Operads and Moduli of Hyperelliptic curves ISI Bangalore

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Organization

- Background on Moduli of Curves
- Introduction to Operads
- Gravity and Hypercommutative operads
- My result and techniques

Algebraic Curves

An algebraic curve is a smooth projective variety of dimension 1. Over \mathbb{C} an Algebraic curve is just a compact Riemann Surface (one dimensional complex manifold).

Genus of a Riemann Surface X is half its first betti number.

$$g(X) = \frac{1}{2} \dim H_1(X)$$

Examples are \mathbb{P}^1 , elliptic curves ...



The Moduli of n pointed genus g Curves

As usual in a moduli problem we want a parameter space $M_{g,n}$ for isomorphism classes of algebraic curves of genus g with n distinct marked points.

Two such curves (C, p_1, \ldots, p_n) and (C', p'_1, \ldots, p'_n) are isomorphic if there is an isomorphism $f : C \to C'$, such that $f(p_i) = p'_i$. (We require 2g - 2 + n > 0, so that there are only finitely many automorphisms of a curve fixing its marked points.)

Further we ask that there be a universal family $\mathcal{T}_{g,n} \to M_{g,n}$, such that for any family of *n* pointed genus *g* curves $C \to B$ (smooth flat morphism with *n* disjoint sections, such that the geometric fibers are smooth *n* pointed genus *g* curves), there exits a map $B \to M_{g,n}$ under which *C* is a pull back of $\mathcal{T}_{g,n}$. So we should have a fibre square



There is no solution for the moduli problem in the category of schemes, but a fine moduli space exists as a Deligne-Mumford stack (analogue of orbifold in Algebraic Geometry).

Underlying the stack there is a coarse moduli space which is an algebraic variety usually denoted $M_{g,n}$, where as the stack will be denoted $\mathcal{M}_{g,n}$.

Deligne-Mumford compactification

 $M_{g,n}$ is not a complete variety. The reason being, there can be singular curves in the limit of smooth curves as they vary in families.



Deligne and Mumford gave a compactification of $M_{g,n}$, by enlarging the moduli problem to include certain singular curves.

It turns out that the class of curves that can arise as limits of smooth curves are curves with only nodal singularities. The main ingredient here is the *stable reduction theorem* for curves.

Stable Curves

A stable curve C of genus g, and n marked points $\{p_1, \ldots, p_n\}$, is a projective curve with the following properties:

- dim $H^1(C, \mathcal{O}_C) = g$, where \mathcal{O}_C is the structure sheaf.
- 2 The singularities of C are all nodes.
- $\{p_1, \ldots, p_n\}$ are distinct smooth points of the curve.
- Csm \ {p₁,..., p_n} (the smooth locus with the marked points removed) has negative Euler characteristic.



Properties

Some properties of the moduli space:

- $\textcircled{3} \overline{\mathcal{M}}_{g,n+1} \text{ along with the natural map } \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n} \text{ is the universal family.}$
- **③** $\overline{M}_{g,n}$ is an irreducible, projective variety, of dimension 3g 3 + n, and has only mild singularties (finite quotient).
- $M_{g,n}$ is an open dense subvariety of $\overline{M}_{g,n}$ and the complement is a divisor with normal crossings.
- So When g = 0, $M_{0,n}$ and $\overline{M}_{0,n}$ are smooth varieties and fine moduli spaces for their moduli problems.

Elementary examples

- $\overline{M}_{0,3}$ is a point.
- $M_{0,4}\cong \mathbb{P}^1ackslash\{0,1,\infty\}$ and $\overline{M}_{0,4}\cong \mathbb{P}^1$
- $\overline{M}_{0,5}$ can be realised as $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at (0,0), (1,1) and (∞,∞) , this is a Del-pezzo surface of degree 5.
- In general $\overline{M}_{0,n+1}$ can be inductively constructed as a blow up of $\overline{M}_{0,n} \times \mathbb{P}^1$.
- $M_{1,1} \cong M_{0,4}/S_3$ and $\overline{M}_{1,1} \cong \overline{M}_{0,4}/S_3 \cong \mathbb{P}^1/S_3$.

• $\overline{M}_2 \cong \overline{M}_{0,6}/S_6$.

Operad

An Operad (of vector spaces) is a sequence of vector spaces

 $\{\mathcal{V}(n)\mid n\geq 0\}$

with an action of the symmetric group S_n on $\mathcal{V}(n)$ and bilinear operations

$$\circ_i: \mathcal{V}(n) \otimes \mathcal{V}(m) \to \mathcal{V}(m+n-1)$$

for $1 \le i \le n$ satisfying certain axioms of associativity and equivariance.

Intuitively an element a of $\mathcal{V}(n)$ can be thought of as a rooted tree with n input leaves, and one output leaf which is the root.



The product $a \circ_i b$ then corresponds to grafting of trees as follows:



The equivariance and associativity axioms ensure that we can form the products unambiguously.



Example: Little Disks Operad

An elementary example is the little disks operad. This is an operad of topological spaces. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $\mathcal{O}(n)$ be the topological space

$$\mathcal{O}(n) = \left\{ \begin{pmatrix} z_1, \dots, z_n \\ r_1, \dots, r_n \end{pmatrix} \in D^n \times \mathbb{R}^n_+ \mid \text{ disks } r_i D + z_i \text{ are disjoint subsets of } D \right\}$$

 S_n acts on $\mathcal{O}(n)$ by permuting disks. There is a natural operad structure on \mathcal{O} by gluing of disks. If $a = \binom{w_1, \dots, w_m}{s_1, \dots, s_m}$ and $b = \binom{z_1, \dots, z_n}{r_1, \dots, r_n}$ then

$$a \circ_i b = \begin{pmatrix} w_1, \dots, w_{i-1}, w_i + s_i z_1, \dots, w_i + s_i z_n, w_{i+1}, \dots, w_m \\ s_1, \dots, s_{i-1}, s_i r_1, \dots, s_i r_n, s_{i+1}, \dots, s_m \end{pmatrix}$$

Let

$$E_2(n) = H_{\bullet}(\mathcal{O}(n))$$

then E_2 is an operad in graded vector spaces by the virtue of being the homology of an operad in topological spaces.

Product in little disks operad



Operads and Moduli of Curves

There are two operads that are closely related to the Moduli of Curves.

Let $\mathcal{M}(n) = \overline{\mathcal{M}}_{0,n+1}$. There is an action of S_n on $\mathcal{M}(n)$, by permuting the first *n* marked points of $\overline{\mathcal{M}}_{0,n}$.

Further the operations $\circ_i : \mathcal{M}(n) \times \mathcal{M}(k)$ is obtained through the embeddings



Thus \mathcal{M} is an operad in topological spaces. Taking homology we get an operad in the category of graded vector spaces called the **Hypercommutative** operad.

$$\operatorname{Hycom}(n) = \begin{cases} H_{\bullet}(\overline{M}_{0,n+1}) & n \geq 2\\ 0 & n < 2 \end{cases}$$

Similarly taking the homology of $M_{0,n}$ and suitably suspending we get the **Gravity** operad.

$$\operatorname{Grav}(n) = \begin{cases} \Sigma^{2-n} \operatorname{sgn}_n \otimes H_{\bullet}(M_{0,n+1}) & n \ge 2\\ 0 & n < 2 \end{cases}$$

It is a little more involved to see the operad structure of Grav. The operads Grav and Hycom are Koszul dual to each other.

A Filtration

There is a filtration on $\overline{M}_{g,n}$ given by the number of rational components of a curve.

Let $\overline{M}_{g,n}^{\leq k}$ be the open set in $\overline{M}_{g,n}$ parametrizing stable curves with at most k rational components (components of geometric genus 0).



Curve C has 2 rational components where as D has just 1. We have

$$\overline{M}_{g,n}^{\leqslant 0} \subset \overline{M}_{g,n}^{\leqslant 1} \subset \ldots \subset \overline{M}_{g,n}^{\leqslant 2g-2+n} = \overline{M}_{g,n}$$

Constructible Cohomological Dimension

Let X be a complex variety. Recall that an abelian sheaf \mathcal{F} on X is said to be **constructible** if there exists a locally finite partition of X into subsets that are locally closed for the Zariski topology such that the restriction of \mathcal{F} to each member of that partition is locally constant for the Euclidean topology (its stalks may be arbitrary).

Constructible cohomological dimension, ccd(X), of a variety X is the smallest integer d with the property that

$$H^n(X,\mathcal{F}) = 0$$
 for $n > d$

for every constructible sheaf \mathcal{F} on X.

Cohomological Excess

The cohomological excess of a non-empty variety X, denoted ce(X), is

 $ce(X) = max{ccd(W) - dim W | W closed subvariety of X}$

It can be easily seen that

$$0 \leq \operatorname{ce}(X) \leq \dim X$$

If X is affine ce(X) = 0, and when projective $ce(X) = \dim X$.

The cohomological excess has several good properties, for example if ce(X) = a, then X is homotopy equivalent to a CW complex of dimension a or less.

My research was inspired by a conjecture by Looijenga which dates back to 1990.

Conjecture (Looijenga)

 M_g can be covered by g - 1 open affine subvarieties.

Fontanari and Pascolutti recently proved this for genus up to 5.

Later Looijenga introduced cohomological excess and made a weaker conjecture that $ce(M_g) \le g - 2$.

There is a generalization of Looijenga's conjecture due to Roth and Vakil.

Conjecture (Roth and Vakil)

ce
$$\overline{M}_{g,n}^{\leqslant k} \leq g-1+k$$
 for $g>0, k\geq 0$

Sharpness of these bounds are not known.

I studied analogous questions for the hyperelliptic locus.

Hyperelliptic Locus

Let $H_g \subset M_g$ be the hyperelliptic locus, that is the subspace parametrizing isomorphism classes of hyperelliptic curves. A hyper-elliptic curve of genus g is a double cover of \mathbb{P}^1 , ramified over 2g + 2 points. Let \overline{H}_g be the closure of H_g in \overline{M}_g .

There is the filtration on \overline{H}_g , induced by the filtration on \overline{M}_g .

$$\overline{H}_{g}^{\leqslant k} = \overline{H}_{g} \cap \overline{M}_{g}^{\leqslant k}$$

We are interested in the affine stratification number and cohomological excess of the varieties $\overline{H}_{g}^{\leqslant k}$.

Results

We show Looijenga's bounds hold for these varieties.

Theorem (C)

$$\operatorname{asn} \overline{H}_g^{\leqslant k} \leq g-1+k \quad ext{for all } g,k$$

This provides evidence towards the actual conjecture of Roth and Vakil. The effectiveness of the bound is not known, but when k = 0, we show

Theorem (C)

$$\operatorname{ce}(\overline{H}_g^{\leqslant 0}) \geq g-1 \quad \textit{for} \quad g \geq 2$$

This proves

Reduction to questions on $\overline{M}_{0,n}$

We have a surjection,

$$\pi: \overline{\mathcal{H}ur}_{g,2} \cong \overline{M}_{0,2g+2} \to \overline{H}_g$$

Here π is the quotient map under the action of the symmetric group S_{2g+2} which acts by permuting the fixed points. Hence we have

$$\overline{H}_g \cong \overline{M}_{0,2g+2}/S_{2g+2}$$

Let

$$\overline{M}_{0,2g+2}^{(k)} = \pi^{-1} \overline{H}_g^{\leqslant k}$$

An affine stratification of $\overline{M}_{0,2g+2}^{(k)}$ gives an affine stratification of $\overline{H}_g^{\leqslant k}$ of the same length.

Dual Graphs of Stable curves

The dual graph of a marked stable curve is obtained by placing a vertex for each component, an edge for each node and a leg for each marked point. Further the vertices are labelled by the geometric genus of the component it represents.

We denote the set of isomorphism classes of dual graphs of genus g, with n marked points by $\Gamma(g, n)$.

The genus of the curve can be obtained by the formula

$$g = \sum_{v \in \text{ vertices of } G} g_v + b_1(G)$$

For a graph G, denote by M_G , the locus of curves whose dual graph is G, and by \overline{M}_G its closure. Then

$$\overline{M}_{g,n} = \bigcup_{G \in \Gamma(g,n)} M_G$$

 \overline{M}_G is a subvariety of codimension equal to the number of edges of G.

Examples of Dual Graphs



Coloring of graphs

We define **parity** of graph T of type (0, 2k).

- Legs of T are odd.
- Edges are odd if deleting it produces two trees each with odd number of marked points otherwise even.

Let the ramification number of a vertex v of T be the number of legs and odd edges of the vertex. We denote it by $\rho(v)$.

We call a vertex of a graph *internal* if it has more than one edge.

If a graph is not-stable we can stabilize it by suitably deleting the unstable vertexes.



Going between strata

We have a stratification of the Moduli of curves by dual graphs. To go between the strata of $\overline{M}_{0,2g+2}$ and that of \overline{H}_g , we describe the following algorithm.

Algorithm

Given a tree T corresponding to a curve C in $\overline{M}_{0,2g+2}$, the dual graph of $\pi([C]) \in \overline{M}_g$ is the stabilization of the graph G defined as follows.

- There are two edges in G for each even edge of T, and one edge in G for each odd edge of T. (The leaves of T do not contribute flags to G.)
- A vertex v of T contributes a single vertex to G, of genus $(\rho(v) 2)/2$, unless $\rho(v) = 0$, in which case it contributes two vertices of genus 0.

Illustration



Here the even edges are drawn in red, where as the legs and the odd edges are drawn in black.

Strata in $\overline{M}_{0,2g+2}^{(k)}$

To determine which strata are contained in $\overline{M}_{0,2g+2}^{(k)}$, we have the following lemma, which follows easily from the Algorithm we described.

Lemma

Let C be a curve in $\overline{M}_{0,2g+2}$ with corresponding dual graph G. Then the image $\pi([C]) \in \overline{H}_g$ has a rational component if and only if G has an internal vertex v with $\rho(v) \leq 2$. Furthermore the number of rational components of $\pi([C])$ is given by

$$2 \times \#\{v \in V(G) \mid \rho(v) = 0\} + \#\{v \in V(G) \mid v \text{ internal and } \rho(v) = 2\}$$

Hence a curve in $\overline{M}_{0,2g+2}$ belongs to $\overline{M}_{0,2g+2}^{(0)}$ if its dual graph has only internal vertexes of ramification 4 or more. We call such graphs **good** graphs.

Examples of good trees



Sharpness for k = 0

Consider the constant sheaf $\underline{\mathbb{C}}$ on $\overline{M}_{0,2g+2}^{(0)}$, and let $\mathcal{L} = \pi_*\underline{\mathbb{C}}$. Then \mathcal{L} is a constructible sheaf on $\overline{H}_g^{\leq 0}$. Note that

$$H^{i}(\overline{H}_{g}^{\leqslant 0},\mathcal{L})\cong H^{i}(\overline{M}_{0,2g+2}^{(0)},\underline{\mathbb{C}})$$

Since dim $\overline{H}_g^{\leqslant 0} = 2g-1$ if we can show that

$$H^{3g-2}(\overline{H}_{g}^{\leqslant 0},\mathcal{L})\cong H^{3g-2}(\overline{M}_{0,2g+2}^{(0)},\underline{\mathbb{C}})
eq 0$$

that would imply $\operatorname{ce}\left(\overline{H}_{g}^{\leqslant 0}\right)$ is at least g-1.

Lemma (C)

The cohomology group $H^{3g-2}(\overline{M}_{0,2g+2}^{(0)},\underline{\mathbb{C}})$ is non-zero and has a pure Hodge structure of weight 2(2g-1) and $H^k(\overline{M}_{0,2g+2}^{(0)}) = 0$ for k > 3g-2.

A Spectral Sequence

Let $\Gamma_p(0, n)$ be the set of dual graphs of stable curves of genus 0, with n marked points and p nodes. Then we have the spectral sequence.

$${}_{n}E_{1}^{p,q} = \bigoplus_{[T]\in \Gamma_{-p}(0,n)} H_{c}^{p+q}(M_{T})$$

This spectral sequence converges to $H^{p+q}(\overline{M}_{0,n})$. In fact the spectral sequence is in the category of mixed Hodge structures, and by a purity argument for the cohomology of $\overline{M}_{0,n}$, it can be seen that

$$H^{2i}(\overline{M}_{0,n}) \cong {}_{n}E_{2}^{-(n-3-i),n-3+i}$$

and $_{n}E_{2}^{p,q} = 0$ if $p + q \neq 2(n - 3)$.

<i>x</i>	p = -5	<i>p</i> = -4	<i>p</i> = -3	<i>p</i> = −2	p = -1	<i>p</i> = 0	
1	0	0	0	0	0	$\mathbb{C}(5)^1$	<i>q</i> = 10
99	0	0	0	0	$\mathbb{C}(4)^{119}$	$\mathbb{C}(4)^{20}$	<i>q</i> = 9
715	0	0	0	$\mathbb{C}(3)^{1918}$	$\mathbb{C}(3)^{1358}$	$\mathbb{C}(3)^{155}$	<i>q</i> = 8
715	0	0	$\mathbb{C}(2)^{9450}$	$\mathbb{C}(2)^{13902}$	$\mathbb{C}(2)^{5747}$	$\mathbb{C}(2)^{580}$	q = 7
99	0	$\mathbb{C}(1)^{17325}$	$\mathbb{C}(1)^{40950}$	$\mathbb{C}(1)^{33348}$	$\mathbb{C}(1)^{10668}$	$\mathbb{C}(1)^{1044}$	<i>q</i> = 6
1	$\mathbb{C}(0)^{10395}$	$\mathbb{C}(0)^{34650}$	C(0) ⁴⁴¹⁰⁰	$\mathbb{C}(0)^{26432}$	$\mathbb{C}(0)^{7308}$	$\mathbb{C}(0)^{720}$	<i>q</i> = 5

 $\overline{M}_{0,8}$ example $(_8E_1^{\bullet,\bullet})$

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Truncated Spectral Sequence

For a genus 0 graph G with 2g + 2 marked points, either $M_G \subset \overline{M}_{0,2g+2}^{(0)}$, if G is **good** or intersection is empty.

We have a spectral sequence in compactly supported cohomology.

$${}_{g}F_{1}^{p,q} = \bigoplus_{\substack{G \in \Gamma_{-p}(0,2g+2) \\ G \text{ good}}} H_{c}^{p+q}(M_{G})$$

This spectral sequence converges to $H_c^{p+q}(\overline{M}_{0,2g+2}^{(0)})$. The spectral sequence has

- a natural mixed Hodge structure
- an action of S_{2g+2}

We use this spectral sequence to show $H_c^g(\overline{M}_{0,2g+2}^{(0)})$ is non-trivial and use Poincaré duality.

$$H^g_c(\overline{M}^{(0)}_{0,2g+2})\cong H^{3g-2}(\overline{M}^{(0)}_{0,2g+2})^{\vee}$$

The spectral sequence



Figure: Support of ${}_{g}F_{1}^{\bullet,\bullet}$



Table: Genus 2, $_2F_1^{\bullet,\bullet}$

First Proof

From the Spectral Sequence it is clear that

$$H_c^g(\overline{M}_{0,2g+2}^{(0)}) \cong {}_gF_{\infty}^{-g+1,2g-1} \cong {}_gF_2^{-g+1,2g-1}$$

In the first proof which is computational we analyze the S_{2g+2} action ${}_gF_1^{p,q}$.

Ezra Getzler calculated the S_n equivariant cohomology of $M_{0,n}$ an $\overline{M}_{0,n}$.

Using those techniques we can decompose the vector spaces into irreducible representations of S_{2g+2} .

The differential is S_{2g+2} equivariant, and counting dimensions of the isotypic components for the standard representation, of ${}_{g}F_{1}^{-g+1,2g-1}$ and ${}_{g}F_{1}^{-g+2,2g-1}$ we infer that ${}_{g}F_{2}^{-g+1,2g-1} \neq 0$. The computations work only up to genus 5.

Tables

$$\begin{tabular}{|c|c|c|c|c|} \hline $p = -2$ & $p = -1$ & $p = 0$ \\ \hline $q = 5$ & 4 & 1 & 0 \\ \hline \end{tabular}$$

Table: ${}_{3}F_{1}^{p,q}$, multiplicities of $s_{7,1}$

	<i>p</i> = -3	p = -2	p = -1	<i>p</i> = 0
q = 7	11	6	1	0

Table: ${}_{4}F_{1}^{p,q}$, multiplicities of $s_{9,1}$

	<i>p</i> = -4	<i>p</i> = -3	<i>p</i> = -2	p = -1	<i>p</i> = 0
<i>q</i> = 9	37	36	10	1	0

Table: ${}_{5}F_{1}^{p,q}$, multiplicities of $s_{11,1}$

Second Proof

This proof came out of carefully analyzing the results of the computations in the first proof and works for all genus. Consider the following trees $T_{I,g}$.



Clearly

$$\operatorname{Aut}(T_{l,g}) \cong S_{2g-2l+1} \times (S_l \wr S_2) \subset S_{2g+1}.$$

Let

$$W_{l,g} = H_c^{2g-1-l}(M_{{\mathcal T}_{l,g}}) \qquad ext{for } l = 0,\ldots,g+1.$$

and

$$V_{l,g} = (W_{l,g})^{\operatorname{Aut}(T_{l,g})}$$

Second proof cont.

We have the following commutative diagram,



The proof then concludes by showing that dim $V_{g,g} = 1$ and $d_1 : V_{g,g} \to V_{g-1,g}$ is non-zero, hence the kernel of $d_1 : V_{g-1,g} \to V_{g-2,g}$ is non-trivial.

To know more about all of this please read my thesis.

- Investigate similar questions about trigonal or tetragonal locus, and other Brill-Noether loci.
- See whether there are similar operadic interpretation of the (co)homology as in the hyper-elliptic case.

Thank You !

