Operads and Moduli of Curves

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Algebraic Curves

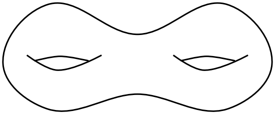
We will be working over $\mathbb{C}$.

A smooth algebraic curve is a smooth projective variety of dimension 1.
Any algebraic curve is a Riemann surface and vice versa.

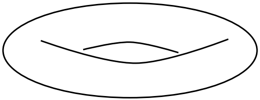
Genus of a smooth curve C is

$$g(C) := \frac{1}{2} \dim H_1(C) = \dim H^0(C, \Omega_C) = \dim H^1(C, \mathcal{O}_C)$$

Examples are $\mathbb{P}^1$, elliptic curves ...



Genus = 2



Genus =1

The Moduli of n pointed genus g Curves

An n pointed genus g curve $(C, p_1, \ldots, p_n)$ is a smooth algerbaic curve of genus g, with the additional data of n distinct points on it.

Two such curves $(C, p_1, \ldots, p_n)$ and $(C', p'_1, \ldots, p'_n)$ are isomorphic if there is an isomorphism $f : C \to C'$, such that $f(p_i) = p'_i$. (We require $2g - 2 + n > 0$, so that there are only finitely many automorphisms of a curve fixing its marked points.)

Parameter Space: We want a parameter space $\mathcal{M}_{g,n}$ for isomorphism classes of n pointed genus g curve

Universal family: Further we require that there is a universal family $\mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$, such that for any family of n pointed genus g curves $C \to B$ there exits a map $B \to \mathcal{M}_{g,n}$ under which C is a pull back of $\mathcal{T}_{g,n}$.

So we should have a fiber square

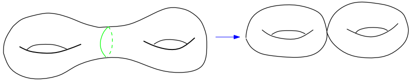
$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{T}_{g,n} \arrow[d] \\ B \arrow[r] & \mathcal{M}_{g,n} \end{tikzcd}$$

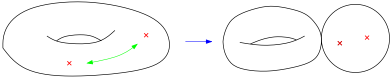
There is no solution for the moduli problem in the category of schemes, but a fine moduli space exists as a Deligne-Mumford stack (analogue of orbifold in Algebraic Geometry).

Underlying the stack there is a coarse moduli space which is an algebraic variety usually denoted $M_{g,n}$, where as the stack will be denoted $\mathcal{M}_{g,n}$.

Deligne-Mumford compactification

$M_{g,n}$ is not a complete variety. The reason being, there can be singular curves in the limit of smooth curves as they vary in families.





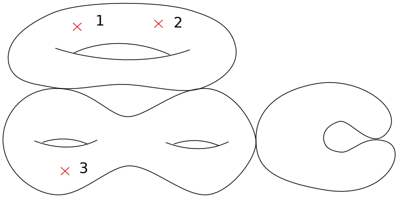
Deligne and Mumford gave a compactification of $M_{g,n}$, by enlarging the moduli problem to include certain singular curves.

It turns out that the class of curves that can arise as limits of smooth curves are curves with only nodal singularities and finite automorphism group. Such curves are called **stable curves**.

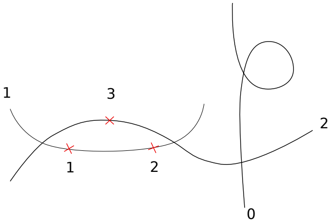
Stable Curves

A stable curve C of genus g, and n marked points $\{p_1, \ldots, p_n\}$, is a projective curve with the following properties:

1. $\dim H^1(C, \mathcal{O}_C) = g$, where $\mathcal{O}_C$ is the structure sheaf.
2. The singularities of C are all nodes.
3. $\{p_1, \ldots, p_n\}$ are distinct smooth points of the curve.
4. $C^{\mathrm{sm}} \setminus \{p_1, \ldots, p_n\}$ (the smooth locus with the marked points removed) has negative Euler characteristic.



Analytic space



Algebraic Curve

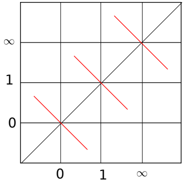
Properties

There exists a D-M stack $\overline{\mathcal{M}}_{g,n}$ which is the moduli space of n pointed stable curves. We denote the corresponding coarse moduli space by $\overline{M}_{g,n}$. Some properties are:

1. $\overline{\mathcal{M}}_{g,n}$ is irreducible, smooth, projective of dimension $3g-3+n$.
2. It is the quotient of a smooth projective variety by the action of a finite group. (Pikaart-Dejong)
3. $\overline{\mathcal{M}}_{g,n+1}$ along with the natural map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family.
4. $\overline{M}_{g,n}$ is an irreducible, projective variety, of dimension $3g-3+n$, and has only mild singularties (finite quotient).
5. $M_{g,n}$ is an open dense subvariety of $\overline{M}_{g,n}$ and the complement is a divisor with normal crossings.
6. When $g=0$, $\overline{M}_{0,n}$ is a smooth projective varieties and $\overline{\mathcal{M}}_{0,n} \cong \overline{M}_{0,n}$.

Examples

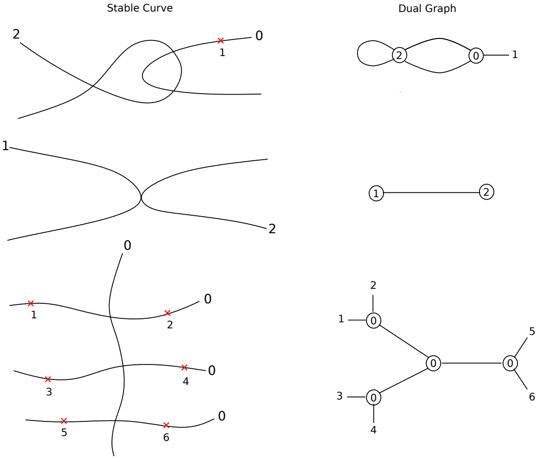
- $\overline{M}_{0,3}$ is a point.
- $M_{0,4} \cong \mathbb{P}^1 \backslash \{0, 1, \infty\}$ and $\overline{M}_{0,4} \cong \mathbb{P}^1$.
- $\overline{M}_{0,5}$ is congruent to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at $(0,0), (1,1)$ and (∞, ∞).



- In general $\overline{M}_{0,n+1}$ can be inductively constructed as a blow up of $\overline{M}_{0,n} \times \mathbb{P}^1$. It is a smooth projective variety.
- $M_{1,1} \cong M_{0,4}/S_3$ and $\overline{M}_{1,1} \cong \overline{M}_{0,4}/S_3 \cong \mathbb{P}^1/S_3$.
- $\overline{M}_2 \cong \overline{M}_{0,6}/S_6$.

Dual Graphs

To a stable curve we can associate a dual graph.



The dual graphs depend only on the topological type of the curve.

Stratification by dual graphs

There is a stratification of $\overline{M}_{g,n}$ by the dual graphs. Let G be a graph and M_G the locus of curves in $\overline{M}_{g,n}$ whose dual graph is G.

Then M_G is locally closed and $\overline{M}_G$ contains $M_{G'}$ if G can be obtained from G' by contracting some edges of G'.

Let $\Gamma(g,n)$ be the set of dual graphs to stable curves of genus g with n marked points, then

$$\overline{M}_{g,n} = \bigsqcup_{G \in \Gamma(g,n)} M_G$$

$\overline{M}_G$ is a subvariety of co-dimension equal to the number of edges of G.

Operads

An $\mathbb{S}$ module in a symmetric monoidal category $\mathcal{E}$ is a collection of objects

$$\mathcal{V} = \{\mathcal{V}(n) \mid n \geq 0\}$$

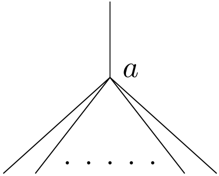
with an action of the symmetric group S_n on $\mathcal{V}(n)$.

An **Operad** $\mathcal{V}$ in $\mathcal{E}$ is an $\mathbb{S}$ module along with bilinear operations

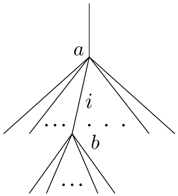
$$\circ_i : \mathcal{V}(n) \otimes \mathcal{V}(m) \to \mathcal{V}(m+n-1)$$

for $1 \leq i \leq n$ satisfying certain axioms of associativity and equivariance.

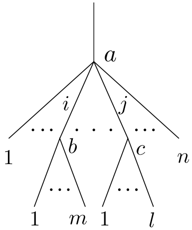
Intuitively an element a of $\mathcal{V}(n)$ can be thought of as a rooted tree with n input leaves, and one output leaf which is the root.

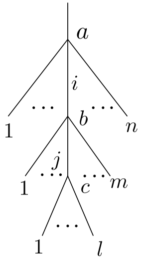


The product $a \circ_i b$ then corresponds to grafting of trees as follows:



The equivariance and associativity axioms ensure that we can form the products unambiguously.





First examples of Operads

Commutative operad *Com*, has the trivial representation $Com(n)$ of S_n for each n, with basis μ_n, and

$$\mu_n \circ_i \mu_m = \mu_{m+n-1}$$

Associative operad $Ass(n)$ is the regular representation consisting of a basis of words in the letters $\{x_1, \ldots, x_n\}$ and

$$(x_1 x_2) \circ_2 (x_2 x_3 x_1) = x_2 x_3 x_4 x_1$$

Lie operad $Lie(n)$ consists of Lie words in the letters $\{x_1, \ldots, x_n\}$ each letter occurring exactly once.

Free Operads

Starting with any $\mathbb{S}$ module $\mathcal{V}$ we can form the free operad $\mathbf{F}\mathcal{V}$, by summing over trees. Let $\mathbb{T}$ be the set of (isomorphism classes of) rooted trees and $\mathbb{T}(n)$ the trees with n input leaves. Then

$$\mathbf{F}\mathcal{V}(n) = \bigoplus_{T \in \mathbb{T}(n)} \mathcal{V}(T)$$

where

$$\mathcal{V}(T) = \bigotimes_{v \in T} \mathcal{V}(v)$$

This has a natural product structure.

Free operads are useful since other operads can be presented as the quotient of some free operad by an operadic ideal.

They are also useful in the cobar construction as we shall see later.

Little Disks Operad

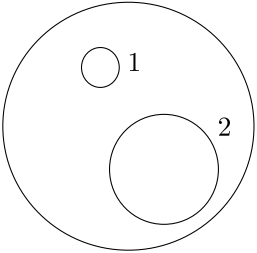
An elementary example is the little disks operad. This is an operad of topological spaces. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $\mathcal{D}(n)$ be the topological space

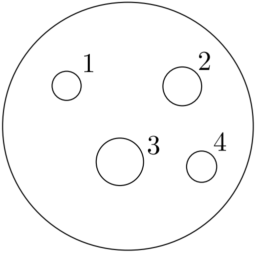
$$\mathcal{D}(n) = \left\{ \begin{pmatrix} z_1, \ldots, z_n \\ r_1, \ldots, r_n \end{pmatrix} \in D^n \times \mathbb{R}_+^n \mid \text{ disks } r_i D + z_i \text{ are disjoint subsets of } D \right\}$$

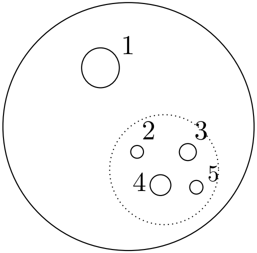
S_n acts on $\mathcal{O}(n)$ by permuting disks. There is a natural operad structure on $\mathcal{O}$ by gluing of disks. If $a = \binom{w_1, \ldots, w_m}{s_1, \ldots, s_m}$ and $b = \binom{z_1, \ldots, z_n}{r_1, \ldots, r_n}$ then

$$a \circ_i b = \begin{pmatrix} w_1, \ldots, w_{i-1}, w_i + s_i z_1, \ldots, w_i + s_i z_n, w_{i+1}, \ldots, w_m \\ s_1, \ldots, s_{i-1}, s_i r_1, \ldots, s_i r_n, s_{i+1}, \ldots, s_m \end{pmatrix}$$

Product in little disks operad

$a =$ 

$b =$ 

$a \circ_2 b =$ 

The E_2 operad

The E_2 operad is an operad in graded vector spaces. It is obtained by taking the homology of the little disks operad.

$$E_2(n) = H_*(\mathcal{D}(n))$$

Note that $\mathcal{D}(n)$ is homotopic to $\mathbb{C}_0^n$ the configuration of n points in $\mathbb{C}$. Thus $E_2(n) = H_*(\mathbb{C}_0^n)$.

From Cohen's work on the cohomology of $\mathbb{C}_0^n$ it turns out that

$$E_2 \cong Com \circ \Lambda^{-1} Lie \quad (*)$$

Here $\circ$ is an operation on $\mathbb{S}$ modules and Λ is suspension

$$\Lambda \mathcal{V}(n) = \Sigma^{1-n} \operatorname{sgn}_n \otimes \mathcal{V}(n)$$

Gravity operad

The moduli space $M_{0,n}$ parametrizes n points on $\mathbb{P}^1$ upto projective transformations. Hence

$$M_{0,n+1} \cong \mathbb{C}_0^n / \mathrm{Aff}(\mathbb{C})$$

where

$$\mathrm{Aff}(\mathbb{C}) \cong \mathbb{C} \rtimes \mathbb{C}^\times$$

Thus

$$M_{0,n+1} \cong \mathbb{C}_0^n / S^1$$

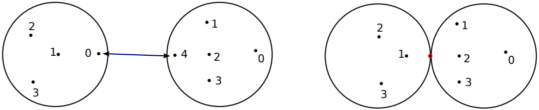
Consequently the homology of $M_{0,n}$ suitably suspended forms a (cyclic) operad, called the gravity operad.

$$Grav(n) = \Sigma^{3-n} \operatorname{sgn}_n \otimes H_*(M_{0,n})$$

Hypercommutative operad

The spaces $\overline{M}_{0,n}$ form a topological (cyclic) operad. Note that there is an embedding as a boundary divisor (clutching morphism)

$$\overline{M}_{0,m+1} \times \overline{M}_{0,n+1} \to \overline{M}_{0,m+n}$$



Hence we get a (cyclic) operad in graded vectorspaces by taking the homology.

$$Hycom(n) = H_*(\overline{M}_{0,n})$$

Cobar construction

Let $\mathcal{V}$ be an $\mathbb{S}$ module of chain complexes, and $\mathcal{V} * (n)$ the graded dual vector space. Then we define the $\mathbb{S}$ module $\mathcal{V}^\vee$ as

$$\mathcal{V}^\vee(n) = \Sigma^{n-3}\,\mathrm{sgn}_n \otimes \mathcal{V}^*(n)$$

Recall the free operad generated by $\mathcal{V}^\vee$ is given by

$$\mathbf{F}\mathcal{V}^\vee(n) = \bigoplus_{T \in \mathbb{T}(n)} \mathcal{V}^\vee(T)$$

This has a natural filtration, $\mathbf{F}_i\mathcal{V}(n)$ is the direct sum over trees with i vertices. If $\mathcal{V}$ is an operad then there is a differential

$$\partial : \mathbf{F}_i\mathcal{V}(n) \to \mathbf{F}_{i+1}\mathcal{V}(n)$$

which is obained by the operadic composition.

Cobar construction

The cobar operad of $\mathcal{V}$ is the operad $\mathcal{BV}$ with the underlying $\mathbb{S}$ module of $\mathbf{F}\mathcal{V}^\vee$ and the differential obtained by twisting the differential of $\mathbf{F}\mathcal{V}^\vee$ with ∂.

Theorem

$\mathcal{B}$ is a homotopy functor, and $\mathcal{BBV}$ is homotopy equivalent to $\mathcal{V}$.

We will see that $\mathcal{B}\,\mathit{Grav} \simeq \mathcal{B}\,\mathit{Hycom}$.

A Spectral sequence

Let X_p be the p dimensional strata in $\overline{M}_{0,n}$. Then

$$\emptyset \subset X_0 \subset \ldots \subset X_{n-3} = \overline{M}_{0,n}$$

There is a spectral sequence associated to this filtration

$$E_1^{p,q} = H^{p+q}(X_p/X_{p-1}) \Longrightarrow H^{p+q}(\overline{M}_{0,n})$$

Note that $H^{p+q}(X_p/X_{p-1}) \cong H_c^{p+q}(X_p \backslash X_{p-1})$.

Let $\Gamma_k(0,n)$ be the dual graphs with k edges, then

$$X_p \backslash X_{p-1} = \bigsqcup_{G \in \Gamma_{n-3-p}(0,n)} M_G$$

and

$$E_1^{p,q} = \bigoplus_{G \in \Gamma_{n-3-p}(0,n)} H_c^{p+q}(M_G)$$

Duality of *Grav* and *Hycom*

For any graph G, $M_G \cong \prod_{v \in \mathrm{Vert}(G)} M_v$. Thus summing up all the rows of E_1, we exactly get

$$\mathcal{B}\, Grav(n)$$

The above spectral sequence carries a mixed Hodge structure and and an action of S_n and the differentials are compatible with both.

It follows from the purity of the MHS of the cohomology of $M_{0,n}$ that $d_2 = 0$ and the Spectral sequence collapses in the second page.

$$E_2 = E_\infty$$

Further $E_2^{p,p} \cong H^{2p}(\overline{M}_{0,n})$ and $E_2^{p,q} = 0$ if $p \neq q$.

This proves $\mathcal{B}\, Grav$ is homotopy equivalent to *Hycom*.

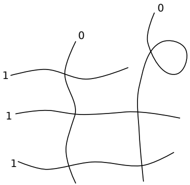
	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$q=5$	0	0	0	0	0	$\mathbb{C}(5)^{1}$
$q=4$	0	0	0	0	$\mathbb{C}(4)^{119}$	$\mathbb{C}(4)^{20}$
$q=3$	0	0	0	$\mathbb{C}(3)^{1918}$	$\mathbb{C}(3)^{1358}$	$\mathbb{C}(3)^{155}$
$q=2$	0	0	$\mathbb{C}(2)^{9450}$	$\mathbb{C}(2)^{13902}$	$\mathbb{C}(2)^{5747}$	$\mathbb{C}(2)^{580}$
$q=1$	0	$\mathbb{C}(1)^{17325}$	$\mathbb{C}(1)^{40950}$	$\mathbb{C}(1)^{33348}$	$\mathbb{C}(1)^{10668}$	$\mathbb{C}(1)^{1044}$
$q=0$	$\mathbb{C}(0)^{10395}$	$\mathbb{C}(0)^{34650}$	$\mathbb{C}(0)^{44100}$	$\mathbb{C}(0)^{26432}$	$\mathbb{C}(0)^{7308}$	$\mathbb{C}(0)^{720}$
$q=-1$	0	0	0	0	0	0

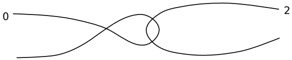
Table : Spectral sequence for $\overline{M}_{0,8}$

A Filtration

There is a filtration on $\overline{M}_{g,n}$ given by the number of rational components of a curve.

Let $\overline{M}_{g,n}^{\leqslant k}$ be the open set in $\overline{M}_{g,n}$ parametrizing stable curves with at most k rational components (components of geometric genus 0).





Curve C has 2 rational components where as D has just 1. We have

$$\overline{M}_{g,n}^{\leqslant 0} \subset \overline{M}_{g,n}^{\leqslant 1} \subset \ldots \subset \overline{M}_{g,n}^{\leqslant 2g-2+n} = \overline{M}_{g,n}$$

Original conjecture by Looijenga.

Conjecture (Looijenga)

M_g can be covered by $g - 1$ open affine subvarieties.

Fontanari and Pascolutti proved this for genus up to 5 (2012).

There is a generalization of Looijenga's conjecture due to Roth and Vakil using the affine stratification number.

Conjecture (Looijenga Roth and Vakil)

$$\operatorname{asn} \overline{M}_{g,n}^{\leqslant k} \leq g - 1 + k \quad \textit{for} \quad g > 0, k \geq 0$$

Sharpness of these bounds are not known.

Hyperelliptic locus

A hyperelliptic curve is the smooth completion of an affine curve of the form:

$$y^2 = P(x)$$

where P is a polynomial of degree $2g+2$ with distinct roots.

Let $H_g \subset M_g$ be the locus of hyper elliptic curves, and $\overline{H}_g$ its closure. We have an induced filtration on $\overline{H}_g$ from the filtration on $\overline{M}_g$.

$$\overline{H}_g^{\leqslant k} = \overline{H}_g \cap \overline{M}_g^{\leqslant k}$$

We show

Theorem

$\mathrm{asn}(\overline{H}_g^{\leqslant k}) \leq g - 1 + k$ *and when* $k = 0$, $\mathrm{asn}(\overline{H}_g^{\leqslant 0}) = g - 1$.

We have the following corollary

Corollary

$\mathrm{asn}(\overline{M}_g^{\leqslant 0}) \geq g - 1.$

which shows that the upper bound of Roth and Vakil's conjecture is sharp in certain cases.

The main ingredient of the proof of the previous theorem is following lemma.

Lemma

There is a constructible sheaf Ł on $\overline{H}_g$ such that $H^{3g-2}(\overline{H}_g^{\leqslant 0}, Ł)$ is non-trivial.

Proof Techniques

We begin by the observing that as a coarse moduli space,

$$\overline{H}_g \cong \overline{M}_{0,2g+2}/S_{2g+2}$$

This isomorphism uses the theory of admissible covers introduced by Harris and Mumford.

Let $\pi : \overline{M}_{0,2g+2} \to \overline{H}_g$ be the quotient map; define

$$\overline{M}^{(k)}_{0,2g+2} = \pi^{-1}(\overline{H}_g^{\leqslant k})$$

Let $Ł = \pi_* \underline{\mathbb{C}}$, then

$$H^k(\overline{H}_g^{\leqslant 0}, Ł) \cong H^k(\overline{M}^{(0)}_{0,2g+2}, \mathbb{C})$$

Hence the sharpness result boils down to showing a certain cohomology group of $\overline{M}^{(0)}_{0,2g+2}$ is non-trivial.

Truncated spectral sequence

Just as the stratification of $\overline{M}_{0,n}$ by dual graphs there is a stratification of $\overline{M}^{(0)}_{0,2g+2}$ by dual graphs too, but there is a combinatorial condition on the graphs now (good graph).

Corresponding to this stratification there is a spectral sequence as before

$$F_1^{p,q} = \bigoplus_{\substack{G \text{ good with} \\ n-3-p \text{ edges}}} H_c^{p+q}(M_G) \Longrightarrow H_c^{p+q}(\overline{M}^{(0)}_{0,2g+2})$$

By Poincaré duality we have

$$H_c^g(\overline{M}^{(0)}_{0,2g+2}) \cong H^{3g-2}(\overline{M}^{(0)}_{0,2g+2})$$

From the support of the Spectral sequence it is clear that

$$\overline{H}_c^g(\overline{M}^{(0)}_{0,2g+2}) \cong F_\infty^{g,0} \cong F_2^{g,0}$$

Method 1 Count dimensions: Fails after genus 3.

Method 2 Count multiplicity of irreducible representations: Fails after genus 6.

Method 3 Please read my paper.

Further research

- Analyse other cases of the conjecture. So far only partial results for small g.
- What can be said about the trigonal or tetragonal locus instead of the hyperelliptic locus?