Presentabilty of Mapping Class Group

M499 Project-II Presentation

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Outline

Recap

- We consider surfaces which are a connect sum of $g \geq 0$ tori with a 2-sphere, with $b \geq 0$ boundary components and $n \geq 0$ punctures and we denote our surface as $S_{g,n}^b$.

- We define the **Mapping Class Group** of S, $Mod(S)$ as,

$$\boxed{Mod(S) = \frac{Homeo^+(S, \partial S)}{Homeo_0^+(S, \partial S)} = \frac{Homeo^+(S, \partial S)}{\sim}}$$

Where $Homeo_0^+(S, \partial S)$ is the path component of the identity in $Homeo^+(S, \partial S)$ and "$\sim$" denotes the isotopy relation.

Recap

- A closed curve α is simple if it has no self intersections. α is **essential** if it is not homotopic to a point, a puncture or a boundary component.
- We define the **geometric intersection number** between free homotopy classes a and b of simple closed curves in a surface as,

$$i(a,b) = \min\{|\alpha \cap \beta| : \alpha \in a, \beta \in b\}$$

- Two representatives α, β of homotopy classes a, b of simple closed curves are in **minimal position** if $|\alpha \cap \beta| = i(a,b)$.

Here, we state the change of coordinates theorem.

Theorem (Change of Coordinates)

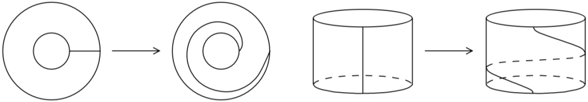
Two essential simple closed curves α, β on a surface S have the same topological type if and only if there is an orientation-preserving homeomorphism $\phi : S \longrightarrow S$ that fixes ∂S, with $\phi \circ \alpha = \beta$.

Dehn Twists

Let $A = S^1 \times [0,1]$ be the annulus. Consider the map $T : A \longrightarrow A$,

$$T(\theta, t) = (\theta + 2\pi t, t)$$

The map T is an orientation-preserving homeomorphism that fixes ∂A pointwise. Here are two pictorial representations of the map T.



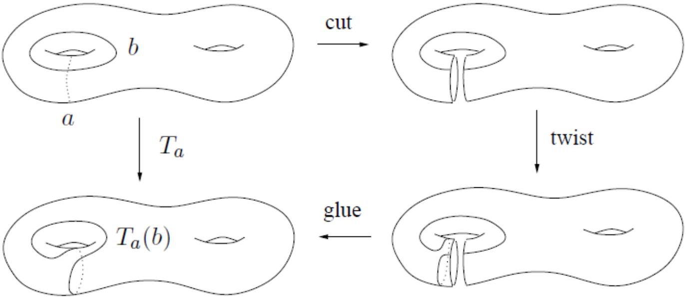
Now let S be an arbitrary (oriented) surface and let α be a simple closed curve in S and N be a regular neighborhood of α, and choose an orientation preserving homeomorphism $\phi : A \longrightarrow N$.

Dehn Twists

Define $T_\alpha : S \longrightarrow S$, called a **Dehn twist** about α, as follows:

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & x \in N \\ x & x \in S \backslash N \end{cases}$$

This can be represented as,



Outline

The Dehn Lickorish Theorem

The main theorem that we will prove for finite generation of $Mod(S_g)$ is the following.

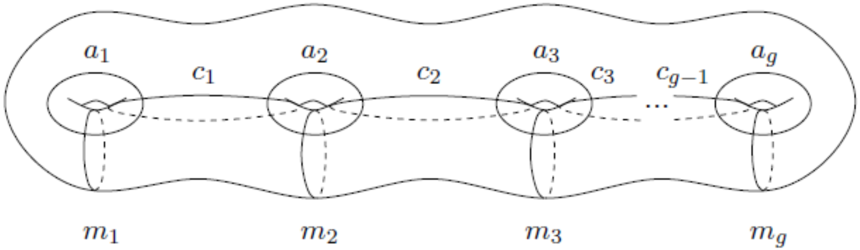
Theorem (Dehn-Lickorish)

For $g \geq 0$ the group $Mod(S_g)$ is generated by finitely many Dehn twists about nonseparating simple closed curves.

- In the 1920's Dehn proved that $Mod(S_g)$ is generated by $2g(g-1)$ Dehn twists.
- Mumford, building on Dehn's work, showed in 1967 that only Dehn twists about nonseparating curves were needed.

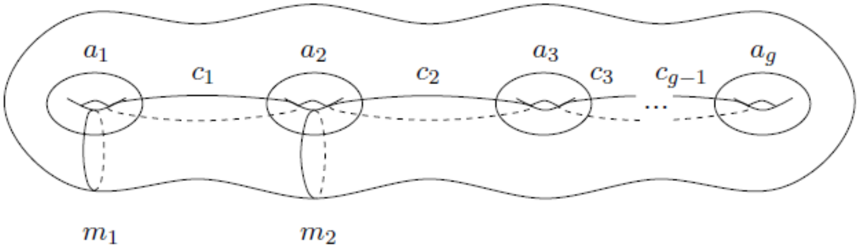
Explicit Generators

In 1964 Lickorish gave an independent proof that $Mod(S_g)$ is generated by the Dehn twists about the $3g-1$ nonseparating curves given below.



Explicit Generators

In 1979 Humphries proved that the twists about the $2g + 1$ curves given below suffice to generate $Mod(S_g)$.



These generators are called the Humphries generators. Humphries further showed that any set of Dehn twist generators for $Mod(Sg)$ must have at least $2g + 1$ elements.

Outline

The Setup

Definition (Pure Mapping Class Group)

For a surface $S_{g,n}$, the **pure mapping class group**, $PMod(S_{g,n})$, is defined as the subgroup of $Mod(S_{g,n})$ consisting of elements that fix each puncture individually.

This gives us a short exact sequence,

$$1 \to PMod(S_{g,n}) \xrightarrow{i} Mod(S_{g,n}) \xrightarrow{\phi} \Sigma_n \to 1$$

where i is the inclusion map and ϕ maps an element of $Mod(S_{g,n})$ to the corresponding action on the n punctures in Σ_n. Note that $PMod(S_{g,0}) = Mod(S_{g,0})$ and $PMod(S_{g,1}) = Mod(S_{g,1})$.

Outline of the proof

In order to prove the Dehn-Lickorish Theorem, we will actually prove the more general result given below.

Theorem

Let $S_{g,n}$ be a surface of genus $g \geq 1$ with $n \geq 0$ punctures. Then the group $PMod(S_{g,n})$ is finitely generated by Dehn twists about nonseparating simple closed curves in $S_{g,n}$.

This proof will follow by double induction on the genus and punctures respectively, with base cases $Mod(S_{1,0}) \cong Mod(S_{1,1}) \cong SL(2, \mathbb{Z})$.

Outline of the proof

- Induction on genus: The main ingredient of this part of the proof is the modified complex of curves that we will define after this section along with the lemma given below.

Lemma (Braid Relation)

If a and b are isotopy classes of simple closed curves that satisfy $i(a, b) = 1$, then $T_a T_b(a) = b$.

- Induction on no. of punctures: For this part of the proof, we will use the Birman exact sequence (which we will define later) to show that the difference between $Mod(S_{g,n})$ and $Mod(S_{g.n+1})$ is finitely generated.

Outline

Complex of curves

The **complex of curves**, $\mathcal{C}(S)$, associated with a surface S is defined to be a simplicial complex whose 1-skeleton is given by the following data.

- **Vertices** correspond to isotopy classes of essential simple closed curves on the surface.
- There is an **edge** between two isotopy classes a, b if $i(a, b) = 0$.

We will use a modified subcomplex of this complex of curves for the inductive step on the genus. The main results that we will prove will be about the connectedness of these complexes.

Complex of curves

Theorem

If $3g + n \geq 5$, then $\mathcal{C}(S_{g,n})$ is connected.

Note that the condition implies $g \geq 2, n \geq 0$ or $g \geq 1, n \geq 2$ or $g \geq 0, n \geq 5$. We will be using the following lemma from our previous seminar.

Lemma (Bigon Criterion)

Two transverse simple closed curves in a surface S are in minimal position if and only if they do not form a bigon.

Complex of curves

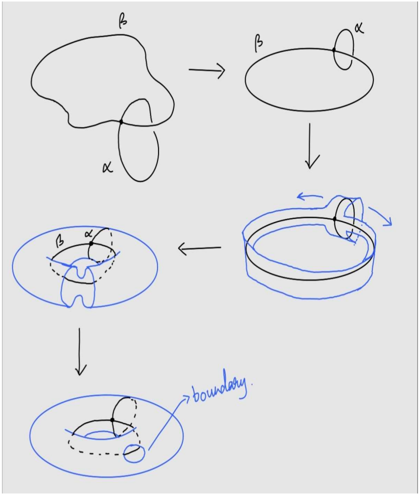
Proof.

Let a, b be two isotopy classes of essential s.c.c.'s on $S_{g,n}$. We induct on $i(a, b)$.

$i(a, b) = 0$ is trivial.

If $i(a, b) = 1$, let $\alpha \in a, \beta \in b$ such that $|\alpha \cup \beta| = 1$.

Let c be the isotopy class of the boundary of $\alpha \cup \beta$. If c is nullhomotopic, this implies $S_{g,n} \simeq S_{1,0}$, if c is homotopic to a puncture, then $S_{g,n} \simeq S_{1,1}$. Both cases contradict $3g + n \geq 5$. □



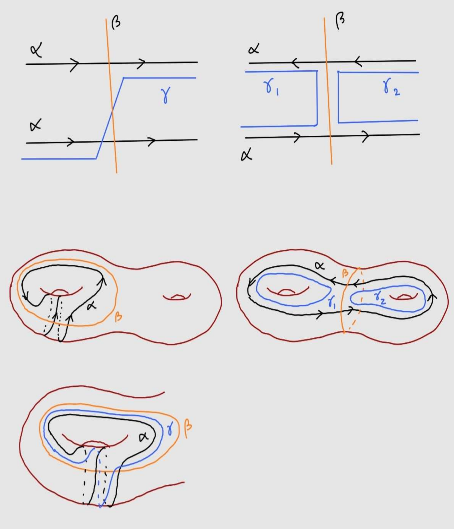
Complex of curves

(contd.)

Suppose $i(a, b) \geq 2$. Then, for minimal position representatives $\in a, \beta \in b$, give α an orientation.

Case 1: γ is essential since $|\alpha \cap \gamma| = 1$. Take $c = \gamma$

Case 2: If either one of γ_1, γ_2 is nullhomotopic, this implies arcs of α and β bound a disc(bigon), then by the bigon criterion, α, β are not in minimal position. □

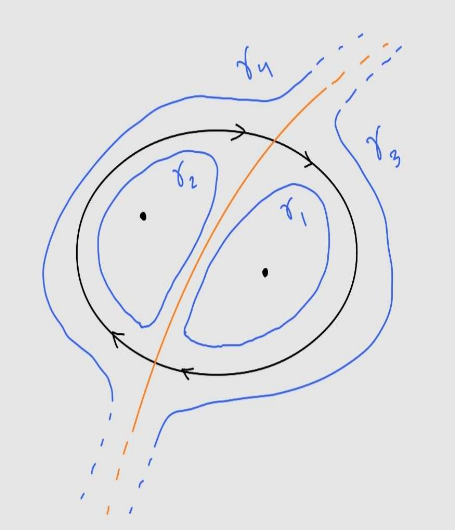


Complex of curves

(contd.)

If both γ_1, γ_2 are homotopic to a puncture, then α bounds a twice punctured disc on one side. Then take similar γ_3, γ_4 on the other side of α. For similar reasons as above, both γ_3, γ_4 are not nullhomotopic.

If both are nullhomotopic, then α bounds a twice punctured disc on both sides which implies $S_{g,n} \simeq S_{0,4}$ which is a contradiction. So we can take $c = \gamma_3$ or $c = \gamma_4$. □



(contd.)

So, by definition, $i(c, a) < i(a, b)$ and $i(c, b) < i(a, b)$. Then, by inductive hypothesis, the result follows. $\square$

Let $\mathcal{N}(S)$ denote the subcomplex of $\mathcal{C}(S)$ consisting of only the isotopy classes of nonseparating essential simple closed curves.

Theorem

If $g \geq 2$, then $\mathcal{N}(S)$ is connected.

Proof.

We first prove the theorem for $g \geq 2$ and $n \leq 1$, and then use induction on n.

Let a, b be two isotopy classes of essential nonseparating s.c.c.'s on $S_{g,n}$. Then, by the previous theorem, there exists a sequence $a = c_1 = ... = c_k = b$.

We will show each intermediate element of this can be replaced by isotopy classes of nonseparating s.c.c.'s.

□

(contd.)

Let c_i, be an element of the above sequence. If c_i is nonseparating, we are done. If not, cut $S_{g,n}$ along (a representative of) c_i and let S' and S'' be the two cut components.

Since $g \geq 2, n \leq 1$, S', S'' have strictly positive genus. Now, if c_{i-1}, c_{i+1} belong to different cut components, then $i(c_{i-1}, c_{i+1}) = 0$ so we can just remove c_i from the sequence.

If they belong to the same component, say S', then replace c_i by an isotopy class of nonseparating s.c.c.'s in S''. Then repeat the above process till we get a sequence of nonseparating elements. This proves the result for $g \geq 2, n \leq 1$. □

(contd.)

Now we induct on n. Cutting the surface along c_i again, if c_{i-1}, c_{i+1} belong to different components, we can just remove c_i from the sequence.

If c_{i-1}, c_{i+1} lie in S', a problem arises since the genus of S'' can be zero. If this is the case, then S' has $g \geq 2$ and has fewer punctures than $S_{g,n}$(S'' must have $n \geq 2$ else this implies c_i is not essential).

Then, by the inductive hypothesis we can get a sequence of nonseparating elements on S' and we can replace c_i by that sequence and we are done. □

The modified complex

Define the modified complex of curves $\widehat{\mathcal{N}}(S)$ as the one dimensional simplicial complex whose vertices are isotopy classes of nonseparating simple closed curves and there is an edge between two classes a, b is $i(a, b) = 1$.

Theorem

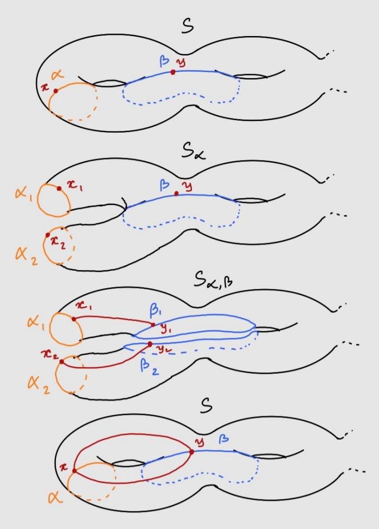
If $g \geq 2$ and $n \geq 0$, then $\widehat{\mathcal{N}}(S)$ is connected.

Proof.

Let a, b be two isotopy classes of s.c.c.'s on $S_{g,n}$. By the previous theorem, there is a sequence $a = c_1 = ... = c_k = b$ such that $i(c_i, c_{i+1}) = 0$ for all $i = 1, ..., k$. Consider c_i, c_{i+1} and let α, β be the respective minimal position representatives. □

The modified complex

(contd.)



The modified complex

(contd.)

Denote the surface obtained after cutting along α as S_α and the surface obtained after cutting along both α, β as $S_{\alpha,\beta}$.

Since $S_{\alpha,\beta}$ may not be connected, the proof boils down to the fact that S_α is connected so α_1 lies in the same component of $S_{\alpha,\beta}$ as one of β_1, β_2 and similarly for α_1.

Let d_i be the corresponding isotopy class. d_i is essential since $i(d_i, c_i) = 1$.

Assume d_i is separating and cut the surface across a representative of d_i. Then α intersects each of the cut boundaries exactly once which implies α maps $[0, 1]$ into two disconnected components which is a contradiction. $\square$

Outline

The Birman Exact Sequqnce

Theorem (Birman exact sequence)

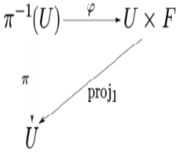
Let S be a surface with $\chi(S) \leq 0$, possibly with punctures and/or boundary. Let (S, x) be the surface obtained from S by marking a point x in the interior of S. Then the following sequence is exact:

$$1 \longrightarrow \pi_1(S, x) \xrightarrow{push} Mod(S, x) \xrightarrow{forget} Mod(S) \longrightarrow 1$$

Definition of Fibre Bundle

A fiber bundle is a structure (E, B, π, F), where E, B, and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying a *local triviality* condition outlined below. The space B is called the **base space** of the bundle, E the **total space**, and F the **fiber**. The map π is called the **projection map** (or **bundle projection**). We shall assume in what follows that the base space B is connected.

We require that for every $x \in B$, there is an open neighborhood $U \subseteq B$ of x (which will be called a trivializing neighborhood) such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ (where $\pi^{-1}(U)$ is given the subspace topology, and $U \times F$ is the product space) in such a way that π agrees with the projection onto the first factor. That is, the following diagram should commute:



where $\mathrm{proj}_1 : U \times F \to U$ is the natural projection and $\varphi : \pi^{-1}(U) \to U \times F$ is a homeomorphism. The set of all $\{(U_i, \varphi_i)\}$ is called a **local trivialization** of the bundle.

Source:- Wikipedia

Proof of Birman Exact Sequence

Proof.

- First we need to show that the following is a fibre bundle

$$\begin{array}{ccc} \varepsilon^{-1}(U) & \xleftarrow{\ h\ } & U \times Homeo^{+}(S, x) \\ \downarrow \varepsilon & \swarrow \text{pr} & \\ x \in U \subseteq S & & \end{array}$$

$\varepsilon(\phi) = \phi(x)\ \forall \phi \in Homeo^{+}(S)$
$\varepsilon^{-1}(U) = \{\phi \in Homeo^{+}(S)\ |\phi(x) \in U\} \subseteq Homeo^{+}(S)$

□

Proof (contd.)

■ To Show that h is a homeomorphism as other properties hold we are give explicit description of h and it's inverse as follows:

$$h(u,\psi) = \phi_u \circ \psi \text{ and } h^{-1}(\Psi) = (\Psi(x), \phi^{-1}_{\Psi(x)} \circ \Psi)$$

Here $(u,\psi) \in U \times Homeo^+(S,x)$ fixes the point x and after that $\phi_u \in Homeo^+(U)$ is defined for each $u \in U$ such that $\phi_u(x) = u$ so $h(u,\psi) = \phi_u \circ \psi(x) = \phi_u(x) = u \in U$ so $h(u,\psi) \in \varepsilon^{-1}(U)$ and h is continuous in compact open topology as ϕ_u varies continuously with u similarly we can say about the inverse of h. So h is a homeomorphism and now it is proven that the diagram commutes. So it is a fibre bundle.

□

Proof of Birman Exact Sequence (continued)

Proof (contd.)

■Now For any other point $y \in S$ we can take $\xi \in Homeo^+(S)$ such that $\xi(y) = x$ then there exist a homemorphism between $\varepsilon^{-1}(U)$ and $\varepsilon^{-1}(\xi(U))$ $[\phi \longrightarrow \phi \circ \xi]$. So it is a fibre bundle. Now We will use two result from Hatcher

Theorem 4.41. *Suppose $p : E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geq 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. Hence if B is path-connected, there is a long exact sequence*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

Proposition 4.48. *A fiber bundle $p : E \to B$ has the homotopy lifting property with respect to all CW pairs (X, A).*

□

Proof of Birman Exact Sequence (continued)

Proof (contd.)

■So we get a Long Exact Sequence of Homotopy groups:
$... \longrightarrow \pi_1(Homeo^+(S)) \longrightarrow \pi_1(S) \longrightarrow \pi_0(Homeo^+(S,x)) \longrightarrow \pi_0(Homeo^+(S)) \longrightarrow \pi_0(S) \longrightarrow ...$
Now S is path-connected so $\pi_0(S) \cong 1$ and by a theorem stated below $\pi_1(Homeo^+(S)) \cong 1$ as $Homeo_0(S)$ [Connected component of the identity in the space of homeomorphisms of a surface S.] is contractible, hence every connected component of $Homeo^+(S)$ is contractible (consider group structure of $Homeo_+(S)$). As each connected component is contractible the whole space must be simply connected.

Theorem

Let S be a compact surface, possibly minus a finite number of points from the interior. Assume that S is not homeomorphic to $S^2, \mathbb{R}^2, D^2, T^2$, the closed annulus, the once-punctured disk, or the once punctured plane. Then the space $Homeo_0(S)$ is contractible.

Proof of Birman Exact Sequence (continued)

Proof (contd.)

■Now From the previous results we get:

$\implies 1 \longrightarrow \pi_1(S) \longrightarrow \pi_0(Homeo^+(S,x)) \longrightarrow \pi_0(Homeo^+(S)) \longrightarrow 1$

$\implies 1 \longrightarrow \pi_1(S,x) \longrightarrow \pi_0(Homeo^+(S,x)) \longrightarrow \pi_0(Homeo^+(S)) \longrightarrow$ 1[as S is path-connected]

$\implies 1 \longrightarrow \pi_1(S,x) \longrightarrow Mod(S,x)) \longrightarrow Mod(S) \longrightarrow$ 1[From Definition]

Now the proof is complete. □

Remark:-To prove the theorem stated in the above slide we need the fact that $\chi(S) \leq 0$

Outline

A short lemma on group action

Lemma

Suppose that a group G acts by simplicial automorphisms on a connected, 1-dimensional simplicial complex X. Suppose that G acts transitively on the vertices of X and that it also acts transitively on pairs of vertices of X that are connected by an edge. Let v and w be two vertices of X that are connected by an edge and choose $h \in G$ so that $h(w) = v$. Then the group G is generated by the element h together with the stabilizer of v in G.

Proof of Lemma

Proof.

■ Let $H \leq G$ generated by the stabilizer of v together with the element h. Now we will try to show that $\forall g \in G \implies g \in H$
Now fixing $g \in G$ and as X is connected we can get a path from v to $g(v)$ let's assume the path be $v = v_1, v_2, ..., v_k = g(v)$

■ Now as G acts transitively on X so we can get a $g_i \in G$ such that $g_i(v) = v_i$. Now we will use induction on i to prove that $g \in H$.The base case is trivial. So let's assume that $g_i \in H$. So in the next step we will show that $g_{i+1} \in H$

■ From the action we get that g_i^{-1} takes the edge between $v_i = g_i(v)$ and $v_{i+1} = g_{i+1}(v)$ to the edge $(v, g_i^{-1}g_{i+1}(v)$ and from transitivity property $\exists r \in G$ such that it takes the previous edge to (v, w)

■ So now we can get $r(v) = v \implies r$ is in to stabilizer of v and $rg_i^{-1}g_{i+1}(v) = w \implies hrg_i^{-1}g_{i+1}(v) = h(w) = v \implies hrg_i^{-1}g_{i+1} \in H$. As $h \in H$ and $r \in H$ from definition and $g_i \in H$ by hypothesis. So $g_{i+1} \in H$.Now $G = H$ as induction implies $g \in H$ for any $g \in G$. □

Finte Generation

Theorem

Let $S_{g,n}$ be a surface of genus $g \geq 1$ with $n \geq 0$ punctures. Then the group $PMod(S_{g,n})$ is finitely generated by Dehn twists about nonseparating simple closed curves in $S_{g,n}$.

Proof.

As per the outline we will use double induction on number of Genus and Punctures.

- Base Cases:-$T^2 = S_{1,0}$ and $S_{1,1}$ has been proved before that it is finitely generated by Dehn twists.
- Induction on punctures:-Let $g \geq 1$and let $n \leq 0$ Assume that $PMod(S_{g,n})$ is generated by finitely many Dehn twists about nonseparating simple closed curves $\{\alpha_i\}$ in $S_{g,n}$, Now we have the Birman-Exact Sequence [Restricting to the Subgroup $PMod$]:
  $1 \longrightarrow \pi_1(S_{g,n}) \longrightarrow PMod(S_{g,n+1}) \longrightarrow PMod(S_{g,n+1}) \longrightarrow 1$

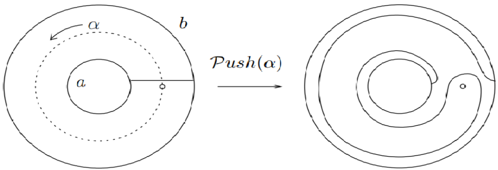
An Useful Fact

Lemma

Let α be a simple loop in a surface S representing an element of $\pi_1(S, x)$. Then

$$Push([\alpha]) = T_a T_b^{-1}$$

where a and b are the isotopy classes of the simple closed curves in (S, x) obtained by pushing α off itself to the left and right, respectively. The isotopy classes a and b are nonseparating in (S, x) if and only if α is nonseparating in S.



The point-pushing map $\mathcal{P}ush$ from the Birman exact sequence.

Proof of Finite Generation (Contd.)

Proof.

- Induction on punctures (Contd.):- Now we can say that $\pi_1(S_{g,n})$ is finitely generated by Van-Kampen Theorem. Now we will use a fact (stated above) on the push map to get that the image of each of these generating loops is a product of two Dehn twists about nonseparating simple closed curves.
  Now given the nonseparating simple curve α_i in $S_{g,n}$, there exists a nonseparating curve in $S_{g,n+1}$ that maps to α_i under the forgetful map $S_{g,n+1} \longrightarrow S_{g,n}$. Thus the Dehn twist T_{α_i} in $PMod(S_{g,n})$ has a preimage in $PMod(S_{g,n+1})$ that is a Dehn twist about a nonseparating simple closed curve in $S_{g,n+1}$.
  So the proof for the induction on punctures is done.

□

Proof of Finite Generation (Contd.)

Remark:-From the inductive step on the number of punctures that, for any $n \geq 0$, the group $PMod(S_{1,n})$ is generated by finitely many Dehn twists about nonseparating simple closed curves as $Mod(S_{1,1}), Mod(S_{1,0})$ are finitely generated from base case so can assume that $g \geq 2$ in your next Inductive step.

Proof (Contd.)

- Induction on genus:-Now let's assume that $PMod(S_{g-1,n})$is finitely generated by Dehn twists about nonseparating simple closed curves for any $n \geq 0$. Now we can use the result proved before:

Theorem

If $g \geq 2$ and $n \geq 0$, then $\widehat{\mathcal{N}}(S)$ is connected.

and also we can check that (S_g) acts transitvely on both set of vertices and edges of $\widehat{\mathcal{N}}(S_g)$ [From Corollary stated below]

□

Change of Co-ordinates Principle

Theorem (Change of Co-ordinates Principle)

Two essential simple closed curves α, β on a surface S have the same topological type if and only if there is an orientation-preserving homeomorphism $\phi \colon S \longrightarrow S$ that fixes ∂S, with $\phi \circ \alpha = \beta$

Corollary

If α is a non-separating simple closed curve on S, then there is a non-separating simple closed curve β with $i(\alpha, \beta) = 1$.

Proof of Finite Generation (Contd.)

Proof (Contd.)

- Induction on genus (Contd.):-Let a be an arbitrary isotopy class of nonseparating simple closed curves in S_g and let b be an isotopy class with $i(a, b) = 1$. Let $Mod(S_g, a)$ denote the stabilizer in $Mod(S_g)$ of a. By

Lemma (Braid Relation)

If a and b are isotopy classes of simple closed curves that satisfy $i(a, b) = 1$, then $T_a T_b(a) = b$.

We have $T_b T_a(b) = a$. Thus, by Lemma proved before this result, $Mod(S_g)$ is generated by $Mod(S_g, a)$ together with T_a and T_b. Thus it suffices to show that $Mod(S_g, a)$ is finitely generated by Dehn twists about nonseparating simple closed curves.

□

Proof of Finite Generation (Contd.)

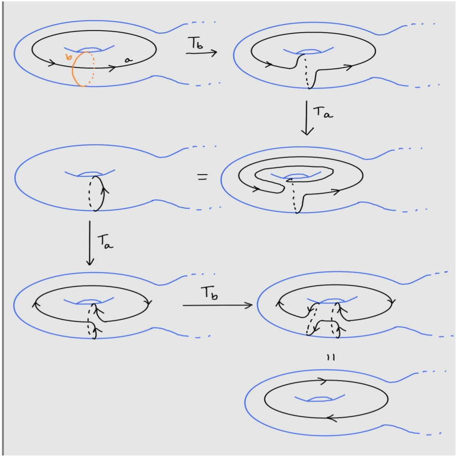
Proof (Contd.)

- Induction on genus (Contd.):-Let $Mod(S_g, \vec{a}) \leq Mod(S_g, a)$ consisting of elements that preserve the orientation of a. from this we get:
  $1 \longrightarrow Mod(S_g, \vec{a}) \longrightarrow Mod(S_g, a) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$
  Since $T_b T_a^2 T_b$ switches the orientation of a (by corollary of change of coordinates), it represents the nontrivial coset of $Mod(S_g, \vec{a})$ in $Mod(S_g, a)$. Thus the proof reduces to show that $Mod(S_g, \vec{a})$ is finitely generated by Dehn twists about nonseparating simple closed curves in S_g.

□

Coset $T_b T_a^2 T_b$

By corollary of change of coordinates we only need t show for a fix a and b with intersection number 1. So



Proof of Finite Generation (Contd.)

Proof (Contd.)

- Induction on genus (Contd.):- Now by a Theorem:

Theorem (The cutting homomorphism)

Let S be a closed surface with finitely many marked points. Let $\alpha_1, ..., pha_n$ be a collection of pairwise disjoint, homotopically distinct essential simple closed curves in S. There is a well-defined homomorphism

$$\zeta : Mod(S, \{[\alpha_1], ..., [\alpha_n]\}) \longrightarrow Mod(S - \bigcup \alpha_i)$$

with kernel $\langle T_{\alpha_1}, ..., T_{\alpha_n} \rangle$.

We get another short exact sequence:
$1 \longrightarrow \langle T_a \rangle \longrightarrow Mod(S_g, \vec{a}) \longrightarrow PMod(S_g - \alpha) \longrightarrow 1$,
where $S_g - \alpha$ is the surface obtained from S_g by deleting a representative α of a.

Proof of Finite Generation (Contd.)

Proof (Contd.)

- Induction on genus (Contd.):- So Now we get a surface$S_g - \alpha$ which is homeomorphic to $S_{g1,n+2}$.
  So by our inductive hypothesis, $PMod(Sg - \alpha)$ is generated by finitely many Dehn twists about nonseparating simple closed curves. Since each such Dehn twist has a preimage in $Mod(S_g, \vec{a})$ that is also a Dehn twist about a nonseparating simple closed curve, it follows that $Mod(S_g, \vec{a})$ is generated by finitely many Dehn twists about nonseparating curves, and we are done.

□

Outline

Alexander Method

Theorem (Alexander Method)

Let S be a compact surface, possibly with marked points, and let $\phi \in Homeo^+(S, \partial S)$. Let $\{\gamma_1, \ldots, \gamma_n\}$ a collection of essential simple proper arcs and closed curves on S with the following properties:
1. The γ_i are pairwise in minimal position.
2. The γ_i are pairwise nonisotopic.
3. For distinct i, j, k, at least one of $\gamma_i \cap \gamma_j, \gamma_i \cap \gamma_k,$ or $\gamma_j \cap \gamma_k$ is empty.
(1) If there is a permutation σ of $1, ..., n$ so that $\phi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ relative to ∂S for each i, then $\phi(\cup\gamma_i)$ is isotopic to $\cup\gamma_i$ relative to ∂S.

Outline

Strategy of the proof of Finite Presentable

- Proof is suggested by Andrew Putnam

- We will show the arc complex $\mathcal{A}(S)$ is contractibe.

- Then we will use the action of **Mod(S)** on $\mathcal{A}(S)$ to build **K(Mod(S),1)** with finite 2-skeleton.

- From this it immediately follows that **Mod(S)** is finitely presented.

Outline

Disadvantage of the proof

This proof is simple proof of finite presentability, we do not know what explicit finite presentation comes out of this approach.

Outline

Let **S** be a compact surface that either has nonempty bondary or has at least one marked point.

The Arc Complex

The arc complex $\mathcal{A}(S)$ is a an abstract simplicial **flag** complex described by the following data :-

Vertices: There is one vertex for each free isotopy class of essential simple proper arcs in **S**

Edges: Vertices are connected by an edge if the corresponding free isotopy classes have disjoint representatives.

Contractibility of the Arc Complex

Theorem

Let ***S*** *be any compact surface with finitely many marked points. If $A(S)$ is nonempty, then it is contractible. (****Hatcher****) [Hat91]*

Proof

Strategy:-
First we choose a base vertex $\mathbf{v}$ of $\mathcal{A}(S)$.
We will define a flow of $\mathcal{A}(S)$ which will take a point on $\mathcal{A}(S)$ to the simplicial star associated to $\mathbf{v}$.

Contractibility of the Arc Complex

Realization of a point on $\mathcal{A}(S)$ on the surface S:-

Let **p** is an arbitrary point of A(S) spanned by $v_1, v_2, ...v_n$ is given by barycentric coordinates, that is, a formal sum $\sum c_i v_i$ where $\sum c_i = 1$ and $c_i \geq 0 \ \forall i$.

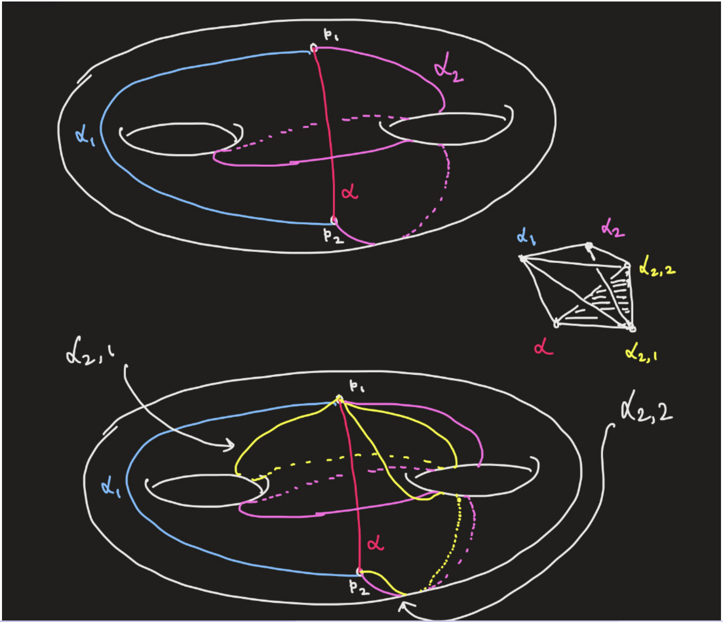
Let α be a fixed representative of **v**.

We can realize **p** in **S** as follows:-

■ First realize the v_i as disjoint arcs in **S**, each in minimal position with α,

■ Then thicken each v_i-arc to a band which is declared to have width c_i.

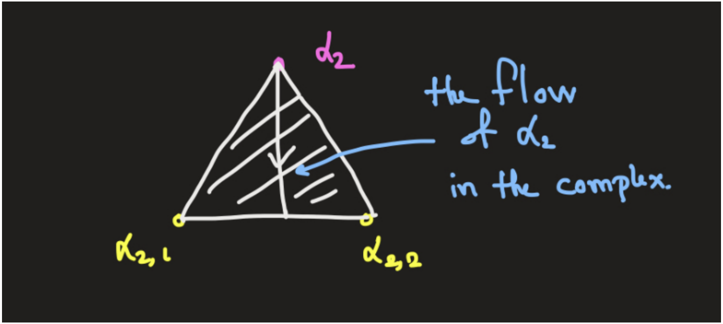
Contractibility of the Arc Complex



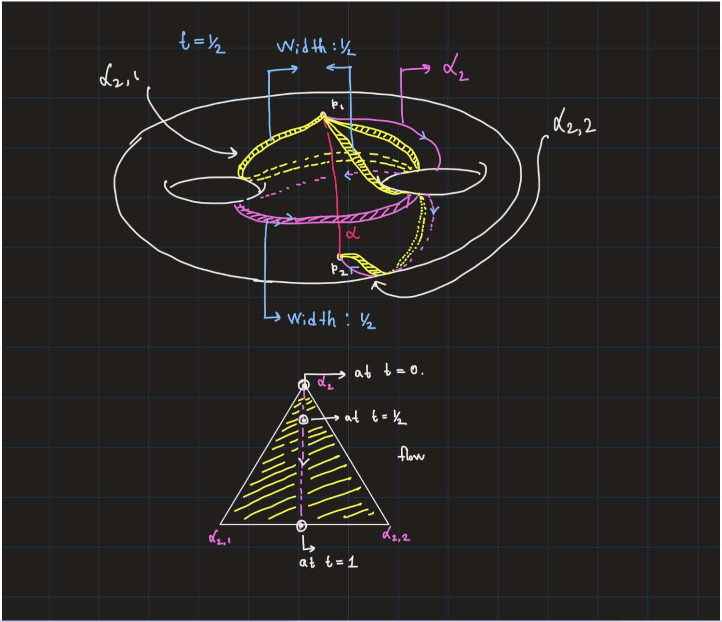
Definition of the flow

At time $\mathbf{t}$, we push a total band width of $\mathbf{t}\theta$ in some prechosen direction along the arc α.
At time 1, all of the bands are disjoint from the arc α, and we are in the star of $\mathbf{v}$.

Contractibility of the Arc Complex



Contractibility of the Arc Complex



Outline

Definition

A group $\mathbf{G}$ acts on a **CW**-complex $\mathbf{X}$ without rotations if, whenever an element $g \in \mathbf{G}$ fixes a cell $\sigma \subset \mathbf{X}$, then g fixes σ pointwise.

Remark:

Any action of a group on **CW**-complex can be turned into an action without rotations by baricentrically subdividing the complex.

Finite Presentability via Group Actions on Complexes

Theorem

Let G be a group acting on a contractible CW-complex X without rotations.
Suppose that each of the following conditions holds.

1. *The quotient X is finite.*
2. *Each vertex stabilizer is finitely presented.*
3. *Each edge stabilizer is finitely generated.*

Then G is finitely presented.

Outline

Proof That The Mapping Class Group Is Finitely Presented

Theorem

*If **S** is a compact surface with finitely many marked points, then the group **Mod(S)** is finitely presented.*

Proof That The Mapping Class Group Is Finitely Presented

Proof.

Assumption:

The proof **holds** for $S_{g,n}$ with $n \geq 0$ marked points

■ With this assumption we will show that the theorem holds for the case when S has **nonempty boundary**

■ Then we will prove for the case when S is **closed**.

□

Proof(continues)

Let S be a compact surface with $n \geq 0$ boundary components and assume that S is not the disc D^2.
Induction Hypothesis:
For any compact surface with $n-1$ boundary components, the mapping class group is **finitely presented**.
Base Case: We took that as our assumption.

The Capping Homomorphism

Let S' be the surface obtained from a surface S by capping the boundary component β with a **once-marked disk**;
call the marked point in this disk $_0$. Denote by $Mod(S, p_1, ..., p_k)$ the subgroup of $Mod(S)$ consisting of elements that fix the punctures $p_1, ..., p_k$, where $k \geq 0$.
Let $Mod(S, p_0, ..., p_k)$ denote the subgroup of $Mod(S)$ consisting of elements that fix the marked points $p_0, ..., p_k$ and then let $Cap : Mod(S, p_1, ..., p_k) \to Mod(S, p_0, ..., p_k)$ be the induced homomorphism. Then the following sequence is **exact**:

$$1 \to \langle T_\beta \rangle \to Mod(S, p_1, ..., p_k) \xrightarrow{Cap} Mod(S, p_0, ..., p_k) \to 1.$$

Induction Step

Suppose S^* is the surface obtained from a surface S by capping a **boundary component** β with a once marked disk, then the following sequence is **exact:**

$1 \to \langle T_\beta \rangle \to Mod(S) \xrightarrow{Cap} Mod^*(S^*) \to 1$,

where $Mod^*(S^*)$ is the subgroup of $Mod(S^*)$ consisting of elements that **fix** the marked point coming from the **capping operation**.

By the **inductive** hypothesis, we have that $Mod(S^*)$ is **finitely presented**.

Since $Mod^*(S^*)$ has finite index in $Mod(S^*)$, it is also **finitely presented.** [WM66]

Since the extension of a finitely presented group by a finitely presented group is finitely presented, it follows by the above exact sequence that $Mod(S)$ is **finitely presented.**

Proof(Continues)

Consider the **Birman Exact Sequence**

$1 \to \pi_1(S_{g,n}) \to PMod(S_{g,n+1}) \to PMod(S_{g,n}) \to 1$

So, $Mod(S_{g,0})$ is finitely presented if $Mod(S_{g,1})$ is, since the quotient of a finitely presented group by a finitely generated group is finitely presented. [WM66]

We have reduced the proof to showing that $Mod(S_{g,n})$ is finitely presented when $n > 0$.
We may assume that $(g, n) \neq (0, 1)$ because we already know $Mod(S_{0,1}) = 1$.

Proof(continues)

Since a group is finitely presented if and only if any of its finite-index subgroups are finitely presented, it suffices to prove that $PMod(S_{g,n})$ is finitely presented.
We proceed by **induction.**

Induction Hypothesis: $PMod(S_{g',n'})$ is finitely presented when $g' < g$ or when $g' = g$ and $n' < n$.

Proof(continues)

We know that the arc complex $\mathcal{A}(S_{g,n})$ is **contractible.**

Now, $PMod(S_{g,n})$ acts without rotation on its barycentric subdivision,$\mathcal{A}^{'}(S_{g,n})$.

Note that the vertices of $\mathcal{A}^{'}(S_{g,n})$ correspond to simplices of $\mathcal{A}(S_{g,n})$.

It follows from the **change of coordinates principle** that the quotient of $\mathcal{A}^{'}(S_{g,n})$ by $PMod(S_{g,n})$ is finite.

Proof(G_v is finitely presented)

Let v be a vertex of $\mathcal{A}'(S_{g,n})$ and let G_v be its stabilizer in $PMod(S_{g,n})$. We will show that G_v is finitely presented.

■ v corresponds to a simplex of $\mathcal{A}(S_{g,n})$, that is, the isotopy class of a collection of disjoint simple proper arcs α_i in $S_{g,n}$.

■ We cut $S_{g,n}$ along the α_i, to obtain a (possibly disconnected) compact surface with boundary S_α, possibly with marked points in its interior.

■ We may pass from the cut surface S_α to a surface with marked points but no boundary by collapsing each boundary component to a marked point (or, what will have the same effect, capping each boundary component with a once-marked disk).

■ Denote the connected components of the resulting surface by R_i. Each R_i has marked points coming from the marked points of $S_{g,n}$ and/or marked points coming from $\cup\alpha_i$.

Proof(G_v is finitely presented)

Note: Each $PMod(R_i)$ falls under the inductive hypothesis.

Proof(G_v^0 is finitely presented)

Let G_v^0 denote the subgroup of G_v consisting of elements that **fix** each isotopy class $[\alpha_i]$ with **orientation.**

■ Then these elements necessarily fix each R_i as well.

■ Since G_v^0 has finite index in G_v, it suffices to show that G_v^0 is finitely presented.

Proof(G_v^0 is finitely presented)

There is a map:
$\eta : G_v^0 \to \Pi PMod(R_i)$

η is a well defined map

Lemma:
Let $\alpha_1, ..., \alpha_n$ be a collection of homotopically distinct simple closed curves in a surface S, each not homotopic to a point in S.
Let β and β' be simple closed curves in S that are both disjoint from $\cup\alpha_i$ and are homotopically distinct from each α_i.
If β and β' are isotopic in S, then they are isotopic in $S - \cup\alpha_i$.

Proof of the Lemma

■ It suffices to find an isotopy from β to β' in S that avoids $\cup\alpha_i$.
■ First, we may modify β so that it is transverse to β' and is still disjoint from $\cup\alpha_i$.
■ If $\beta \cap \beta' = \phi$ then β and β' form the boundary of an annulus A in S.
■Since β (and β') is not homotopic to any α_i, it cannot be that any α_i are contained in A. The annulus A gives the desired isotopy from β and β'.
■If $\beta \cap \beta' \neq \phi$, then by the bigon criterion they form a bigon. Since the α_i are not homotopic to a point and $(\cup\alpha_i) \cap (\beta \cup \beta') = \phi$, the intersection of $\cup\alpha_i$ with the bigon is empty. We can thus push across the bigon, keeping β disjoint from $\cup\alpha_i$ throughout the isotopy.
■ By induction, we reduce to the case where β and β' are disjoint. This completes the proof.

η is a well defined map

For η to be **well-defined homomorphism**, one needs the fact that if two homeomorphisms of $S_{g,n}$ fixing $\cup\alpha_i$ are homotopic,
then they are homotopic through homeomorphisms that **fix** $\cup\alpha_i$.

η is a surjective map

■ Choose an element of $\Pi PMod(R_i)$.

■Then one can choose a representative homeomorphism that is the identity in a neighborhood of the marked points.

■ And then one can lift this to a representative of an element of G_v^0 that is the identity on a neighborhood of the union of the marked point with the α_i.

Kernel of the map η

Lemma:The inclusion Homomorphism:
Let S be a closed subsurface of a surface $S^{'}$.
Assume that S is not homeomorphic to a closed annulus and that no component of $S^{'} - S$ is an open disk.
Let $\eta : Mod(S) \to Mod(S^{'})$ be the induced map. Let $\alpha_1 ... \alpha_n$ denote the boundary components of S that bound once-punctured disks in $S^{'} - S$ and let $\beta_1, \gamma_1,...,\beta_n, \gamma_n$ denote the pairs of boundary components of S that bound annuli in $S^{'} - S$.
Then the kernel of η is the free abelian group
$\ker(\eta) = \langle T_{\alpha_1}, ..., T_{\alpha_m}, T_{\beta_1} T_{\gamma_1}^{-1}, ..., T_{\beta_n} T_{\gamma_n}^{-1} \rangle$.
In particular, if no connected component of $S^{'} - S$ is an open annulus, an open disk, or an open once-marked disk, then η is injective.

Proof of the lemma

Let $f \in ker(\eta)$ and let $\phi \in Homeo^+(S, \partial S)$ be a representative.
We may extend ϕ by the identity in order to obtain
$\hat{\phi} \in Homeo^+(S^{'}, \partial S^{'})$.
By definition, $\hat{\phi}$ represents $\eta(f)$.
Therefore, $\hat{\phi}$ lies in the connected component of the identity in
$Homeo^+(S^{'}, \partial S^{'})$

Proof of the Lemma (continues)

Let δ be an arbitrary oriented simple closed curve in S. Since $\hat{\phi}$ is isotopic to the identity, we have that $\hat{\phi}(\delta)$ is isotopic to δ in $S^{'}$. Since $\hat{\phi}$ agrees with ϕ on S, we have that $\phi(\delta)$ is isotopic to δ in $S^{'}$. By the previous lemma and the assumption on $S^{'} - S$, we have that $\phi(\delta)$ is isotopic to δ in S.

Proof of the Lemma (continues)

We can choose a collection of simple closed curves$\delta_1, ..., \delta_k$ in S that satisfy the three properties in the statement of the Alexander method **(pairwise minimal position, pairwise nonisotopic, no triple intersections)**
and so that the surface obtained from S by cutting along $\cup\delta_i$ is a collection of disks, once-punctured disks, and closed annular neighborhoods N_i of the boundary components.
Moreover, we can choose δ_i so that any homeomorphism that fixes $\cup\delta_i \cup \partial S$ necessarily preserves the complementary regions.

Proof of the Lemma (continues)

■ By the first statement of the **Alexander method**, ϕ is isotopic (in S) to a homeomorphism of S that fixes $\cup\delta_i \cup \partial S$.
■ Since $Mod(D_2) = 1$ and $Mod(D_2 - point) = 1$, it follows that f has a representative that is supported in the N_i.
■ We know $Mod(Annulus) = \mathbb{Z}$
■ So, it follows that f is a product of Dehn twists about boundary components.
■ **Property of Dehn Twists:**
Let $a_1, ..., a_m$ be a collection of distinct nontrivial isotopy classes of simple closed curves in a surface S and assume that $i(a_i, a_j) = 0 \forall i, j$. Let $b_1, ..., b_n$ be another such collection. Let $p_i, q_i \in \mathbb{Z}0$.
If $T_{a_1}^{p_1} T_{a_2}^{p_2} T_{a_m}^{p_m} = T_{b_1}^{q_1} T_{b_2}^{q_2} T_{b_m}^{q_m}$
in $Mod(S)$, then $m = n$ and the sets $T_{a_i}^{p_i}$and $T_{a_i}^{p_i}$ are equal. (The mapping class $\Pi T_{a_i}^{p_i}$ is called a **multitwist**.
■ So, f must be a trivial multitwist and hence of the required form.

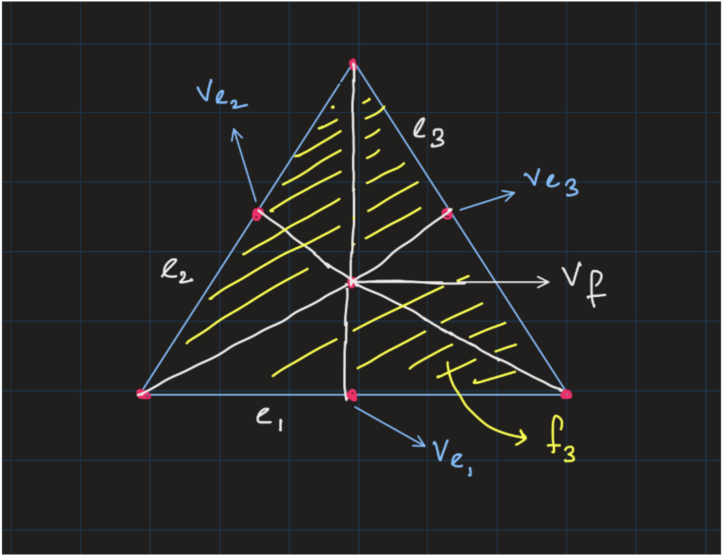
Proof(G_v^0 is finitely presented)

■ By the above lemma, $ker(\eta)$ is **generated** by the Dehn Twists about the components of the boundary of the cut surface S_α

■ Since each $PMod(R_i)$ is finitely presented, their product is as well.

■ By first isomorphism theorem,
$G_v^0/ker(\eta) \cong \Pi PMod(R_i)$
■ So, G_v^0 is finitely presentable.

Proof(Edge stabilizers are finitely Generated)



Proof(Edge stabilizers are finitely Generated)

■ Two vertices of $\mathcal{A}^{'}(S_g, n)$ are connected by an edge if and only if the corresponding simplices of $\mathcal{A}^{'}(S_g, n)$ share a containment relation (i.e., one is contained in the other).
■ It follows that the stabilizer of an edge in $\mathcal{A}^{'}(S_g, n)$ is a finite-index subgroup of the larger of the two stabilizers of its vertices.
■ Thus edge stabilizers are finitely presented, and in particular they are finitely generated. [WM66]

Proof That The Mapping Class Group Is Finitely Presentable

We thus have that $Mod(S_g, n)$ acts on the contractible simplicial complex $\mathcal{A}(S_g, n)$ without rotations, with finitely presented vertex stabilizers and finitely generated edge stabilizers. Applying the second thorem that we mentioned, to this action gives that $Mod(S_g, n)$ is finitely presented.

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Thank You