaslow, Clay Mathematics Monographs, Vol 1 (**HKZ**).

2. Moduli Spaces

2.1. **Fine Moduli.** A (fine) moduli space is a "space" that parametrizes isomorphism classes of certain objects, and has some "more structure". We will always work over complex numbers.

In Algebraic Geometry we want the "space" to be an algebraic variety M. The "more structure" means there should be a universal family over the variety that is a variety U with a map $\pi : U \to M$ such that fibers of π are the objects whose isomorphism classes constitute the points of M.

Our objects will be smooth projective curves of a fixed genus g, that is 1 dimensional smooth projective varieties, these are also compact Riemann surfaces.



Example: *$y^2z - x^3 - xz^2 = 0$ in $\mathbb{P}^2$ is a smooth genus 1 curve.* ♣

Example: *A degree d smooth curve in $\mathbb{P}^2$ has genus $g = (d-1)(d-2)/2$ (Hartshorne).* ♣

More generally we shall be interested in n pointed curves of fixed genus g $(C; p_1, \dots, p_n)$ where C is a smooth genus g curve and $p_i \in C$ distinct points. Two such curves $(C; p_1, \dots, p_n)$ and $(C'; p'_1, \dots, p'_n)$ are isomorphic if there is an isomorphism $\phi : C \to C'$ such that $\phi(p_i) = p'_i$.

A (fine) moduli space $M_{g,n}$ of n pointed genus g curves should be a variety such that

(1) Isomorphism classes of n pointed smooth genus g curves correspond to points of $M_{g,n}$.
(2) There is a variety $C_{g,n}$ with a map $\pi : C_{g,n} \to M_{g,n}$ and sections $\sigma_1, \ldots, \sigma_n : M_{g,n} \to C_{g,n}$ such that for any point $P \in M_{g,n}$ corresponding to the isomorphism class of $(C; p_1, \ldots, p_n)$ $(\pi^{-1}(P); \sigma_1(P), \ldots, \sigma_n(P))$ is isomorphic to $(C; p_1, \ldots, p_n)$.
(3) For any family of n pointed smooth genus curves $C \to B$ there is a morphism $\phi : B \to M_{g,n}$ such that the family C is a pull back by ϕ.

Condition 1 determines $M_{g,n}$ as a set and 3 gives the algebraic structure. $C_{g,n}$ is called the universal family. This set-up is true for any moduli space.

If $g = 0$ and $n \geq 3$ then $M_{0,n}$ is a smooth affine variety. This is because there is no non-trivial automorphism of $\mathbb{P}^1$ fixing 3 or more points. Similarly if n is big enough there is a smooth variety $M_{g,n}$ which is a fine moduli space.

Example: *$M_{0,3}$ is a point. $M_{0,4} \cong \mathbb{P}^1 - \{0, 1, \infty\}$ and in general*

$$M_{0,n} = \left(\mathbb{P}^1 - \bigcup_{i,j} \Delta_{i,j} \right) / \operatorname{PGL}(2, \mathbb{C})$$

where $\Delta_{i,j}$ are the diagonals. ♣

In other cases there is no variety that is a fine moduli space. Non-trivial automorphisms fixing marked points obstruct the existence of a fine moduli space in the category of algebraic varieties.

Example: *Let E be an elliptic curve, for instance $y^2z - x^3 - xz^2 = 0$ in $\mathbb{P}^2$. Let $\imath$ be the elliptic involution $\imath[x : y : z] = [x : -y : z]$. The point $[0 : 0 : 1]$ will be called 0 of this elliptic curve. Consider $E \times \mathbb{C}^\times$, $\mathbb{Z}/2\mathbb{Z}$ acts on this space by $n((P, z)) = (\imath^{-n}P, (-1)^n z)$, and let C be the quotient space. There is a projection $\pi : C \to \mathbb{C}^\times$ given by $[P, z] \mapsto z^2$ and there is the zero-section to this map $\sigma_1(z) = [0, z] = [\imath 0, z]$. This is an example of a non-trivial family of elliptic curves with 1 marked point all of whose fibers are E with marked point 0, hence this can not be the pull-back of a universal family over a moduli-space.* ♣

In the above example we just used an automorphism of the curve fixing a point, hence this can be generalised to any marked curve which has an automorphism fixing the marked points.

When $g > 0$ the Moduli space is a complex orbifold of the form $[M/G]$ where M is a smooth complex variety, G is a finite group acting smoothly and algebraically on M but the action is not free. There is indeed a variety $X = M/G$ which is not necessarily smooth, however that is not a fine moduli space. It is called the coarse moduli space which we describe below.

Example: *$M_{1,1} \cong \mathbb{H}/\operatorname{SL}(2, \mathbb{Z})$. Let $\Gamma(N)$ be the kernel of the surjective homomorphism $\operatorname{SL}(2, \mathbb{Z}) \to \operatorname{SL}(2, \mathbb{Z}/N)$, then $Y(N) = \mathbb{H}/\Gamma(N)$ is a Riemann surface. If $N \geq 3$ then $M_{1,1} \cong Y(N)/\operatorname{SL}(2, \mathbb{Z}/N)$.* ♣

In the algebraic world the analogue of an orbifold is a Deligne-Mumford stack, see Chapter XII of [**ACG**] for definitions.

2.2. **Coarse moduli.** Since there is no variety which satisfies 1,2,3 and hence no moduli space in the category of algebraic varieties we relax 2 and 3. A course moduli space $M_{g,n}$ is a variety that satisfies

1' Isomorphism classes of n pointed smooth genus g curves correspond to points of $M_{g,n}$.
2' For any family of n pointed smooth genus curves $f : C \to B$ there is a unique morphism $\phi : B \to M_{g,n}$ sending the fibers of f to their isomorphism classes.

There are coarse moduli spaces $M_{g,n}$ which are quasi-projective varieties, but not smooth.

Example: *These moduli spaces are also not proper varieties (ie not compact). To see this we consider the family of genus 1 curves $\chi \to B$ where $B = \mathbb{P}^1 - \{0, 1, \infty\}$ and $\chi \subset \mathbb{P}^2 \times B$, given by $y^2z - x(x-z)(x-\lambda z) = 0$ where $[x : y : z] \in \mathbb{P}^2$ and $\lambda \in B$ with projection map to B and section $\sigma_1(\lambda) = ([0 : 1 : 0], \lambda)$, then there is no extension of χ to all of $\mathbb{P}^1$, that is we can not extend the corresponding map $B \to M_{1,1}$ from the punctured curve to the punctures (even after a base change).* ♣

3. Nodal curves and Compactification

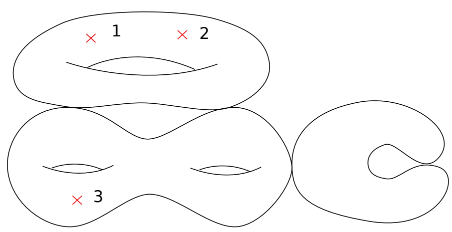
A proper (compact) moduli space is needed for enumerative geometry, since in that case we can do intersection theory, that is intersect algebraic cycles of complimentary dimensions to get numbers. Philosophically, counting objects amounts to intersecting certain cohomology classes on the moduli space of those objects.

We would thus like to get a moduli space which is compact and contains our moduli space. The new moduli space should not be too big, so we impose the condition that the original moduli space is contained in the compact one as a dense open set. Typically to get such a compactification we have to parametrize more objects than we started with.

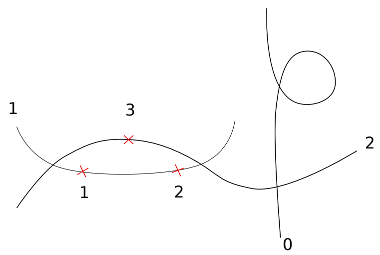
So instead of just restricting to smooth curves we now allow certain singular curves. Turns out that it suffices to consider curves with mildest possible singularity, that is a node.

3.1. **Stable Curves.** A stable n pointed genus g curve $(C; p_1, \dots, p_n)$ is such that

- C has at worst nodes as singularities.
- Arithmetic genus of C is g.
- $p_1, \dots, p_n$ are distinct smooth points on C.
- C has a finite automorphism group fixing the marked points (any rational component would have to have atleast 3 special points and any elliptic component at least 1 special point).



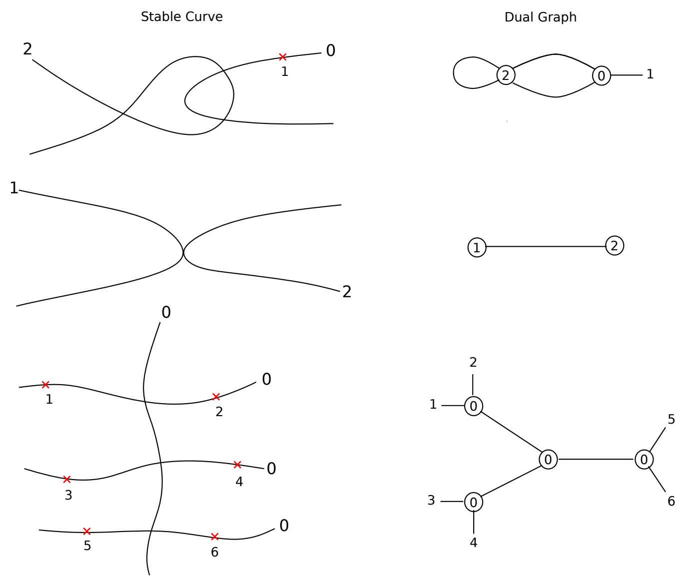
Analytic space



Algebraic Curve

Example: *The curves* $y^2z - x^2(x-z) = 0$ *and* $y^2z - x(x-z)^2$ *in* $\mathbb{P}^2$ *are nodal curves of genus 1 and occur as a limiting curves in the family in the example of subsection 2.2.* ♣

Associated to a stable curve there is a combinatorial object called the dual graph.



3.2. **Compactified moduli space.** There is a smooth, proper Deligne-Mumford stack which is the fine moduli space of stable n pointed genus g curves and we denote it by $\overline{\mathcal{M}}_{g,n}$. This stack is of the form

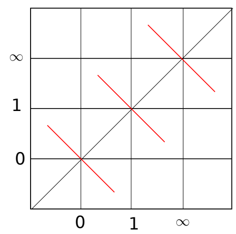
$[M/G]$ where M is a smooth projective variety and G a finite group acting smoothly and algebraically on M. It is also thus a smooth, compact, complex orbifold if we forget the algebraic structure.

There is also a course moduli space which is a projective variety (hence compact), but this is not smooth and has finite quotient singularities, we denote this by $\overline{M}_{g,n}$. It is just the variety quotient M/G.

Instead of going into the construction let us list some properties of the coarse moduli space which is of interest to us.

- $\overline{M}_{g,n}$ is an irreducible projective variety of dimension $3g-3+n$ with only finite quotient singularities. The orbifold $\overline{\mathcal{M}}_{g,n}$ is smooth and compact of the same dimension.
- The boundary, that is $\overline{M}_{g,n} \setminus M_{g,n}$ is a normal crossings divisor.
- When $g=0$, $\overline{M}_{0,n}$ is also the fine moduli space, and in this case the variety is smooth.
- There is a stratification of the moduli space using their topological type which is completely determined by the dual graph. All curves with at least k nodes form a sub-variety of codimension k. This sub-variety is completely contained in the boundary.
- There is the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ which is the universal curve in orbifold sense. Here we forget the last marked point and stabilise the curve.
- There are clutching morphisms $\overline{\mathcal{M}}_{g,n+1} \times \overline{\mathcal{M}}_{h,k+1} \to \overline{\mathcal{M}}_{g+h,n+k}$ obtained by identifying marked points. Similarly we have $\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$. These land on the boundary.

Example: *The genus zero moduli spaces are combinatorial objects. $\overline{M}_{0,3} \cong \overline{M}_{0,3}$ is a point. $\overline{M}_{0,4} \cong \mathbb{P}^1$. $\overline{M}_{0,5}$ is congruent to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at $(0,0)$, $(1,1)$ and (∞,∞), it is thus a Del-Pezzo surface. $\overline{M}_{0,n+1}$ can in general be constructed from by $\overline{M}_{0,n} \times \overline{M}_{0,n}$ along certain sub-varieties. See ACG section.* ♣



Example: *As course moduli spaces $\overline{M}_{1,1} \cong M_{0,4}/S_3$ and $\overline{M}_{0,6}/S_6$. This is not true as stacks.* ♣

3.3. **Stable reduction theorem.** The main ingredient in showing that $\overline{\mathcal{M}}_{g,n}$ is compact is to show that for any family of smooth pointed curves over a punctured disk the family can be extended to the whole disk possibly after base change by a covering map so that the fiber over 0 is a stable curve. This is guaranteed by the stable reduction theorem. Let $\Delta = \{z \in \mathbb{C} \mid |z| < \epsilon\}$ for a positive real number ϵ and $\dot{\Delta} = \Delta - \{0\}$.

Theorem 1 (Stable reduction)**.** *Suppose there is a family $f : C \to \dot{\Delta}$, and sections $\sigma_1, \ldots, \sigma_n$ of stable n-pointed genus g curves, then there exits a family*

$$\bar{f} : \bar{C} \to \Delta; \quad \bar{\sigma}_1, \ldots, \bar{\sigma}_n$$

of stable curves such that $\bar{f}$ restricted to $\dot{\Delta}$ is the pull back of f by a map $\dot{\Delta} \to \dot{\Delta}$ given by $\zeta \mapsto \zeta^k$.

For a proof see [**ACG**] , Chapter X, section 4. See [**HM**] chapter 3 section C for many examples.

Example. Here are a few example, left as exercises for the reader.

(1) Let C be a smooth curve, and $p, q : \Delta \to C$ holomorphic such that $p(0) = q(0)$ but $p(t) \neq q(t)$). Find the stable limit of the family $(C; p(t), q(t)$ at the origin.
(2) Let C be a nodal curve of genus g and $p : \Delta \to C$ holomorphic such that $p(0)$ is a node but $p(t)$ is a smooth point for all $t \neq 0$. What is the stable limit at 0?

(3) Let Q be a smooth conic in $\mathbb{P}^2$ and F a generic smooth quartic. Consider the family $C \subset \mathbb{P}^2 \times \Delta$ where C_t is given by $Q^2 + tF = 0$. Show that the stable limit at 0 is a hyper-elliptic genus 3 curve.

4. Moduli curves as Orbifolds

Associated to any Riemann compact surface S of genus $g > 1$ there is a unique hyperbolic metric on S (this can be seen via the uniformization theorem for instance). The Moduli space of Riemann surfaces is the same as moduli of hyperbolic surfaces.

Let us fix X, a smooth compact surface of genus g. A marked hyperbolic surface of genus g is a pair (S, ϕ) where S is a hyperbolic surface and ϕ a diffeomorphism $X \to S$. Two such surfaces (S, ϕ) and (T, ψ) are isomorphic if there is an isometry $m : S \to T$ such that $\psi^{-1} \circ m \circ \phi$ is isotopic to the identity.

The Teichmüller space $\mathcal{T}_g$ is the moduli space of marked genus g hyperbolic surfaces. Intuitively since all closed surfaces of genus g are diffeomorphic, so we fix one smooth structure on the surfaces via the marking and look at different complex structures or correspondingly hyperbolic metrics. While the moduli space identifies isometric hyperbolic surfaces or biholomorphic Riemann surfaces, the Teichmüller space identifies two such Riemann surfaces only if the biholomorphism is isotopic to the identity, or in the case of hyperbolic surfaces the isometry is isotopic to the identity.

Let $\mathfrak{M}(X)$ be the space of all hyperbolic metrics on X. Let $\mathrm{Diff}^+(X)$ be the group of orientation preserving self-diffeomorphisms of X and $\mathrm{Diff}^0(X)$ the path component of the identity in the group. Define the mapping class group of X by

$$\Gamma_g = \mathrm{Diff}^+(X)/\,\mathrm{Diff}^0(X).$$

Then we have the following

$$\mathcal{T}_g = \mathfrak{M}(X)/\,\mathrm{Diff}^0(X) \quad \text{and} \quad \mathcal{M}_g = \mathfrak{M}(X)/\,\mathrm{Diff}^+(X).$$

It is thus clear that Γ_g acts on $\mathcal{T}_g$ an the quotient is $\mathcal{M}_g$. $\mathcal{T}_g$ has a natural complex structure via this description and is isomorphic to an open ball in $\mathbb{C}^{3g-3}$, see for instance Ahlfors's work.

Γ_g is a discrete group and the action on $\mathcal{T}_g$ has finite isotropy groups. The isotropy group of a point is exactly the group of hyperbolic automorphisms of the corresponding surface.

It can be shown that there is a normal finite index sub-group $G \subset \Gamma_g$ acting freely on $\mathcal{T}_g$ and the quotient is a smooth complex manifold with an action of Γ_g/G. The quotient by that action is $\mathcal{M}_g$, showing that $\mathcal{M}_g$ is a complex orbifold.

There are many ways of giving coordinates on the Teichmüller space. The Fenchel-Nielsen coordinates are obtained as follows: choose $3g-3$ closed geodesics on S cutting along which decomposes S into a set of pair of pants, precisely $2g-2$. For each such curve we can record the length and twist parameters $(l_1, \tau_1, \ldots, l_{3g-3}, \tau_{3g-3})$. This gives a homemorphism $\mathcal{T}_g \to \mathbb{R}^{3g-3} \times \mathbb{R}_+^{3g-3}$.

For $\mathcal{M}_{g,n}$ one looks at smooth compact genus g surfaces with n punctures, such that the Euler characteristic is negative. See [**ACG**] Chapter XV for a survey and references.

5. Some line bundles and cohomology classes

Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve. Let ω_π be the relative dualising sheaf, that is ω_π restricted to each fiber is the cotangent bundle of the corresponding curve. Then we define the Hodge bundle

$$\mathbb{H} = \pi_* \omega_\pi.$$

At the point $[(C; p_1, \ldots, p_n)]$ the fiber of $\mathbb{H}$ is the vector space $H^0(\Omega_C, C)$ the space of global sections of the cotangent sheaf. Hence $\mathbb{H}$ is a rank g vector bundle.

5.1. **Lambda classes.** Let

$$\lambda_1 = c_1(\mathbb{H}) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Similarly we define

$$\lambda_i = c_i(\mathbb{H}) \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

5.2. **Psi classes.** Let $\sigma_1, \dots, \sigma_n$ be the sections of π corresponding to the marked points. Then define the line bundles

$$\mathbb{L}_i = \sigma_i^* \omega_\pi.$$

At the point $[(C; p_1, \dots, p_n)]$ the fiber of $\mathbb{L}_i$ is the cotangent space to C at p_i. We define

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

5.3. **Mumford (kappa) classes.** Let $D_1, \dots, D_n$ be the images of $\sigma_1, \dots, \sigma_n$. The generic curve of D_i has one node and two irreducible components of genus 0 and g, such that the genus 0 component has the marked points $i, n+1$ and the genus g component has the rest of the marked points. Let $D = D_1 + \cdots + D_n$ and

$$K_\pi = c_1(\omega_\pi(D)) \in A^1(\overline{\mathcal{M}}_{g,n+1}, \mathbb{Q}).$$

Note that we take the Chern class in the Chow ring in this case. Define

$$\kappa_i = \pi_*(K_\pi^{i+1}) \in A^i(\overline{\mathcal{M}}_{g,n+1}, \mathbb{Q}).$$

We can also consider the kappa classes in the cohomology ring via the cycle class map and Poincaré duality. Note that $\kappa_0 = 2g - 2 + n$.

5.4. **Boundary divisors.** As we mentioned the boundary $\partial\mathcal{M}_{g,n}$ is a normal crossings divisor. The irreducible components of the boundary consist of curves with one node. This node is called separating or non-separating depending on whether the deletion of the node creates two components or just a single component.

Let Δ_0 be the boundary component consisting of curves with a non-separating node.

Let $A \subset \{1, \dots, n\}$ be the subset such that $1 \in A$ and $0 \leq h \leq g$. Denote by $\Delta_{h,A}$ the boundary divisor consisting of curves with a separating node such that one of the irreducible components has genus h and marked points labelled by A and the other has genus $g - h$ and the rest of the marked points, stability conditions dictate that $|A| \geq 2$ if $h = 0$ and $|A| \leq n - 2$ if $h = g$.

These are all the boundary divisors.