Maryam Mirzakhani memorial talk

IISER Pune, Maths Club

November 17, 2017

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She did her graduate studies at Harvard University, under the supervision of Fields medallist Curtis T. McMullen and obtained her PhD in 2004.

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She died of breast cancer on 14 July 2017 at the age of 40.

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This conjecture has deep consequences in quantum gravity, a field of theoretical physics, that seeks to describe gravity according to the principles of quantum mechanics.

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Angles: Same as Euclidean angles.

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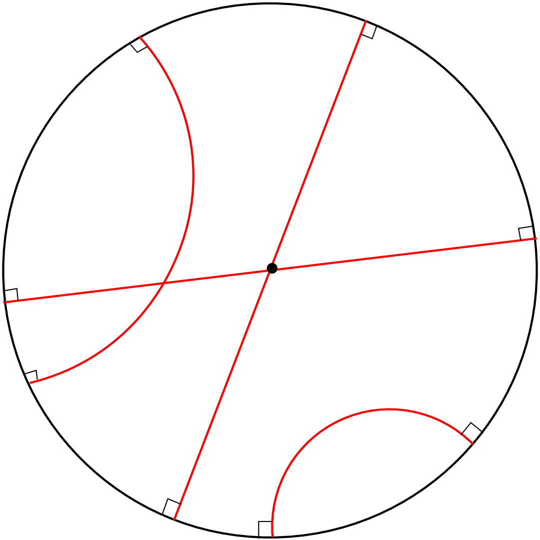

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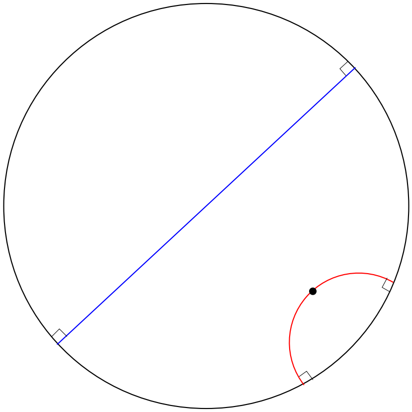
However it does not satisfy the 5th postulate, or the parallel postulate:

5 Given any straight line and a point not on it, there "exists one and only one straight line which passes" through that point and never intersects the first line.

Infinite Parallels



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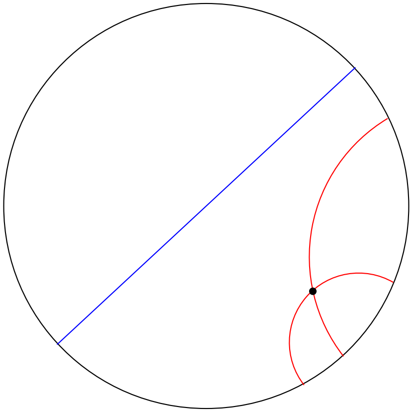

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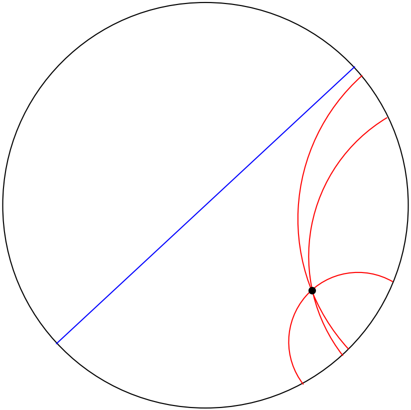

Hyperbolic distance

Recall in Euclidean plane if $\gamma : [0,1] \to \mathbb{R}^2$ is a curve, and $\gamma(t) = (x(t), y(t))$ then its length is

$$\ell(\gamma) = \int_0^1 |\gamma'(t)| dt = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

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In the hyperbolic plane we measure lengths of curves differently. If

$$\gamma : [0,1] \to D, \quad \gamma(t) = (x(t), y(t))$$

then length of γ is

$$\ell(\gamma) = \int_0^1 \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} dt = \int_0^1 \frac{2\sqrt{(x'(t))^2 + (y'(t))^2}}{1 - x^2(t) - y^2(t)} dt.$$

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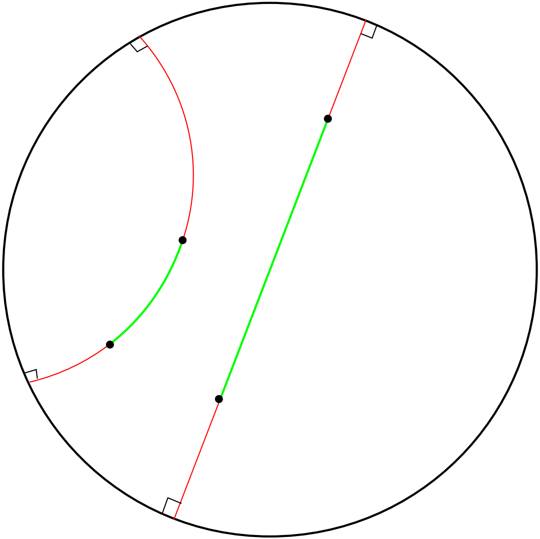
The distance between any two points in the Hyperbolic plane is the length of the shortest curve joining the two points. Hence the length of the unique geodesic between those points.

Geodesics



Distance

Geodesic between the origin $O = (0,0)$ and the point $A = (a,0)$ in D is the straight line OA, parametrized by

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All distances can be calculated using this, since there are isometries of D that take any two points to the origin and a point on the x-axis.

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- Distances go off to infinity as we approach the boundary so this is a complete metric space.
- D is a Riemannian manifold of dimension 2, since it is an (open) subset of $\mathbb{R}^2$ and we can measure lengths of curves.
- The geometry of D is a type of non-euclidean geometry since it does not satisfy the parallel postulate of Euclid.

Closed Hyperbolic surfaces

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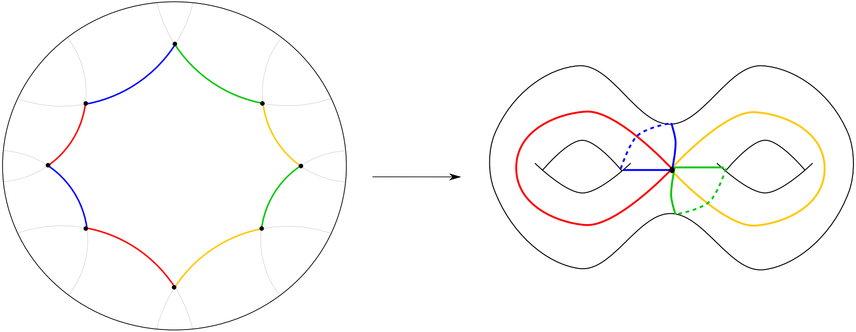

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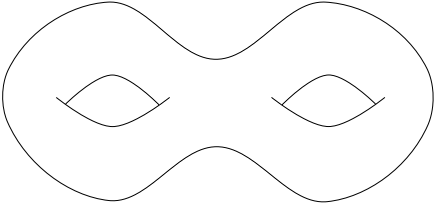
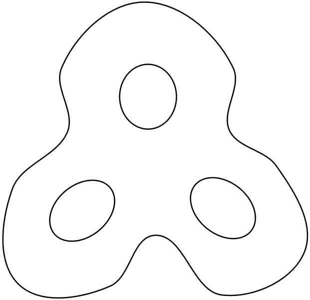
These manifolds arise naturally in nature and also in physics. For instance there is something called the world sheet in String theory. It is the 2 dimensional manifold traced out by a string moving in space and can be realised as a hyperbolic surface in certain cases.

Genus

The genus of a closed surface is just the number of holes it has. A surface is hyperbolic if it has genus at least 2.



Genus 2



Genus 3

Moduli Space

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$\mathcal{M}_g$ is called the moduli space of genus g hyperbolic surfaces and this is a major topic of study in mathematics, investigated by numerous mathematicians like Riemann, Mumford, Deligne, Kontsevic, Okounkov to name a few.

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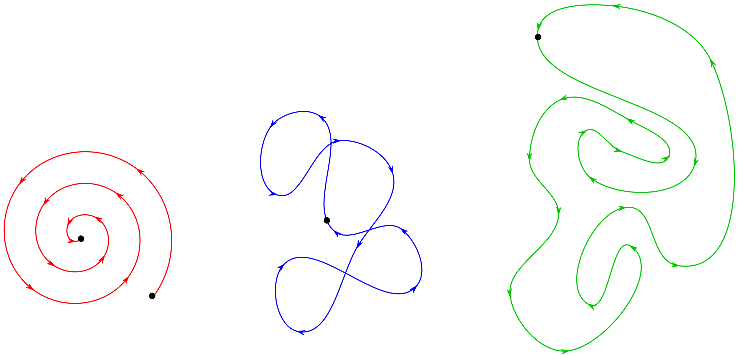
In words γ has the same starting and ending points, which is also a geodesic and which does not cross itself.

Curves



Red curve is not closed, blue curve is closed but not simple, green curve is simple and closed.

Mirzakhani's result

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Let $s_X(L)$ be the number of simple closed geodesics in X whose length is at most L. Then Mirzakhani proves that asymptotically

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Moreover $\eta : \mathcal{M}_g \to \mathbb{R}_+$ is a continuous function.