Hodge Decomposition of Compact Kahler Manifolds

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Write about the people and the things you are indebted to in fulfilling this project

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Contents

Abstract

This thesis hopes to elucidate results needed to understand the Hodge Decomposition for Compact Kahler Manifolds and discuss the Hodge diamond associated with some interesting examples. We will start from the basics , introducing what a complex manifold is and move on to discuss Hodge Theory on Compact Kahler Manifolds.

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Chapter 1

Introduction

Broadly speaking, Complex Geometry is concerned with spaces (analytic varieties) and their geometric objects which are modelled on the complex plane.Complex Geometry sits at the intersection of Algebraic Geometry, Differential Geometry, and Complex Analysis, and uses tools from all three areas. We will be closely following [**1**] (with occasional references to [**2**] as our main reference).

Basic Definition of a Complex Manifold

To begin defining a Complex Manifold, we first need to look at what a Smooth manifold is.

Definition 1.0.1. A **Smooth (n) Manifold** M is a Hausdorff, Second Countable Topological space with the property that it has a collection of homeomorphisms $\{\ \phi_i : U_i \to \phi_i(U_i) \subseteq \mathbb{R}^n\ \}$ with the properties $\{U_i\}$ is a cover of M and the transition maps ie: $\phi_i \circ \phi_j^{-1}$ is a smooth map whenever $U_i \cap U_j \neq \emptyset$.

Remark 1.0.2.
- This collection of these special maps (commonly called charts) satisfying **all the above properties is known as a smooth atlas**.
- It is common practice to denote the charts as a tuple $(\phi_i$,$U_i)$.
- **Any pair of atlases $\{\ (\phi_i, U_i)\ \}$ and $\{(\psi_k, V_k)\ \}$ are equivalent** if the maps $\phi_i \circ \psi_k^{-1}$ is smooth whenever $U_i \cap V_k \neq \emptyset$

Definition 1.0.3. A **Complex (n) Manifold** X is a Smooth (2n) Manifold with a holomorphic atlas ie: collection of homeomorphisms $\{\ \phi_i : U_i \to \phi_i(U_i) \subseteq \mathbb{C}^n\ \}$ with the properties $\{U_i\}$ is a cover of X and the transition maps ie: $\phi_i \circ \phi_j^{-1}$ is are holomorphic whenever $U_i \cap U_j \neq \emptyset$.

Definition 1.0.4. Given two complex Manifolds X and Y then, a continuous map $f : X \to Y$ is **holomorphic** if for each p $\in$ X ; $\exists$ are charts (ψ_j, U_j) and , (ϕ_i, V_i) :

$$\begin{tikzcd}
p \in U_j \subseteq f^{-1}(V_i) \subseteq X \arrow[rr, "f"'] & & V_i \subseteq Y \arrow[d, "\phi_i"'] \\
\psi_j(U_j) \arrow[u, "\psi_j"'] \arrow[rr, "\phi_i \circ f \circ \psi_j^{-1}"] & & \phi_i(V_i)
\end{tikzcd}$$

Remark 1.0.5.

- f $= (f_1, ..., f_n) : U \subseteq \mathbb{C}^m \to V \subseteq \mathbb{C}^n$ is holomorphic if each f_i is holomorphic in each variable
  ie: $\dfrac{\partial f_i}{\partial \bar{z}_j} \equiv 0$; j = 1 , ... , m i = 1 , ... ,n
- Homeomorphisms ϕ_i (described in definition 0.3) are known as **Holomorphic charts** and it is common practice to denote the charts as a tuple (ϕ_i, U_i).
- A **holomorphic function on a Complex Manifold X**, if a holomorphic function f: X $\to \mathbb{C}$ and, the space of all holomorphic functions on open subset U $\subseteq$ X is given by $\mathcal{O}_X(U)$ (or) $\Gamma(U, \mathcal{O}_X)$
- $\mathcal{O}_X$ is the sheaf of all Holomorphic functions. $\mathcal{O}_X^*$ is the sheaf of all non -vanishing Holomorphic functions (Function is always non-zero).
- **Any pair of holomorphic atlases { (ϕ_i, U_i) } and {(ψ_k, V_k) } are equivalent** if the maps $\phi_i \circ \psi_k^{-1}$ is holomorphic whenever $U_i \cap V_k \neq \emptyset$
- It is interesting to note that the definitions of Smooth and Complex manifolds differ only on the holomorphicity or smoothness condition on their atlas. However, this simple change makes a huge difference. For Example, Holomorphicity of these charts implies that Complex Manifolds are orientable unlike Smooth Manifold counterparts like $\mathbb{RP}^2$

Examples of Complex Manifolds and Holomorphic Functions

Let us try to understand more about Complex manifolds and Holomorphic Maps between these spaces using some non-trivial examples.

1.1 Affine Hyper-surfaces

An Affine Hypersurface X is the zero set of a holomorphic function f : $\mathbb{C}^n \to \mathbb{C}$. Then, as a consequence Holomorphic Implicit function theorem (If f: U $\subset \mathbb{C}^n$ $\to \mathbb{C}$ holomorphic whose Jacobian has maximal rank at p $\in$ U .Then, $\exists$ a neighbourhood V in U and g: V $\to$ V' $\subset \mathbb{C}$ biholomorphic such that f(g($z_1, ..., z_n$)) = z_1 $\forall (z_1, ..., z_n) \in$ V.) We have a atlas (U_i, g_i) of X , $g_i : U_i \to C^{n-1}$ and $g_i \circ g_j^{-1}$ is holomorphic and $\bigcup_i U_i$ = X

1.2 Complex Projective Spaces

$\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim \mathbb{C}^*$
where z $\sim \lambda$ z $\forall z \in \mathbb{C}^{n+1} \setminus \{0\} and, \lambda \in \mathbb{C}^*$
Each element of $\mathbb{P}^n$ is represented by [z_0 : : ... : z_n]
Then, the standard cover of $\mathbb{P}^n$ is $\{U_i | 0 \leq i \leq n\}$ where $U_i := \{[z_0 : ... : z_n] | z_i \neq 0\}$
We define charts $\phi_i : U_i \to \mathbb{C}^n$ where, $\phi_i([z_0 : ... : z_n]) = (z_0/z_i, ..., \hat{z_i/z_i}, ..., z_n/z_i)$
[Hat coordinate is excluded in the map]
$\phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \subset \mathbb{C}^n \to \phi_i(U_i \cap U_j) \subset \mathbb{C}^n$

$$\phi_{ij}(z_1, ..., z_n) = (z_1/z_i, ..., z_{j-1}/z_i, 1/z_i, z_j/z_i, ..., z_n/z_i)$$

which is holomorphic for points in the domain as , 1/z is holomorphic when z $\neq$ 0

1.3 Smooth Projective Varieties

Let f be a homogeneous polynomial in n + 1 variables z_0 , . . . , z_n.
Assume that $0 \in \mathbb{C}$ is a regular value for the induced holomorphic map

$$f : \mathbb{C}^{n+l} \setminus \{0\} \to \mathbb{C}$$

.
By the previous example of Affine Hypersurfaces, we know that $f^{-1}(0) = V(f)$ is a complex manifold. X is covered by the open subset $X \cap U_i$, where the U_i are the standard charts of $\mathbb{P}^n$. Using the charts ϕ_i defined previously, the set $X \cap \ U_i$ is identified with the fibre over $0 \in \mathbb{C}$ of the map
$f_i : (w_1 ... w_n) \to f(w_1, ..., w_{i-1}, 1, w_i, ..., w_n)$

1.4 Veronese and Segre Embedding

The degree d Veronese Embedding is given by,
$v_{n,d} : P^n \to P^{\binom{n+d}{d}}$ $v_{n,d}([z_0 : ... : z_n]) = [z_0^d : z_0^{d-1} z_1 : ... : z_n^d]$ each of the components are degree d polynomials in each of the n + 1 variables.
Locally on the chart where $z_0 \neq 0$ We have,

$$\phi_0 \circ v_{n,d} \circ \phi_0^{-1}(z_1, ..., z_n) = \phi_0 \circ v_{n,d}[1 : z_1 : ... : z_n] = \phi_0[1 : z_1 : ... : z_n^d] = (z_1, z_2, ..., z_n^d)$$

which is a smooth immersion and it is easy to check it is a homeomorphism onto its image.

The Segre Embedding is given by

$$s_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$

$$s_{n,m}([z_0 : ... : z_n], [w_0 : ... : w_m]) = [z_0 w_0 : z_n w_m]$$

This map is in fact an embedding and verifies that the product of projective spaces is a closed smooth manifold in a higher dimensional projective space.

1.5 Hopf (n -) Manifold

Let $\mathbb{Z}$ act $\mathbb{C}^n \setminus \{0\}$ by $(z_1, ..., z_n) \to (\lambda^k z_1, ..., \lambda^k z_n)$ for $k \in \mathbb{Z}$.
For $0 < \lambda < 1$ the action is free and discrete.

Using the result (Let $G \times X \to X$ be the proper and free action of a Complex Lie group G on a Complex manifold X . Then the quotient X / G is a Complex manifold in a natural way and the quotient map $\pi : X \to X/G$ is holomorphic.) The quotient complex manifold X is called the Hopf Manifold.

The map $\phi : S^{2n-1} \times S^1 \to$X

$$\phi(t, x_1, ..., x_{2n}) \to [\lambda^t(x_1 + ix_2, ..., x_{2n-1} + ix_{2n})]$$

defines an diffeomorphism onto the Hopf Manifold X.

Remark 1.5.1. For n ≥ 2 , we know that H$_2(S^{2n-1} \times S^1) = 0$.
Hence by Poincare Duality , we have that Hopf n-Manifolds are class of examples of Non - Kahler Complex Manifold.

Chapter 2

Holomorphic Vector Bundles

In general, A vector bundle is a topological construction that makes precise the idea of a family of vector spaces parametrized by another space X which we use to generalise vector valued functions.
In our cases this space X, is a Complex Manifold and we will see that there are certain holomorphicity conditions that are imposed.
We will see shortly that hypersurfaces in X and holomorphic line bundles on X are related.

Definition 2.0.1. A **holomorphic vector Bundle of rank r** on a Complex Manifold X is a Complex Manifold E with a holomorphic map $\pi : E \to X$ (commonly called projection map) such that each fibre E(x) := $\pi^{-1}(\{x\})$ is a r - dimensional $\mathbb{C}$ vector space.In addition, there is an open cover $\{ U_i \}$ and biholomorphisms $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ such that first component, $\pi_{U_i} \circ \psi_i = \pi|_{U_i}$ and the induced map on each fibre E(x) is $\mathbb{C}$ - Linear isomorphism (such a map ψ_i is called a local trivialization).

What the definition basically encodes is that there is a surjective holomorphic projection from E to X such that locally, E is of the form $U \times \mathbb{C}^r$

Remark 2.0.2.

- A holomorphic line Bundle is a holomorphic vector bundle of rank one
- The induced transition functions $\psi_{ij} : U_i \cap U_j \to$ {Invertible $\mathbb{C}$ - Linear Maps of Rank r} ψ_{ij}(x):= $\psi_i \circ \psi_j^{-1}(x, \_) : \mathbb{C}^r \to \mathbb{C}^r$ is a holomorphic map.
- We think of this induced transitions function as, $\psi_{ij} : U_i \cap U_j \to GL_r(\mathbb{C})$

- A holomorphic vector bundle should not be confused with a Complex vector bundle. **Complex Vector Bundle is a smooth vector bundle** whose fibers are $\mathbb{C}$ spaces and the transition maps are $\mathbb{C}$ linear ie: the local trivializations are diffeomorphism not just biholomorphisms.

Definition 2.0.3. Given two holomorphic vector bundles E and F on X with projection maps π_E and π_F respectively. **A vector bundle homomorphism from E to F of rank k** is a holomorphic map $\phi : E \to F$ such that $\pi_F \circ \phi = \pi_E$ and the induced map between fibers $\phi : E(x) \to F(x)$ is a rank k $\mathbb{C}$- linear map $\forall x \in$ X

Remark 2.0.4. We say that two vector bundles E and F are isomorphic if $\exists$ bijective vector bundle homomorphism $\phi : E \to F$

2.1 Tautological Line Bundle O(-1)

An important example that will come up later on is the Line Bundle.
$\mathcal{O}(-1) := \{(l, z) | z \in [l]\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ called the Tautological line bundle on $\mathbb{P}^n$
The projection map $\pi : \mathcal{O}(-1) \to \mathbb{P}^n$ taking (l,z) $\mapsto$ l .
Thus, $\mathcal{O}(-1)(\mathrm{l}) = \{\mathrm{l}\} \times [\ \mathrm{l}\ ] \subset P^n \times \mathbb{C}^{n+1}$ is a $\mathbb{C}$ vector space of rank 1 $\forall\ l \in \mathbb{P}^n$.
Using the standard open charts $\{U_i\}$, $U_i := \{[w_0, ..., w_n] \mid w_i \neq 0\ \} \subset \mathbb{P}^n$
We get the local trivializations $\phi_i : \pi^{-1}(U_i) \to U_i{\times}\mathbb{C}$, $([w_0, ..., w_n], z) \mapsto ([w_0, ..., w_n], z_i w_i)$
where, z = $(z_0, ..., z_n) = z_i(w_0/w_i, ..., w_{i-1}/w_i, 1, w_{i+1}/w_i, ..., w_n/w_i)$
The transition functions are $\phi_{ij}(z)$ is given by the matrix $[z_i/z_j]$

2.2 Operations on Vector Bundles

Similair to Smooth Vector Bundles, holomorphic Vector Bundles have a corresponding result.

Theorem 2.2.1. Meta Theorem for Holomorphic Vector Bundles Given any canonical construction in linear algebra gives rise to a geometric version for holomorphic vector bundles. (In category theoretic language this result reads as, Given any functor between the category of vector spaces there is an associated functor between the category of holomorphic vector bundles)

This result is important as it helps ensures that the following operations in fact give rise to holomorphic vector bundles.
Given two holomorphic vector bundles E and F on X, with local trivializations $\{\phi_i : \pi_E^{-1}(U_i) \to U_i \times \mathbb{C}^k\ \}$ and $\{\varphi_j : \pi_F^{-1}(V_j) \to V_j \times \mathbb{C}^l\ \}$ respectively. For ease of

calculation let $A_i = \pi_{\mathbb{C}^k} \circ \phi_i$ and $B_j = \pi_{\mathbb{C}^l} \circ \varphi_j$

1. Direct sum: E $\oplus$ F is the holomorphic vector bundle over X with fibres isomorphic to E(x) $\oplus$ F(x)
   For a simple construction,
   E $\oplus$ F := $\{\ (p, v) \mid p \in X, \quad v \in E(p) \oplus F(p)\}$
   $\pi : E \oplus F \to X \quad \pi(p, v) \mapsto p$ is the projection map
   $\psi_{(i,j)} : \pi^{-1}(U_i \cap V_j) \to (U_i \cap V_j) \times \mathbb{C}^{k+l}$ are all trivialisations of $E \oplus F$
   $\psi_{(i,j)}(p, x) \mapsto (p, (A_i(u), B_j(v))) \in (U_i \cap V_j) \times \mathbb{C}^{k+l}$
   where, $u \in E(p) \quad v \in F(p) \quad u + v = x$
   (It is easy to check that this map is well defined and satisfies all the required conditions as a transition function)
   From simple computation of transition functions gives that,
   $\psi_{(i,j)(i',j')}(p) = \begin{bmatrix} \phi_{(i,i')}(p) & 0 \\ 0 & \varphi_{(j,j')(p)} \end{bmatrix} \forall p \in U_i \cap U_{i'} \cap V_j \cap V_{j'}$

2. Tensor Product: E $\otimes$ F is the holomorphic vector bundle over X with fibres isomorphic E(x) $\otimes$ F(x)
   For a simple construction,
   E $\oplus$ F := $\{\ (p, v) \mid p \in X, \quad v \in E(p) \otimes F(p)\}$
   $\pi : E \otimes F \to X \quad \pi(p, v) \mapsto p$ is the projection map
   $\psi_{(i,j)} : \pi^{-1}(U_i \cap V_j) \to (U_i \cap V_j) \times \mathbb{C}^{kl}$ are all trivialisations of $E \oplus F$
   $\psi_{(i,j)}(p, x) \mapsto (p, (A_i(u) \otimes B_j(v))) \in (U_i \cap V_j) \times \mathbb{C}^{kl}$
   for, $u \in E(p), \quad v \in F(p) \quad x = u \otimes v$
   (It is easy to check that this map is well defined and satisfies all the required conditions as a transition function)
   From simple computation of transition functions gives that,
   $\psi_{(i,j)(i',j')}(p) = \phi_{(i,i')}(p) \otimes \varphi_{(j,j')(p)} \quad \forall p \in U_i \cap U_{i'} \cap V_j \cap V_{j'}$
   (A $\otimes$ B is the tensor product for matrices)

3. Dual Bundle : E* is the the holomorphic vector bundles over X with fibres isomorphic to E(x)*

4. ith Symmetric Power: S^iE is the holomorphic vector bundle with fibres isomorphic to S^iE(x)

5. ith Exterior Power: $\wedge^i$E is the holomorphic vector bundle with fibres isomorphic to $\wedge^i$E(x)

6. Determinant Line Bundle: If E is holomorphic vector bundle of rank k then, det(E) := $\wedge^k$E is a holomorphic line bundle.

7. If $\phi : E \to F$ is a vector bundle homomorphism then, Ker(ϕ) and Coker(ϕ) with fibres isomorphic to Ker($\phi(x) : E(x) \to F(x)$) and Coker($\phi(x) : E(x) \to F(x)$) respectively.

Proposition 2.2.2. If $0 \to E \xrightarrow{f} F \xrightarrow{g} G \to 0$ is a short exact sequence of vector bundle homomorphisms. We have, $det(F) \cong det(E) \otimes det(G)$

Proof. We have from the exactness of the vector bundle homomorphisms that f is injective , g is a surjective vector bundle homomorphism with Coker(f) $\cong$ G and E $\cong$ Image(f) = f(E) $\cong$ Ker(g)
Then we have $\psi : E \to \text{Ker}(g)$ and $\phi : G \to \text{Coker}(g)$ be these vector bundle isomorphisms
This gives vector bundle isomorphisms,
$det(\psi) : det(E) \to det(\text{Ker}(g))$ and $det(\phi) : det(G) \to det(\text{Coker}(g))$
$\det(\psi)(v_1, ..., v_m) \mapsto det(\psi(v_1), ..., \psi(v_m))$
$\det(\phi)(v_1, ..., v_m) \mapsto det(\phi(v_1), ..., \phi(v_m))$
$\varphi : det(E) \otimes det(G) \to det(\text{Ker}(g)) \oplus \text{Coker}(g)) \cong det(F)$
$\varphi(v \otimes w) \mapsto det(\psi)(v).\det(\phi)(w)$ which we view as an element of det(F).
This is the required isomorphism and completes the proof.

□

2.2.1 Relation Between Transition Functions and Vector Bundles

It is interesting to note the correspondence between transition functions and Vector Bundles.

If have already seen that the induced transition function are holomorphic maps, but it is interesting to note that under matrix composition (or composition of linear transformations),

$$[\phi_{ij}].[\phi_{jk}] = [\phi_{ik}] \text{ on } U_i \cap U_j \cap U_k$$

This is known as the cocycle condition.
On the other hand if we have an open cover $\{ (U_i)_{i\in\lambda}\}$ of the complex manifold X with a collection of holomorphic maps $\{(g_{ij} : U_i \cap U_j \to GL_n(\mathbb{C}))\} \quad \forall i, j \in \lambda$ satisfying the cocycle condition as shown above and $g_{ii} = I_n \quad \forall i \in \lambda$
Then, the E := $\lambda \times X \times \mathbb{C}^n \;/\!\sim\; (i, p, v) \sim (j, p, g_{ij}(p)v)$
with projection map $\pi : E \to X$ and trivializations $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n \quad \forall i \in \lambda$

$$\pi([i, p, v]) = p \quad \& \quad \phi_i([j, p, v]) = (p, g_{ij}(p)v) \quad \forall i \in \lambda$$

forms a holomorphic rank k bundle over X with induced transition functions

$$\phi_{ij} = g_{ij} \quad \forall i \neq j \in \lambda$$

Definition 2.2.3. **The Picard Group of a Complex Manifold X** is the set of all isomorphism classes of line bundles on X and denoted by Pic(X).

Proposition 2.2.4. Pic(X) is a group under tensor Product with the trivial bundle (X×$\mathbb{C}$) as identity and dual as the inverse map.

Theorem 2.2.5.

$$Pic(X) \cong H^1(X, \mathcal{O}_X^*)((\text{Group Isomorphism})$$

H^1(X,$\mathcal{O}_X^*$) is the space of all holomorphic cocycles upto product by a non-vanishing global holomorphic function

Proof. From the local definition it is clear that [E] $\in$ Pic(X) is determined by a unique (Upto scaling by a global holomorphic function) collection of cocycles $\{\phi_{ij}, U_i \cap U_j\}$.
Conversely, given a collection $\{\phi_{ij}, U_i \cap U_j\} \in H^1(X, \mathcal{O}_X^*)$.
The map is clearly a group homomorphism as tensoring of line bundles correspondes to the product of its cocycles.
Finally, the Kernel of the homomorphism is all line bundles that are isomorphic to the trivial line bundle. This is possible only when there is a global non-vanishing holomorphic function from X to $\mathbb{C}$ □

2.3 Tangent Bundle and Adjunction Formula

Tangent Bundle corresponding to the Holomorphic atlas is infact a Holomorphic Vector Bundle. We will denote this by $\mathcal{T}_X$

The Holomorphic Tangent Bundle has cocycles $\{ \ (J(\phi_{ij}) \circ \phi_j, U_i \cap U_j \ ) \ \}$

Let $\Omega_X := \mathcal{T}_X^*$ is the Holomorphic Cotangent Bundle and $K_X := det(\Omega_X)$ is the canonical bundle.
Finally, If Y $\subset$ X is a Complex Submanifold, Similair to the smooth case we have,

$$\mathcal{T}_X|_Y = \mathcal{T}_Y \bigoplus N_{Y/X}$$

Where, $\mathcal{T}_X|_Y$ is the restriction of the Holomorphic Tangent Bundle of X on Y and $N_{Y/X}$ is the Holomorphic Normal Bundle of Y in X.

This produces an Vector Bundle exact sequence,
$0 \to \mathcal{T}_Y \xrightarrow{i} \mathcal{T}_X|_Y \to N_{Y/X} \to 0$ By our previous proposition, we get

Theorem 2.3.1. (Adjunction Formula) If Y is a Complex submanifold of a Complex Manifold X then, $K_Y \cong K_X|_Y \otimes det(N_{Y/X})$

Chapter 3

Divisors and Line Bundles

An important object that while come up often in future discussions are divisors. In order to describe a divisor it is best that we learn a bit about analytic varieties particularly, analytic varieties.

$\mathcal{O}_{\mathbb{C}^n,z} := \{(U, f) \mid f : U \to \mathbb{C}$ holomorphic $U \subseteq \mathbb{C}^n$ open $\} \ / \sim$
where, (U,f) $\sim$ (V,g) if there is an open subset W $\subseteq$ U $\cap$ V such that f = g on W

It is easy to check that under the operations (U,f) + (V,g) : = (U $\cap$ V , f + g) and, (U,f).(V,g) : = (U $\cap$ V , f.g) is a ring.
There is an interesting class of functions in this ring that are of particular interest to us. Weierstrass polynomials refers to the subring $\mathcal{O}_{\mathbb{C}^{n-1},z}[z_1]$ in $\mathcal{O}_{\mathbb{C}^n,z}$.

Proposition 1.1.15 , 1.1.17 , 1.1.18 and 1.1.19 in [**1**] gives us that,

Proposition 3.0.1.

1. The ring $\mathcal{O}_{\mathbb{C}^n,z}$ is a local Noetherian ring and a UFD.
2. Let f $\in \mathcal{O}_{\mathbb{C}^n,z}$ and let g be a Weierstrass polynomial of degree d. Then there exist Weierstrass polynomial of degree < d and h $\in \mathcal{O}_{\mathbb{C}^n,z}$, such that f = g$\cdot$h + r. The functions h and r are uniquely determined.
3. Let g $\mathcal{O}_{\mathbb{C}^n,z}$ be an irreducible function. If f $\in \mathcal{O}_{\mathbb{C}^n,z}$ vanishes on Z(g) := { z | g(z) = 0 }, then g divides f.

This **ring** $\mathcal{O}_{\mathbb{C}^n,z}$ is important as the zero sets of functions f $\in \mathcal{O}_{\mathbb{C}^n,z}$ help define an analytic subset of $\mathbb{C}^n$

Definition 3.0.2. **The germ of a set in the origin 0** $\in \mathbb{C}^n$ is given by a subset X $\subseteq \mathbb{C}^n$. Two subsets X, Y $\subseteq \mathbb{C}^n$ define the same germ if there exists an open neighbourhood 0 $\in$ U $\subseteq \mathbb{C}^n$ with U $\cap$ X= U $\cap$ Y.

Definition 3.0.3. **A germ X** $\subseteq \mathbb{C}^n$ **in 0** is called analytic if there exist elements f_1, ... , $f_k \in \mathcal{O}_{\mathbb{C}^n,0}$, such that X and Z($f_1$, . . . , f_k) := { z | f_i(z) = 0 i = 1 , ... , k } define the same germ.

Definition 3.0.4. Let U $\subseteq \mathbb{C}^n$ be an open subset.
An analytic subset of U is a closed subset X $\subseteq U$ such that for any x $\in$ X there exists an open neighbourhood x $\in$ V $\subseteq$ U and holomorphic functions $f_1, ..., f_k$: $V \to \mathbb{C}^n$ such that X $\cap$ V = $\{z \mid f_1(z) = ... = f_k(z) = 0\}$.

Remark 3.0.5. From now on the set Z(f) denotes the zero set of the function f (which is usually holomorphic).

Definition 3.0.6. Let X $\subseteq \mathbb{C}^n$ be a germ in the origin. Then I(X) denotes the set of all elements f $\in \mathcal{O}_{\mathbb{C}^n,0}$ with X $\subseteq$ Z(f).

Definition 3.0.7. An analytic germ is irreducible if the following condition is satisfied:
Let X = $X_1 \cup X_2$, where X_1 and X_2 are analytic germs. Then X = X_1 or X = X_2.

Definition 3.0.8. Let U $\subseteq \mathbb{C}^n$ be open. **A meromorphic function f on U** is a function on the complement of a nowhere dense subset S $\subset$ U with the following property:
There exist an open cover U = $\bigcup_i U_i$ and holomorphic functions $g_i, h_i : U_i \to \mathbb{C}^n$ with $h_i \mid_{U_i \setminus S} .f_i \mid_{U_i \setminus S} = g_i \mid_{U_i \setminus S}$.
The set of all holomorphic functions on U is denoted by K(U).

Now we can better appreciate an analytic hypersurface.

Definition 3.0.9. **An analytic hypersurface of X** is an analytic subvariety Y $\subset$ X of codimension one, i.e. dim(Y) = dim(X) - 1.

A hypersurface Y $\subset$ X is locally given as the zero set of a non-trivial holomorphic function. That is, it locally defines Y $\subset$ X (induces germs of codimension one and any such germ is the zero set of a single holomorphic function

Remark 3.0.10. In general any analytic hypersurface can be expressed as a union (which we will assume to be locally finite) of Y_i is a zero set of an irreducible function in $\mathcal{O}_{\mathbb{C}^n,z}$ and such sets are called the irreducible components of the hypersurface.

Definition 3.0.11. An analytic hypersurface of X is an analytic subvariety Y$\subset$ X. **A divisor D on X** is a formal $\mathbb{Z}$ - linear combination of [Y$_i$] which is locally finite (ie:

For each point x has a neighbourhood U , with finitely many Y_i with non - zero $\mathbb{Z}$ - coefficients a_i and satisfying $U \cap Y_i \neq \emptyset$) where, Y_i are irreducible components of Y.

$$\text{i.e.} \quad D = \sum_i a_i[Y_i] \quad a_i \in \mathbb{Z}$$

The set of divisors is given by Div(X).

Remark 3.0.12. Every hypersurface defines a divisor $\sum_i [Y_i] \in$ Div(X), where Y_i are the irreducible components of Y. Conversely, to any divisor $\sum_i a_i[Y_i] \in$ Div(X) with $a_i \neq 0 \forall i$ one can associate the hypersurface, but this construction is clearly not very natural.

Definition 3.0.13. A **divisor D** $=$ $\sum_i a_i[Y_i]$ **is called effective** if $a_i \geq 0 \forall i$. In this case, one writes D ≥ 0.

Definition 3.0.14. Let Y $\subseteq$ X be a hypersurface and let x $\in$ Y.
Suppose that Y defines an irreducible germ in x. Hence, this germ is the zero set of an irreducible g $\in \mathcal{O}_{X,z}$. If f be a meromorphic function in a neighbourhood of z $\in$ Y. Then the **order** $ord_{Y,x}(f)$ **(of f in x with respect to Y)**is given by the equality, f $=$ g$^{ord_{Y,x}(f)}$ h in $\mathcal{O}_{X,z}$

Remark 3.0.15. In fact one can show that the order does not depend on the defining irreducible function g and locally, the order is the same for all regular points.

Definition 3.0.16. If D $=$ $\sum_i a_i[Y_i]$ is a divisor then, we associate an element of $H^0(X, K_X^*/\mathcal{O}_X^*)$ (which we will think of as a locally defined meromorphic function which agrees upto a non-vanishing holomorphic function on the intersection) with the divisor called $O(D)$.

Let X be covered $\{U_j\}$. Let $g_{ij} \in \mathcal{O}^*(U_j)$ be irreducible defining equation of $Y_i \cap U_j$.

$$f_j := \prod_{Y_i \cap U_j \neq \emptyset} g_{ij}^{a_i}$$

is Meromorphic function (which is uniquely defined upto a non vanishing holomorphic function which depends on the defining equation).
Then, f $=$ $[(f_j, U_j)] = O(D)$

Chapter 4

Differential Forms on a Complex Manifold

In this section, we will try to develop some general results for Differentiable Manifolds with Almost Complex Structure M and observe what happens if M is replaced by a Complex Manifold X. This section introduces several notions familiar to those who have studied differential forms and differential operators on differentiable or smooth manifolds.

Hermitian Structure and Almost Complex Manifold

Definition 4.0.1. If M is a differentiable manifold then, an **Almost Complex Structure** is a Vector Bundle Endomorphism I:TM $\rightarrow$ TM satisfying $I^2 = I \circ I = -Id|_{TM}$

Remark 4.0.2.

1. As a consequence of [$T^2 = -Id|_V$ iff dim(V) is even from Linear Algebra], Only even dimensional differentiable manifolds have an almost complex structure

2. (In this section) M refers to a Differentiable Manifold with an almost complex structure I (or) an almost complex manifold.

3. (In this section) X refers to a Complex Manifold.

Proposition 4.0.3. If X is a Complex Manifold, Then X is an almost complex manifold.

Proof. Let $(U_i, \phi_i)_i$ be the holomorphic chart of X.
In each U_k , we will replace ϕ_k by the coordinates $x_j + iy_j$ in $\mathbb{C}^n$
Then on each U$_k$, define $I : T(U_k) \to T(U_k)$, $I(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j} \quad I(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}$
Which is infact $\mathbb{C}$ - linear map on each fiber . I^2 = - Id and full rank real vector bundle endomorphism

□

Remark 4.0.4. It is important to note that the 4 - Dimensional Sphere is a smooth manifold that has no complex structure (non-trivial result) but, has an almost complex structure. Hence, Almost Complex Structure alone does not guarantee smooth manifold has a holomorphic atlas.

Let M be an almost complex manifold. then, we can define a complexification of the tangent bundle $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$ which is a complex vector bundle (not necessarily holomorphic)

$\pi : T_{\mathbb{C}}M \to M$
$\pi(v \otimes \lambda) =$ p $= \pi_{TM}(v)$ ie: v $\in T_pM$
$\varphi_i : \pi^{-1}(U_i \to U_i \times \mathbb{C}$
$\varphi_i(v \otimes \lambda) = (\pi_{TM}(v), \lambda)$ is $\mathbb{C}$ - Linear on each fiber and $\varphi_{ij} \cong J(\phi_i \circ \phi_j^{-1}) \circ \phi_j$

Proposition 4.0.5. Let X be an almost complex manifold. Then the following two conditions are equivalent:
i) $d\alpha = \partial(\alpha) + \bar{\partial}(\alpha)$ for all $\alpha \in \mathcal{A}^*(X)$.
ii) On $\mathcal{A}^{1,0}(X)$ one has $\Pi^{0,2} \circ d = 0$.
Both conditions hold true if X is a complex manifold.

Proof. The last assertion is easily proved by reducing to the local situation .

The implication i) ⇒ ii) is trivial, since $d = \partial + \bar{\partial}$ clearly implies $\Pi^{0,2} \circ d = 0$ on $\mathcal{A}^{1,0}(X)$.

Conversely, $d = \partial + \bar{\partial}$ holds on $\mathcal{A}^{p,q}(X)$ if and only if $d\alpha \in \mathcal{A}^{p+1,q}(X) \oplus \mathcal{A}^{p,q+1}(X)$ for all $\alpha \in \mathcal{A}^{p,q}(X)$. Locally, $\alpha \in \mathcal{A}^{p,q}(X)$ can be written as a sum of terms of the form $fw_{i_1} \wedge \ldots \wedge w_{i_p} \wedge w'_{j_1} \wedge \ldots \wedge w'_{j_q}$ with $w_i \in \mathcal{A}^{1,0}(X)$ and $w'_j \in \mathcal{A}^{0,1}(X)$. Using Leibniz rule the exterior differential of such a form is computed in terms of df, dw_{i_k}, and $dw'_{j_{\ell^*}}$. Clearly, $df \in \mathcal{A}^{1,0}(X) \oplus \mathcal{A}^{0,1}(X)$ and by assumption $dw_i \in \mathcal{A}^{2,0}(X) \oplus \mathcal{A}^{1,1}(X)$, and $dw'_j = \overline{d\bar{w}'_j} \in \mathcal{A}^{1,1}(X) \oplus \mathcal{A}^{0,2}(X)$. For the latter we use that complex conjugating ii) yields $\Pi^{2,0} \circ d = 0$ on $\mathcal{A}^{0,1}$. Thus, $d\alpha \in \mathcal{A}^{p+1,q}(X) \oplus \mathcal{A}^{p,q+1}(X)$. □

Definition 4.0.6. An almost complex structure I on X is called integrable if the condition i) or, equivalently, ii) in Proposition 0.33 is satisfied.

Here is another characterization of integrable almost complex structures.

Proposition 4.0.7. An almost complex structure I is integrable if and only if the Lie bracket of vector fields preserves $T_X^{0,1}$, i.e. $\left[T_X^{0,1}, T_X^{0,1}\right] \subset T_X^{0,1}$.

Proof. Let α be a $(1,0)$-form and let v, w be sections of $T^{0,1}$. Then, using the standard formula for the exterior differential (cf. Appendix A) and the fact that α vanishes on $T^{0,1}$, one finds

$$(d\alpha)(v,w) = v(\alpha(w)) - w(\alpha(v)) - \alpha([v,w]) = -\alpha([v,w]).$$

Thus, $d\alpha$ has no component of type $(0,2)$ for all α if and only if $[v,w]$ is of type $(0,1)$ for all v, w of type $(0,1)$.

$\square$

Corollary 4.0.8. If I is an integrable almost complex structure, then $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$. Conversely, if $\bar{\partial}^2 = 0$, then I is integrable.

Proof. The first assertion follows directly from $d = \partial + \bar{\partial}$ (Proposition 2.6.15), $d^2 = 0$, and the bidegree decomposition.

Conversely, if $\bar{\partial}^2 = 0$ we show that $\left[T_X^{0,1}, T_X^{0,1}\right] \subset T_X^{0,1}$. For v, w local sections of $T_X^{0,1}$ we use again the formula $(d\alpha)(v,w) = v(\alpha(w)) - w(\alpha(v)) - \alpha([v,w])$, but this time for a $(0,1)$-form α. Hence, $(d\alpha)(v,w) = (\bar{\partial}\alpha)(v,w)$. If applied to $\alpha = \bar{\partial} f$ we obtain

$$\begin{aligned} 0 = \left(\bar{\partial}^2 f\right)(v,w) &= v((\bar{\partial} f)(w)) - w((\bar{\partial} f)(v)) - (\bar{\partial} f)([v,w]) \\ &= v((df)(w)) - w((df)(v)) - (\bar{\partial} f)([v,w]), \text{ since } v, w \in T_X^{0,1} \\ &= \left(d^2 f\right)(v,w) + (df)([v,w]) - (\bar{\partial} f)([v,w]) \\ &= 0 + (\partial f)([v,w]), \quad \text{since } d = \partial + \bar{\partial} \text{ on } \mathcal{A}^0 \end{aligned}$$

$\square$

Corollary 4.0.9. There exists a natural direct sum decomposition

$$\bigwedge_{\mathbb{C}}^{k} X = \bigoplus_{p+q=k} \bigwedge^{p,q} X \text{ and } \mathcal{A}_{X,\mathrm{C}}^{k} = \bigoplus_{p+q=k} \mathcal{A}_X^{p,q}$$

Moreover, $\overline{\Lambda^{p,q} X} = \Lambda^{q,p} X$ and $\overline{\mathcal{A}_X^{p,q}} = \mathcal{A}_X^{q,p}$

Definition 4.0.10. Let X be an almost complex manifold. If $d : \mathcal{A}^k_{X,\mathrm{C}} \to \mathcal{A}^{k+1}_{X,\mathrm{C}}$ is the $\mathbb{C}$-linear extension of the exterior differential, then we can define

$$\partial := \Pi^{p+1,q} \circ d : \mathcal{A}^{p,q}_X \longrightarrow \mathcal{A}^{p+1,q}_X, \bar{\partial} := \Pi^{p,q+1} \circ d : \mathcal{A}^{p,q}_X \longrightarrow \mathcal{A}^{p,q+1}_X$$

The Leibniz rule for the exterior differential d implies the Leibniz rule for ∂ and $\bar{\partial}$, e.g. $\partial(\alpha \wedge \beta) = \partial(\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge \partial(\beta)$ for $\alpha \in \mathcal{A}^{p,q}(X)$

Proposition 4.0.11. Let $f : X \to Y$ be a holomorphic map between complex manifolds. Then the pull-back of differential forms respects the above decom positions, i.e. it induces natural $\mathbb{C}$-linear maps $f^* : \mathcal{A}^{p,q}(Y) \to \mathcal{A}^{p,q}(X)$. These maps are compatible with ∂ and $\bar{\partial}$.

Proof. As for any differentiable map $f : X \to Y$ there exists the natural pull-back map $f^* : \mathcal{A}^k(Y) \to \mathcal{A}^k(X)$ which satisfies $f^* \circ d_Y = d_X \circ f^*$.

If f is holomorphic, then the pull-back f^* satisfies,
$f^*\left(\mathcal{A}^{p,q}(Y)\right) \subset \mathcal{A}^{p,q}(X)$ and $\Pi^{p+1,q} \circ f^* = f^* \circ \Pi^{p+1,q}$. Thus, for $\alpha \in \mathcal{A}^{p,q}(Y)$ one has

$$\begin{aligned} \partial_X\left(f^*\alpha\right) &= \Pi^{p+1,q}\left(d_X\left(f^*(\alpha)\right)\right) = \Pi^{p+1,q}\left(f^*\left(d_Y(\alpha)\right)\right) \\ &= f^*\left(\Pi^{p+1,q}\left(d_Y(\alpha)\right)\right) = f^*\left(\partial_Y(\alpha)\right) \end{aligned}$$

Analogously, we have $\bar{\partial}_X \circ f^* = f^* \circ \bar{\partial}_Y$.

□

Chapter 5

Kahler Manifolds, their identities and Dolbeault Cohomology

This section verifies identities specific to the Kähler manifold. Many complex manifolds, (but by far not all) possess a Kähler metric. This section also contains a detailed discussion of the most important examples of compact Kähler manifolds.

Following the discussion in the previous section we have a Hermitian Manifold X with a metric g compatible almost complex structure J.
Then, we can define a Fundamental form $\omega(\alpha, \beta) := g(J(\alpha), \beta)$
Clearly, ω is a smooth 2 - form.

Proposition 5.0.1. ω is an alternating (1,1) - form which is locally given by $\dfrac{i}{2}\sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\bar{z}_j$ in local coordinates $\{z_i, \bar{z}_i\}$. Where $[h_{ij}]$ is a hermitian positive definite matrix.

Proof. $\omega(v, v)$ = g(J(v),v) = - g(v,J(v)) = - g(v,J(v)) , $\implies$ $\omega(v, v) = 0$ $\forall v \in \Omega^1(X)$
It is sufficient to verify the claim locally,
h : = g - i ω ,
h is a non-degenerate hermitian metric :
(i)h(v,v) = g(v,v) $\forall v \in \Omega^1(X)$
(ii)h(μ v , λ w) = $\mu\bar{\lambda}$ h(v,w) $\forall v, w \in \Omega^1(X)$
$\partial_{x_i} = \partial_{z_i} + \bar{\partial_{z_j}}$
Finally, - i $\omega(\partial_{z_i}, \bar{\partial_{z_j}})$ = - i $\omega((\partial_{x_i}, \partial_{x_j})$ = g$(\partial_{x_i}, \partial_{x_j})$
h$(\partial_{x_i}, \partial_{x_j})$ = - 2i $\omega(\partial_{x_i}, \partial_{x_j})$ = - 2i $\omega(\partial_{z_i}, \bar{\partial_{z_j}})$
The matrix associated to a hermitian metric h is hermitian positive definite.
Hence Proved ☐

As a consequence if proposition 0.40 the set of closed positive real $(1,1)$-forms $\omega \in \mathcal{A}^{1,1}(X)$ is the set of all Kähler forms.

Corollary 5.0.2. The set of all Kähler forms on a compact complex manifold X is an open convex cone in the linear space $\{\omega \in \mathcal{A}^{1,1}(X) \cap \mathcal{A}^2(X) \mid d\omega = 0\}$.

Proof. The positivity of a hermitian matrix $(h_{ij}(x))$ is an open property and, since X is compact, the set of forms $\omega \in \mathcal{A}^{1,1}(X) \cap \mathcal{A}^2(X)$ that are locally of the form $\omega = \frac{i}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j$ with (h_{ij}) positive definite at every point is open. The differential equation $d\omega = 0$ ensures that the metric associated to such an ω is Kähler.

In order to see that Kähler forms form a convex cone, one has to show that for $\lambda \in \mathbb{R}_{>0}$ and Kähler forms ω_1, ω_2 also $\lambda \cdot \omega_i$ and $\omega_1 + \omega_2$ are Kähler forms. Both assertion follow from the corresponding statements for positive definite hermitian matrices.

□

We can now define some operators
i) The Lefschetz operator : $L : \bigwedge^k X \longrightarrow \bigwedge^{k+2} X, \alpha \longmapsto \alpha \wedge \omega$ is an operator of degree two.

ii) The Hodge *-operator: $* : \bigwedge^k X \longrightarrow \bigwedge^{2n-k} X$ is induced by the metric g and the natural orientation of the complex manifold X. Here, $2n$ is the real dimension of X.

ie: As there is a nowhere vanishing volume form.
There is a non-degenerate pairing, $\lambda^k X \times \lambda^{2n-k} X \to \mathbb{R}$

$$(\alpha, \beta) \to \alpha \wedge \beta$$

Then, we define an inner product on $\Lambda^k(X)$
As inner product is a non - degenerate pairing ,
$* : \Lambda^k(X) \cong (\Lambda^{(2n-k)}(X))^* \cong (\Lambda^{(2n-k)}(X))$ satisfying,

$$g(\alpha, \beta) Vol = \alpha \wedge *(\bar{\beta})$$

Locally, If $e_1, ..., e_n$ is an orthonormal local frame of TX
We have, $*(e_{i_1} \wedge ... \wedge e_{i_k}) = e_{j_1} \wedge ... \wedge e_{j_{n-k}}$
where, Vol $= (e_{i_1} \wedge ... \wedge e_{i_k}) \wedge (e_{j_1} \wedge ... \wedge e_{j_{n-k}})$

iii) The dual Lefschetz operator:
$\Lambda := *^{-1} \circ L \circ * : \bigwedge^k X \longrightarrow \bigwedge^{k-2} X$
is an operator of degree -2 and depends on the Kähler form ω and the metric g (and, therefore, on the complex structure J).

All three operators can be extended $\mathbb{C}$-linearly to the complexified bundles $\bigwedge_{\mathbb{C}}^k X$. By abuse of notation, those will again be called $L, *$, and Λ, respectively.

Proposition 5.0.3. Let (X, g) be an hermitian manifold. Then there exists a direct sum decomposition of vector bundles

$$\bigwedge^k X = \bigoplus_{i \geq 0} L^i \left(P^{k-2i} X\right)$$

where $P^{k-2i}X := \operatorname{Ker}\left(\Lambda : \bigwedge^{k-2i} X \to \bigwedge^{k-2i-2} X\right)$ is the bundle of primitive forms.

$$P_{\mathbb{C}}^k X = \bigoplus_{p+q=k} P^{p,q} X$$

where, $P^{p,q}X := P_{\mathbb{C}}^{p+q} X \cap \bigwedge^{p,q} X$.

We now define the adjoint operators (these are infact adjoint under an inner product which we will see later)

$$d^* = (-1)^{m(k+1)+1} * \circ d \circ * : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^{k-1}(M)$$

and the Laplace operator is given by

$$\Delta = d^* d + d d^*$$

If the dimension of M is even, e.g. if M admits a complex structure, then $d^* = - * \circ d \circ *$. Analogously, one defines ∂^* and $\bar{\partial}^*$ as follows.

Definition 5.0.4. If (X, g) is an hermitian manifold, then

$$\partial^* := - * \circ \bar{\partial} \circ * \text{ and } \bar{\partial}^* = - * \circ \partial \circ *.$$

Proposition 5.0.5. Hodge $*$-operator maps $\mathcal{A}^{p,q}(X)$ to $\mathcal{A}^{n-q,n-p}(X)$. Thus,

$$\begin{tikzcd}
A^{p,q}(X) \arrow[r, "\partial^*"] \arrow[d, "*"] & A^{p-1,q}(X) \arrow[d, "-*"'] \\
A^{n-p,n-q}(X) \arrow[r, "\bar{\partial}"] & A^{n-q,n-p+1}(X)
\end{tikzcd}$$

and, similarly, $\bar{\partial}^* \left(\mathcal{A}^{p,q}(X)\right) \subset \mathcal{A}^{p,q-1}(X)$.

The following Proposition is an immediate consequence of the decomposition $d = \partial + \bar{\partial}$ which holds because the almost complex structure on a complex manifold is integrable.

Lemma 5.0.1. *If (X, g) is an hermitian manifold then $d^* = \partial^* + \bar{\partial}^*$ and $\partial^{*2} = \bar{\partial}^{*2} = 0$.*

Definition 5.0.6. If (X, g) is an hermitian manifold, then the Laplacians associated to ∂ and $\bar{\partial}$, respectively, are defined as

$$\Delta_{\partial} := \partial^*\partial + \partial\partial^* \quad \text{and} \quad \Delta_{\bar{\partial}} := \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$$

$\Delta_{\partial}, \Delta_{\bar{\partial}} : \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p,q}(X)$

All these linear and differential operators behave especially well if a further compatibility condition on the Riemannian metric and the complex structure is imposed. This is the famous Kähler condition formulated for the first time by Kähler.

Definition 5.0.7. A Kähler structure (or Kähler metric) is an hermitian structure g for which the fundamental form ω is closed, i.e. $d\omega = 0$. In this case, the fundamental ω form is called the Kähler form.

The complex manifold endowed with the Kähler structure is called a Kähler manifold. However, sometimes a complex manifold X is called Kähler if there exists a Kähler structure without actually fixing one. More accurately, one should speak of a complex manifold of Kähler type in this case.

Remark 5.0.8. Hermitian structures exist on any complex manifold but, as we will see shortly, Kähler structures does not always exist.

A simple example is the Hopf 2 - Manifold (As the 2nd Homology Group is Trivial, it cannot be Kahler).

The local version of a Kähler metric has been studied in detail in Section 1.3 of [**1**]. Where we see that the condition $d\omega = 0$ is equivalent to the fact that the hermitian structure g osculates in any point to order two to the standard hermitian structure (see Proposition 1.3 .12 in [**1**] for the precise statement).

5.1 Examples of Kahler Manifolds

5.1.1 Fubini Study Metric on Projective Spaces

The Fubini-Study metric is a canonical Kähler metric on the projective space $\mathbb{P}^n$. Let $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ be the standard open covering and $\varphi_i : U_i \cong \mathbb{C}^n, (z_0 : \ldots : z_n) \mapsto \left(\frac{z_0}{z_i}, \ldots, \frac{\widehat{z_i}}{z_i}, \ldots, \frac{z_n}{z_i}\right)$. Then one defines

$$\omega_i := \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{\ell=0}^n \left|\frac{z_\ell}{z_i}\right|^2\right) \in \mathcal{A}^{1,1}(U_i)$$

which under φ_i corresponds to

$$\frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{k=1}^n |w_k|^2 + 1\right)$$

Observe that ,

$$\log\left(\sum_{\ell=0}^n \left|\frac{z_\ell}{z_i}\right|^2\right) = \log\left(\left|\frac{z_j}{z_i}\right|^2 \sum_{\ell=0}^n \left|\frac{z_\ell}{z_j}\right|^2\right) = \log\left(\left|\frac{z_j}{z_i}\right|^2\right) + \log\left(\sum_{\ell=0}^n \left|\frac{z_\ell}{z_j}\right|^2\right).$$

Thus, it suffices to show that $\partial\bar{\partial}\log\left(\left|\frac{z_j}{z_i}\right|^2\right) = 0$ on $U_i \cap U_j$. Since $\frac{z_j}{z_i}$ is the j-th coordinate function on U_i, this follows from

$$\partial\bar{\partial}\log|z|^2 = \partial\left(\frac{1}{z\bar{z}}\bar{\partial}(z\bar{z})\right) = \partial\left(\frac{zd\bar{z}}{z\bar{z}}\right) = \partial\left(\frac{d\bar{z}}{\bar{z}}\right) = 0$$

Next, we observe that ω_{FS} is a real $(1,1)$-form. Indeed, $\overline{\partial\bar{\partial}} = \bar{\partial}\partial = -\partial\bar{\partial}$ yields $\omega_i = \bar{\omega}_i$ (As Kahler Manifolds are intergrable). Moreover, ω_{FS} is closed, as $\partial\omega_i = 0$.

It remains to show that ω_{FS} is positive definite, i.e. that ω_{FS} really is the Kähler form associated to a metric. This can be verified on each U_i separately. A straightforward computation yields

$$\begin{aligned}\partial\bar{\partial}\log\left(1 + \sum_{i=1}^n |w_i|^2\right) &= \frac{\sum dw_i \wedge d\bar{w}_i}{1 + \sum |w_i|^2} - \frac{\left(\sum \bar{w}_i dw_i\right) \wedge \left(\sum w_i d\bar{w}_i\right)}{\left(1 + \sum |w_i|^2\right)^2} \\ &= \frac{1}{\left(1 + \sum |w_i|^2\right)^2} \sum h_{ij} dw_i \wedge d\bar{w}_j,\end{aligned}$$

with $h_{ij} = \left(1 + \sum |w_i|^2\right)\delta_{ij} - \bar{w}_i w_j$. The matrix (h_{ij}) is positive definite, since for $u \neq 0$ the Cauchy-Schwarz inequality for the standard hermitian product $(,)$ on $\mathbb{C}^n$ yields

$$
\begin{aligned}
u^t\left(h_{ij}\right)\bar{u} &= (u,u)+(w,w)(u,u)-u^t\bar{w}w^t\bar{u} \\
&= (u,u)+(w,w)(u,u)-(u,w)(w,u) \\
&= (u,u)+(w,w)(u,u)-\overline{(w,u)}(w,u) \\
&= (u,u)+(w,w)(u,u)-|(w,u)|^2>0.
\end{aligned}
$$

5.1.2 Complex Curve

ii) Any complex curve admits a Kähler structure. In fact, any hermitian metric is Kähler, as a two-form on a complex curve is always closed.

5.2 Projective Manifolds are Kahler

Proposition 5.2.1. Let g be a Kähler metric on a complex manifold X. Then the restriction $\left.g\right|_Y$ to any complex submanifold $Y\subset X$ is again Kähler.

Proof. Clearly, $\left.g\right|_Y$ is again a Riemannian metric on Y. Since $T_xY\subset T_xX$ is invariant under the almost complex structure I for any $x\in Y$ and the restriction of it to T_xY is the almost complex structure I_Y on Y, the metric $\left.g\right|_Y$ is compatible with the almost complex structure on Y. Thus, $\left.g\right|_Y$ defines an hermitian structure on Y. By definition, the associated Kähler form ω_Y is given by $\omega_Y=\left.g\right|_Y(I_Y(),())=\left.g(I(),())\right|_Y=\left.\omega\right|_Y$. Therefore, $d_Y\omega_Y=d_Y\left(\left.\omega\right|_Y\right)=\left.(d_X\omega)\right|_Y=0$ ☐

Corollary 5.2.2. Any projective manifold is Kähler.

Proof. By definition a projective manifold can be realized as a submanifold of $\mathbb{P}^n$. Restricting the Fubini-Study metric yields a Kähler metric.

The following Proposition calculates the mixed commutators of linear operators, and differential operators explicitly. The Kähler condition $d\omega=0$ is crucial for this.

5.3 Kahler Identities

Proposition 5.3.1 (Kähler identities). Let X be a complex manifold endowed with a Kähler metric g. Then the following identities hold true:

i) $[\bar{\partial},L]=[\partial,L]=0$ and $\left[\bar{\partial}^*,\Lambda\right]=[\partial^*,\Lambda]=0$.

ii) $\left[\bar{\partial}^*,L\right]=i\partial,[\partial^*,L]=-i\bar{\partial}$ and $[\Lambda,\bar{\partial}]=-i\partial^*,[\Lambda,\partial]=i\bar{\partial}^*$.

iii) $\Delta_\partial=\Delta_{\bar{\partial}}=\frac{1}{2}\Delta$ and Δ commutes with $*,\partial,\bar{\partial},\partial^*,\bar{\partial}^*,L$, and Λ.

The theorem will be proved in terms of yet another operator d^c.

Definition 5.3.2.
$$d^c := \mathbf{I}^{-1} \circ d \circ \mathbf{I} \text{ and } d^{c^*} := -* \circ d^c \circ *$$

d^c is a real operator which is extended C-linearly. Equivalently, one could define $d^c = -i(\partial - \bar{\partial})$

Indeed, if $\alpha \in \mathcal{A}^{p,q}(X)$ then
$\mathbf{I}(\partial - \bar{\partial})(\alpha) = i^{p+1-q}\partial(\alpha) - i^{p-q-1}\bar{\partial}(\alpha) = i^{p+1-q}d(\alpha) = id(\mathbf{I}(\alpha))$

Also, $dd^c = 2i\partial\bar{\partial}$

Assertion ii) implies $[\Lambda, d] = i\left(\bar{\partial}^* - \partial^*\right) = -i * (\partial - \bar{\partial})* = -d^{c*}$. In fact, using the bidegree decomposition one easily sees that $[\Lambda, d] = -d^{c^*}$ is equivalent to the assertions of ii).

Proof. Let us first prove i). By definition

$$[\bar{\partial}, L](\alpha) = \bar{\partial}(\omega \wedge \alpha) - \omega \wedge \bar{\partial}(\alpha) = \bar{\partial}(\omega) \wedge \alpha = 0,$$

for $\bar{\partial}(\omega)$ is the (1,2)-part of $d\omega$, which is trivial by assumption.
Similairly, $[\partial, L](\alpha) = \partial(\omega) \wedge \alpha = 0$.

For $\alpha \in \mathcal{A}^k(X)$ we have,

$$\begin{aligned}\left[\bar{\partial}^*, \Lambda\right](\alpha) &= - * \partial * *^{-1} L * (\alpha) - *^{-1} L * (- * \partial *)(\alpha)\\ &= - * \partial L * (\alpha) - (-1)^k *^{-1} L\partial * (\alpha) = -(*\partial L * - * L\partial *)(\alpha)\\ &= - * [\partial, L] * (\alpha) = 0.\end{aligned}$$

Here, we used that $*^2 = (-1)^\ell$ on $\mathcal{A}^\ell(X)$.

The last assertion can be proved analogously. It can also be verified by just complex conjugating: $[\partial^*, \Lambda] = \overline{[\bar{\partial}^*, \bar{\Lambda}]} = \overline{[\bar{\partial}^*, \Lambda]} = 0$, where one uses that $*$ and Λ are $\mathbb{C}$-linear extensions of real operators.

ii) Using the Lefschetz decomposition, it is enough to prove the assertion for forms of the type $L^j\alpha$ with α a primitive k-form. Then $d\alpha \in \mathcal{A}^{k+1}(X)$ using the Lefschetz decomposition can be written as,

$$d\alpha = \alpha_0 + L\alpha_1 + L^2\alpha_2 + \dots$$

with $\alpha_j \in P^{k+1-2j}(X)$. Since L commutes with d and $L^{n-k+1}\alpha = 0$, this yields

$$0 = L^{n-k+1}\alpha_0 + L^{n-k+2}\alpha_1 + L^{n-k+3}\alpha_2 + \dots$$

As the Lefschetz decomposition is a direct sum decomposition, this implies

$$L^{n-k+j+1}\alpha_j = 0, \text{ for } j = 0, 1, \ldots.$$

On the other hand, L^ℓ is injective on $\mathcal{A}^i(X)$ for $\ell \leq n - i$.
Hence, since $\alpha_j \in \mathcal{A}^{k+1-2j}(X)$, one finds $\alpha_j = 0$ for $j \geq 2$.
Thus, $d\alpha = \alpha_0 + L\alpha_1$ with $\Lambda\alpha_0 = \Lambda\alpha_1 = 0$.

Let us first compute $[\Lambda, d]\left(L^j\alpha\right)$ for $\alpha \in P^k(X)$. Using $[d, L] = 0, \Lambda\alpha_i = 0$,we get,

$$\begin{aligned} \Lambda dL^j\alpha = \Lambda L^j d\alpha = \Lambda L^j\alpha_0 + \Lambda L^{j+1}\alpha_1 \\ = -j(k+1-n+j-1)L^{j-1}\alpha_0 - (j+1)(k-1-n+j)L^j\alpha_1 \end{aligned}$$

and

$$\begin{aligned} d\Lambda L^j\alpha &= -j(k-n+j-1)L^{j-1}d\alpha \\ &= -j(k-n+j-1)\left(L^{j-1}\alpha_0 + L^j\alpha_1\right). \end{aligned}$$

Therefore,

$$[\Lambda, d]\left(L^j\alpha\right) = -jL^{j-1}\alpha_0 - (k-n+j-1)L^j\alpha_1$$

On the other hand we have,

$$\begin{aligned} &- d^{c*}L^j\alpha = *\mathbf{I}^{-1}d\mathbf{I} * L^j\alpha \\ &= *\mathbf{I}^{-1}d\mathbf{I}\left((-1)^{\frac{k(k+1)}{2}}\frac{j!}{(n-k-j)!}\cdot L^{n-k-j}\mathbf{I}(\alpha)\right) \\ &= (-1)^{\frac{k(k+1)}{2}+k}\frac{j!}{(n-k-j)!}\cdot\left(\mathbf{I}^{-1} * L^{n-k-j}d\alpha\right) \quad \text{ using } \mathbf{I}^2\big|_{\Lambda^k} = (-1)^k \\ &= (-1)^{\frac{k(k+1)}{2}+k}\frac{j!}{(n-k-j)!}\cdot\left(\mathbf{I}^{-1}\left(*L^{n-k-j}\alpha_0 + *L^{n-k-j+1}\alpha_1\right)\right) \\ &= (-1)^{\frac{k(k+1)}{2}+k+\frac{(k+1)(k+2)}{2}}j\cdot\left(L^{j-1}\alpha_0\right) \\ &+ (-1)^{\frac{k(k+1)}{2}+k+\frac{k(k-1)}{2}}(n-k-j+1)\cdot\left(L^j\alpha_1\right) \\ &= -jL^{j-1}\alpha_0 - (k-n+j-1)L^j\alpha_1. \end{aligned}$$

This yields $[\Lambda, d] = -d^{c^*}$.
iii) We first show that $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$. Indeed, assertion ii) yields $i\left(\partial\bar{\partial}^* + \bar{\partial}^*\partial\right) = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = \partial\Lambda\partial - \partial\Lambda\partial = 0$. Next,

$$
\begin{aligned}
\Delta_\partial &= \partial^*\partial + \partial\partial^* \\
&= i[\Lambda, \bar\partial]\partial + i\partial[\Lambda, \bar\partial] \\
&= i(\Lambda\bar\partial\partial - \bar\partial\Lambda\partial + \partial\Lambda\bar\partial - \partial\bar\partial\Lambda) \\
&= i(\Lambda\bar\partial\partial - (\bar\partial[\Lambda, \partial] + \bar\partial\partial\Lambda) + ([\partial, \Lambda]\bar\partial + \Lambda\partial\bar\partial) - \partial\bar\partial\Lambda) \\
&= i\left(\Lambda\bar\partial\partial - i\bar\partial\bar\partial^* - \bar\partial\partial\Lambda - i\bar\partial^*\bar\partial + \Lambda\partial\bar\partial - \partial\bar\partial\Lambda\right) \\
&= \Delta_{\bar\partial}.
\end{aligned}
$$

In order to compare Δ with Δ_∂, write

$$
\begin{aligned}
\Delta &= (\partial + \bar\partial)\left(\partial^* + \bar\partial^*\right) + \left(\partial^* + \bar\partial^*\right)(\partial + \bar\partial) \\
&= \Delta_\partial + \Delta_{\bar\partial} + \left(\partial\bar\partial^* + \bar\partial^*\partial\right) + \overline{\left(\partial\bar\partial^* + \bar\partial^*\partial\right)} \\
&= \Delta_\partial + \Delta_{\bar\partial} + 0 + 0 \\
&= 2\Delta_\partial.
\end{aligned}
$$

Using $dd^c = 2i\partial\bar\partial = -d^c d$, one computes $\Lambda\Delta = \Lambda dd^* + \Lambda d^* d = d\Lambda d^* - id^{c*}d^* + d^*\Lambda d = dd^*\Lambda + id^*d^{c^*} + d^*d\Lambda - id^*d^{c^*} = \Delta\Lambda$. $\square$

5.4 Dolbeault Cohomology

X is a Complex Manifold. The (p,q) - Dolbeault Cohomology is given by:

$$
H^{(p,q)}(X) := \frac{Ker(\bar\partial : \mathcal{A}^{(p,q)}(X) \to \mathcal{A}^{(p,q+1)}(X))}{Im(\bar\partial : \mathcal{A}^{(p,q-1)}(X) \to \mathcal{A}^{(p,q)}(X))}
$$

The Dolbeault cohomology of X computes the cohomology of the sheaf,

$$
H^{(p,q)}(X) = H^q(X, \Omega^p_X)
$$

Chapter 6

Hodge Theory on Kähler Manifolds

In this chapter we will focus on Compact hermitian and Kähler manifolds. The compactness allows us to apply Hodge theory, or the theory of elliptic operators on compact manifolds.

If X is a complex manifold with an hermitian structure g, we denote the hermitian extension of the Riemannian metric g by $g_{\mathbb{C}}$. It naturally induces hermitian products on all form bundles.

Definition 6.0.1. Let (X, g) be a compact hermitian manifold. Then one defines an hermitian product on $\mathcal{A}^*_{\mathbb{C}}(X)$ by

$$(\alpha, \beta) := \int_X g_C(\alpha, \beta) * 1.$$

Lemma 6.0.1. *Let X be a compact hermitian manifold. Then with respect to the hermitian product $(,)$ the operators ∂^* and $\bar{\partial}^*$ are the formal adjoints of ∂ and $\bar{\partial}$, respectively.*

Proof. The proof is similar for ∂ and $\bar{\partial}$.
Let $\alpha \in \mathcal{A}^{p-1,q}(X)$ and $\beta \in \mathcal{A}^{p,q}(X)$.
By definition,

$$\begin{aligned}(\partial\alpha, \beta) &= \int_X g_{\mathbb{C}}(\partial\alpha, \beta) * 1 = \int_X \partial\alpha \wedge *\bar{\beta} \\ &= \int_X \partial(\alpha \wedge *\bar{\beta}) - (-1)^{p+q-1} \int_X \alpha \wedge \partial(*\bar{\beta}).\end{aligned}$$

The first integral of the last line vanishes due to Stokes' theorem, as $\alpha \wedge *\bar{\beta}$ is a $(p-1, q) + (n-p, n-q) = (n-1, n)$ form and, therefore, $\partial(\alpha \wedge *\bar{\beta}) = d(\alpha \wedge *\bar{\beta})$.

The second integral is computed using $*^2 = (-1)^k$ on $\mathcal{A}^k(X)$:

$$\begin{aligned}\int_X \alpha \wedge \partial(*\bar{\beta}) &= (-1)^{p+q+1} \cdot \int_X g_{\mathbb{C}}(\alpha, *(\bar{\partial}(*\beta))) * 1 \\ &= (-1)^{p+q+1} \cdot \int_X g_{\mathbb{C}}\left(\alpha, -\partial^* \beta\right) * 1 \\ &= (-1)^{p+q+2} \cdot (\alpha, \partial^* \beta)\end{aligned}$$

$\mathcal{H}^k(X,g)$ denotes the space of (d-)harmonic k-forms. Analogously, one defines $\mathcal{H}^{p,q}(X,g)$ as the space of ($d-$)harmonic (p,q)-forms. When the metric is fixed, one often drops g in the notation.

Definition 6.0.2. Let (X,g) be an hermitian complex manifold. A form $\alpha \in \mathcal{A}^k(X)$ is called $\bar{\partial}$-harmonic if $\Delta_{\partial}(\alpha) = 0$. Moreover,

$$\begin{aligned}\mathcal{H}^k_{\bar{\partial}}(X,g) &:= \left\{\alpha \in \mathcal{A}^k_{\mathbb{C}}(X) \mid \Delta_{\bar{\partial}}(\alpha) = 0\right\} \text{ and} \\ \mathcal{H}^{p,q}_{\bar{\partial}}(X,g) &:= \{\alpha \in \mathcal{A}^{p,q}_{\mathbb{C}}(X) \mid \Delta_{\bar{\partial}}(\alpha) = 0\}\end{aligned}$$

Analogously, one defines ∂-harmonic forms and the spaces $\mathcal{H}^k_{\partial}(X,g)$ and $\mathcal{H}^{p,q}_{\partial}(X,g)$

Lemma 6.0.2. *Let (X,g) be a compact hermitian manifold (X,g). A form α is $\bar{\partial}$-harmonic (resp. ∂-harmonic) if and only if $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$ (resp. $\partial\alpha = \partial^*\alpha = 0$)*

Proof. The assertion follows from

$$(\Delta_{\bar{\partial}}(\alpha), \alpha) = \left\|\bar{\partial}^*(\alpha)\right\|^2 + \|\bar{\partial}(\alpha)\|^2.$$

Thus, $\Delta_{\bar{\partial}}(\alpha) = 0$ implies the vanishing of both terms on the right hand side, i.e. $\bar{\partial}(\alpha) = \bar{\partial}^*(\alpha) = 0$. The converse is clear. A similar argument proves the assertion for Δ_{∂}.

Proposition 6.0.3. Let (X,g) be an hermitian manifold, not necessarily compact. Then i) $\mathcal{H}^k_{\bar{\partial}}(X,g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\partial}(X,g)$ and $\mathcal{H}^k_{\partial}(X,g) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\partial}(X,g)$.
ii) If (X,g) is Kähler then both decompositions coincide

6.1 Serre Duality

As X is a Compact Kahler Manifold, Observe that there is a non-degenerate pairing ,

$$H_{\partial}^{(p,q)}(X) \times H_{\partial}^{(n-p,n-q)}(X) \to \mathbb{C}$$
$$\alpha, \beta \to \int_X \alpha \wedge \beta = (\alpha, *\bar{\beta})$$

The pairing is non-degenerate for

$$*\bar{\beta} = \alpha$$

Hence, $H_{\partial}^{(p,q)}(X) \cong (H_{\partial}^{(n-p,n-q)}(X))^*$

6.2 Conjugation Isomorphism

We know that $\partial(\bar{\alpha}) = \bar{\partial}(\alpha)$ and $*(\bar{\alpha}) = \bar{*(\alpha)}$ Hence, $\alpha \in H_{\partial}^{(p,q)}$(X) iff $\overline{\alpha} \in H_{\bar{\partial}}^{(q,p)}$(X)

$H_{\partial}^{(p,q)}(X) \cong H_{\partial}^{(q,p)}(X)$ by the Complex Conjugation Isomorphism.

6.3 Hodge Decomposition Theorem

We have,

Theorem 6.3.1. (Hodge decomposition) Let (X, g) be a compact hermitian manifold. Then there exist two natural orthogonal decompositions

$$\mathcal{A}^{p,q}(X) = \partial\mathcal{A}^{p-1,q}(X) \oplus \mathcal{H}_{\partial}^{p,q}(X, g) \oplus \partial^*\mathcal{A}^{p+1,q}(X)$$

and

$$\mathcal{A}^{p,q}(X) = \bar{\partial}\mathcal{A}^{p,q-1}(X) \oplus \mathcal{H}_{\partial}^{p,q}(X, g) \oplus \bar{\partial}^*\mathcal{A}^{p,q+1}(X)$$

The spaces $\mathcal{H}^{p,q}(X, g)$ are finite-dimensional. If (X, g) is assumed to be Kähler then $\mathcal{H}_{\partial}^{p,q}(X, g) = \mathcal{H}_{\partial}^{p,q}(X, g)$.

The orthogonality of the decomposition is easy to verify.The crucial fact is the existence of the direct sum decomposition which is analogous to the proof of Hodge Decomposition theorem in [**5**].

Corollary 6.3.2. Let (X, g) be a compact hermitian manifold. Then the canonical projection $\mathcal{H}_{\partial}^{p,q}(X, g) \to H^{p,q}(X)$ is an isomorphism.

Proof. Since any $\alpha \in \mathcal{H}_{\partial}^{p,q}(X, g)$ is $\bar{\partial}$-closed, mapping α to its Dolbeault cohomology class $[\alpha] \in H^{p,q}(X)$ defines a map $\mathcal{H}_{\partial}^{p,q}(X, g) \to H^{p,q}(X)$.

Moreover, $\operatorname{Ker}\left(\bar{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X)\right) = \bar{\partial}\left(\mathcal{A}^{p,q-1}(X)\right) \oplus \mathcal{H}_{\bar{\partial}}^{p,q}(X, g)$, as $\bar{\partial}\bar{\partial}^{*}\beta = 0$ if and only if $\bar{\partial}^{*}\beta = 0$. Indeed, $\bar{\partial}\bar{\partial}^{*}\beta = 0$ implies $0 = \left(\bar{\partial}\bar{\partial}^{*}\beta, \beta\right) = \left\|\bar{\partial}^{*}\beta\right\|^{2}$. Thus, $\mathcal{H}_{\partial}^{p,q}(X, g) \to H^{p,q}(X)$ is an isomorphism.

Corollary 6.3.3. Let (X,g) be a Compact Kahler Manifold then, $H^{k}(X, \mathbb{C}) = \oplus_{p+q=k} H^{(p,q)} X$

Proof.

$$H^{k}(X, \mathbb{C}) = \oplus_{p+q=k} \mathcal{H}^{(p,q)}(X, g) = \oplus_{p+q=k} H^{(p,q)}(X) \quad \square$$

Chapter 7

Hodge Diamond of Compact Kahler Manifolds

Finally, we are in a position to compute the Hodge Diamond of certain varieties of Compact Kahler Manifolds.The Hodge Diamond is an elegant way to encode the Hodge numbers $h^{(p,q)}(X) = Dim_{\mathbb{C}}(H^{(p,q)})$ of a Compact Kahler Manifold X.
From the previous few sections we have,

- $h^{(p,q)}(X) = h^{(n-p,n-q)}(X)$ (Serre Duality)

- $h^{(p,q)}(X) = h^{(q,p)}(X)$ (Conjugation Isomorphism)

$$\begin{array}{ccccccccccc}
 & & & & h^{0,0} & & & & & & b_0 \\
 & & & h^{1,0} & & h^{0,1} & & & & & b_1 \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & & & b_2 \\
 & & & & \vdots & & & \ddots & & & \vdots \\
h^{n,0} & & \cdots & & \curvearrowleft & & \cdots & & h^{0,n} & \updownarrow \text{ Hodge} & b_n \\
 & \ddots & & & \text{Serre} & & & & & & \vdots \\
 & & & & \vdots & & & & & & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & & & b_{2n-2} \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & & & b_{2n-1} \\
 & & & & h^{n,n} & & & & & & b_{2n} \\
 & & & & \leftrightarrow & & & & & & \\
 & & & & \text{conjugation} & & & & & &
\end{array}$$

Let us now discuss the Hodge Diamond for 3 cases of Compact Kahler Manifolds.

7.1 Compact Complex Curve

Recall that, any Compact Connected 1-Dimensional Complex Manifold X is a Compact Kahler Manifold. So, We have that $h^{(0,0)}$(X) = 1 and this gives the Hodge Diamond,

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

Where g is the geometric genus of X. (Geometric genus of Complex n - manifold is the value of h$^{(n,0)}$)

7.2 K - 3 Surfaces

We will in particular workout the case of a Quartic Hypersurface in $\mathbb{P}^3$
For this case, we will need a result from [**1**] known as the Weak Lefschetz theorem.

Theorem 7.2.1. *Let X be a Compact Kahler manifold of dimension n and let Y $\subset$ X be a smooth hypersurface (such that the induced line bundle O(Y) is positive). Then, the canonical restriction map ,*

$$H^k(X, \mathbb{C}) \to H^K(Y, \mathbb{C})$$

is bijective for k $\leq$ n - 2 and injective for k $\leq$ n - 1 .

We will need the Euler Characteristic of a Hypersurface in $\mathbb{P}^n$ given in [**4**]

Theorem 7.2.2. *Let $V \subset \mathbb{P}^{n+1}$ be a degree d smooth Complex projective hypersurface. Then, the Euler characteristic of V is given by the formula:*

$$\chi(V) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\}.$$

- Since, $\mathbb{P}^3$ is connected and has 1 cell for every even number from 1 to 2n. H^1(X,$\mathbb{C}$) is trivial and $H^0(X, \mathbb{C}) \cong \mathbb{C}$ (Weak Lefschetz)

- By Serre Duality, we have $h^{(1,0)} = h^{(0,1)} = h^{(3,0)} = h^{(0,3)} = 0$

- By the Adjunction Formula, $\Omega^2(X) \cong \mathcal{O}_X$.
  Hence, $H^{(2,0)}(X, \mathbb{C}) \cong H^0(X, \Omega^2(X)) \cong H^0(X, \mathcal{O}_X) \cong \mathbb{C}$
  (Holomorphic functions on a Compact Connected Complex Manifold are constant)

- $h^{(2,0)} = h^{(0,2)} = 1$ (Conjugation Isomorphism)

- Euler Characteristic for Quartic Surface in $\mathbb{P}^3$ is 24 .
  Hence, $h^{(1,1)} = 20$

This gives the Hodge Diamond,

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Remark 7.2.1. It is important to note here that there exists K - 3 Surfaces that are not biholomorphic to the Quartic Hypersurface in $\mathbb{P}^3$ but, have the same Hodge Diamond. In this case, the subspace $\mathrm{H}^{(p,q)}$ are distinct but, isomorphic Hence, they have the same Hodge Diamond.

7.3 Quintic 3 - Fold

Quintic 3 - fold, Y can be embedded as a Quintic Hypersurface in $\mathbb{P}^4$.
We will look at a Quintic Hypersurface in $\mathbb{P}^4$

- Since, $\mathbb{P}^3$ is connected and has 1 cell for every even number from 1 to 2n. H^1(X,$\mathbb{C}$) is trivial and $H^0(X, \mathbb{C}) \cong \mathbb{C}$. By Weak Lefschetz theorem ,$\mathrm{H}^1(Y, \mathbb{C})$ is trivial and H^2(Y, $\mathbb{C}$) $\cong \mathbb{C}$, $H^0(Y, \mathbb{C}) \cong \mathbb{C}$

- By Serre Duality, $h^{(1,0)} = h^{(0,1)} = h^{(5,0)} = h^{(0,5)} = 0$

- By Weak Lefschetz Theorem, $H^2(Y, \mathbb{C}) \cong \mathbb{C}$ and Y is Kahler.
  We have, $h^{(1,1)} = 1$ and, $h^{(2,0)} = h^{(0,2)} = 0$

- By Serre Duality, $\mathrm{h}^{(4,4)} = \mathrm{h}^{(1,1)} = 1$

- By Adjunction Formula, $\Omega^3(Y) \cong \mathcal{O}_Y$.
  Hence, $H^{(3,0)}(Y, \mathbb{C}) \cong H^0(Y, \Omega^3(Y)) \cong H^0(Y, \mathcal{O}_Y) \cong \mathbb{C}$
  (Holomorphic functions on a Compact Connected Complex Manifold are constant)
- By Conjugation Isomorphism, $h^{(3,0)} = h^{(0,3)} = 1$
- Finally, Euler Characteristic for Quintic hypersurface in $\mathbb{P}^4$ is - 200.
  Hence, $h^{(2,1)} = 101 = h^{(1,2)}$

This gives the Hodge Diamond as follows,

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

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