TOPOLOGICAL BOUNDS ON CERTAIN MODULI SPACE OF CURVES

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1. ORGANISATION

(1) Moduli of Curves [15mins]
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We shall work over the complex numbers.

2. Moduli of Curves

2.1. Algebraic Curves. An algebraic curve is a 1 dimensional smooth projective variety. Said differently it is a compact Riemann surface. Examples are $\mathbb{P}^1$, elliptic curves and higher genus curves. Figures

The genus of a Rimann surface X is half its first Betti number:

$$g(X) = \frac{1}{2} \dim H_1(X)$$

An n pointed curve will be an algebraic curve C, with n distinct numbered points $p_1, \ldots, p_n \in C$. Two such curves $(C; p_1, \ldots, p_n)$ and $(C'; p'_1, \ldots, p'_n)$ are isomorphic if there is an isomorphism $C \to C'$ taking $p_i \mapsto p'_i$.

2.2. Moduli Problem. The moduli problem for algebraic curves is to find a "space" $M_{g,n}$, parameterizing isomorphism classes of n pointed genus g curves. It should have the following properties:

(1) Points of $M_{g,n}$should be in bijecton with isomorphism classes of n pointed curves.
(2) There should be a universal $T_{g,n} \to M_{g,n}$ such that any family of n pointed curves $C \to B$ is a pull back of the universal family.

If such a "space" exists it is called the fine moduli space. It is well known that no such moduli space exits in the category of schemes. The obstruction comes from existence of non-trivial automorphisms of curves.

Never the less, the obstruction is not very serious and a fine moduli space can be constructed as a Deligne-Mumford stack, which we shall denote $\mathcal{M}_{g,n}$. These stacks are analogues of orbifolds in the realm of algebraic geometry.

Underlying the moduli stack there is a coarse moduli space which is an algebraic variety. This we shall denote by $M_{g,n}$. It satisfies property (1) above but not (2). Although whenever we have a family of curves $C \to B$, there exists a unique map $B \to M_{g,n}$.

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The moduli space $M_{g,n}$ is not a complete variety. The reason being, there can be singular curves in the limit of smooth curves as they vary in families. (More precisely, if we have a family of smooth curves over $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, then we may not be able to extend this family to $\mathbb{C}$, even after a base change.)

2.3. **Deligne Mumford Compactification.** To compactify the moduli space Deligne and Mumford considered a more general moduli problem by including certain singular curves but only the necessary ones. It turns out that the class of curves that can arise as limits of smooth curves are curves with only nodal singularities. The main ingredient here is the *stable reduction theorem* for curves.

Stable Curves: A stable curve C of genus g, with n marked points $\{p_1, \dots, p_n\}$, is a projective curve with the following properties:

(1) $\dim H^1(C, \mathcal{O}_C) = g$, where $\mathcal{O}_C$ is the structure sheaf.
(2) The singularities of C are all nodes.
(3) $\{p_1, \dots, p_n\}$ are distinct smooth points of the curve.
(4) $C^{\mathrm{sm}} \setminus \{p_1, \dots, p_n\}$ (the smooth locus with the marked points removed) has negative Euler characteristic.

Examples: Figures

Again the fine moduli space of stable n pointed genus g curves exits as a Deligne-Mumford stack, we denote this by $\overline{\mathcal{M}}_{g,n}$. There is a corresponding coarse moduli space $\overline{M}_{g,n}$ which is an algerbaic variety. For the most part, our discussion will involve the course moduli space.

Some properties of the moduli space are:

(1) $\overline{\mathcal{M}}_{g,n}$, is an irreducible, smooth, projective, Deligne-Mumford stack of dimension $3g - 3 + n$.
(2) $\overline{\mathcal{M}}_{g,n+1}$ along with the natural map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family.
(3) $\overline{M}_{g,n}$ is an irreducible, projective variety, of dimension $3g - 3 + n$, and has only mild singularties (finite quotient).
(4) $M_{g,n}$ is an open dense subvariety of $\overline{M}_{g,n}$ and the complement is a divisor with normal crossings.
(5) When $g = 0$, $M_{0,n}$ and $\overline{M}_{0,n}$ are smooth varieties and fine moduli spaces for their moduli problems.

2.4. **Elementary examples.**

- $\overline{M}_{0,3}$ is a point.
  For any three distinct points p_1, p_2, p_3 on $\mathbb{P}^1$ there is an element of $\mathrm{PGL}(2, \mathbb{C})$ sending them to $0, 1, \infty$ respectively.
- $M_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $\overline{M}_{0,4} \cong \mathbb{P}^1$.
  The four distinct points (x_1, x_2, x_3, x_4) can be taken to $\left(0, 1, \infty, \frac{(x_4-x_1)(x_2-x_3)}{(x_4-x_3)(x_2-x_1)}\right)$, by the automorphism of $\mathbb{P}^1$ given by $x \mapsto \frac{(x-x_1)(x_2-x_3)}{(x-x_3)(x_2-x_1)}$. The number $\frac{(x_4-x_1)(x_2-x_3)}{(x_4-x_3)(x_2-x_1)}$, is called the cross ratio of (x_1, x_2, x_3, x_4). Now if two of the points collide, we get a stable genus zero curve with two rational components joined by a single node and each component having two marked points. In this case the cross ratio is 0,1 or ∞.

- Using a similar argument $\overline{M}_{0,5}$ can be identified with the surface obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at $(0,0)$, $(1,1)$ and (∞, ∞). It is a Del-pezzo surface of degree 5. Figure
- In general $\overline{M}_{0,n+1}$, can as constructed as a blow up of $\overline{M}_{0,n} \times \mathbb{P}^1$.
- $M_{1,1} \cong M_{0,4}/S_3 \cong (\mathbb{P}^1 \setminus \{0,1,\infty\})/S_3$, and $\overline{M}_{1,1} \cong \overline{M}_{0,4}/S_3 \cong \mathbb{P}^1/S_3$.
  Any elliptic curve is a double cover of $\mathbb{P}^1$ ramified over 4 points. Lets say the points in $\mathbb{P}^1$ above which ramification occurs are (x_1, x_2, x_2, x_4). We can take the point on the elliptic curve over x_4 as the marked point. So the point x_4 becomes special, but the order of the first three points is irrelevant. Also if (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) only differ by an automorphism of $\mathbb{P}^1$ then the elliptic curves ramified over the first set of points is isomorphic to the one ramified over the second set. When two ramification points come together we get the marked singular genus 1 curve.
- Similarly $\overline{M}_2 \cong \overline{M}_{0,6}/S_6$.

2.5. **Dual graph.** Associated to a stable curve is its dual graph. The graph is obtained by associating a vertex to each component of the curve, labelled by the geometric genus of the component, an edge for each node and leaf for each marked point. Figure

The dual graphs give a decomposition of $\overline{M}_{g,n}$. We call a dual graph to a stable n pointed genus g curve a stable graph of type (g, n). Let $\Gamma(g, n)$ be the set of distinct stable graphs of type (g, n).

For a graph G let M_G be the subvariety of $M_{g,n}$ parameterizing stable curves whose dual graph is G. Then

$$\overline{M}_{g,n} = \sqcup M_G$$

3. Affine Stratification

3.1. **Affine stratification.** An affine stratification of a scheme X is a finite decomposition

$$X = \bigsqcup_{k=0}^{n} \bigsqcup_{i} Y_{k,i}$$

where $Y_{k,i}$ are disjoint, locally closed affine subvarieties such that $\overline{Y}_{k,i} \backslash Y_{k,i} \subset \bigsqcup_{l>k,i}^{n} Y_{l,i}$. The length of the stratification is n, and the affine stratification number $\operatorname{asn} X$ is the minimum of the lengths over all affine stratifications of X.

When the scheme X is equidimensional we can choose an affine stratification to be

$$X = \bigsqcup_{k=0}^{n} Y_k$$

where Y_k is codimension k in X and $n = \operatorname{asn} X$.

The affine stratfication number of X gives bounds on the "topological complexity" of X. Here are a few properties it satisfies.

(1) $\operatorname{asn} X = 0$ if and only if X is affine.
(2) $\operatorname{asn}(X \times Y) \leq \operatorname{asn} X + \operatorname{asn} Y$.
(3) $\operatorname{asn} X \leq \dim X$, if one of the top dimenstional components of X is proper then equality holds.
(4) If D is an effective Cartier divisor of X, then $\operatorname{asn}(X - D) \leq \operatorname{asn} X$.
(5) If $\mathcal{F}$ is a quasi-coherent sheaf on X, then $H^i(X, \mathcal{F}) = 0$ for $i > \operatorname{asn} X$.
(6) If $Y \to X$ is an affine morphism then $\operatorname{asn} Y \leq \operatorname{asn} X$.
(7) If $Y \subset X$ is a proper closed subscheme of X, then $\dim Y \leq \operatorname{asn} X$.

Examples:

- The first non-trivial example is $\mathbb{A}^2 \setminus \{(0,0)\}$, and an affine stratification is

$$(\mathbb{A}^2 \setminus \{x - \text{axis}\}) \sqcup (x - \text{axis} \setminus (0,0))$$

Hence $\operatorname{asn}(\mathbb{A}^2 \setminus \{(0,0)\}) = 1$.

- *Any value between* 0 *and* $\dim X$ *is possible:* Consider $X = \mathbb{P}^n \setminus \{(n-k-1) - \mathit{plane}\}$, then

$$H^k(X, \mathcal{O}_X) \neq 0$$

hence $\operatorname{asn} X \geq k$. On the other hand $X = \mathbb{A}^n \sqcup \ldots \sqcup \mathbb{A}^{n-k}$ is an affine stratification of length k. Thus $\operatorname{asn} X = k$.

- *Moduli of curves:* $\operatorname{asn} M_g = g - 2$ for $2 \leq g \leq 5$. This is shown in Fontanari and Looijenga. It is well known that M_2 is affine.

  $M_3 = (M_3 \setminus H_3) \sqcup H_3$, where H_3 is the hyperelliptic locus of genus 3, and its compliment in M_3 is the moduli of plane quartics.

  A theta characteristic of an algebraic curve is a line bundle whose square is the canonical bundle. A theta characteristic L on C is called even (correspondingly odd) if the dimension of $H^0(C, L)$ is even (respectively odd). For $g \geq 3$, the sub-variety in M_g parameterizing curves which possess an effective even theta characteristic is an irreducible divisor called the *Thetanull* divisor and denoted M_g'.

  M_4 has an affine stratification

$$M_4 = (M_4 \setminus M_4') \sqcup (M_4' \setminus H_4) \sqcup H_4$$

  Let T_g be the locus of trigonal curves in M_g. These are curves that admit a degree 3 map to $\mathbb{P}^1$. We denote the intersection of the trigonal locus with the thetanull divisor by T_g'. Then M_5 has the following affine stratification of length 3.

$$M_5 = (M_5 \setminus M_5') \sqcup (M_5' \setminus T_5') \sqcup (T_5' \setminus H_5) \sqcup H_5$$

  Another stratification can be obtained by replacing the thetanull divisor by the trigonal locus.

3.2. **Cohomological excess.** Let X be a complex variety and $\mathcal{F}$ an abelian sheaf on X. $\mathcal{F}$ is constructible if there is a locally finite partition of X, into locally closed subvarieties $X = \sqcup_i Y_i$ such that $\mathcal{F}|_{Y_i}$ is locally constant in the Euclidean topology.

The constructible cohomological dimension (ccd) of X is the minimum integer d such that $H^i(X, \mathcal{F}) = 0$ for $i > d$ and any constructible sheaf $\mathcal{F}$ on X.

It is well known that $\operatorname{ccd} X \leq 2 \times \dim X$. We define the cohomological excess of X as follows.

Cohomological excess of X denoted $\operatorname{ce}(X)$ is given by

$$\operatorname{ce}(X) = \max\{\operatorname{ccd} W - \dim W \mid W \ \textit{subvariety of} \ X\}$$

We have the following inequality,

$$\operatorname{ce}(X) \leq \operatorname{asn} X$$

3.3. **Questions.** The original question that inspired my thesis work was due to Looijenga. He conjectured that M_g can be covered by $g-1$ open sets. This was recently show for $g \leq 5$ by Fontanari and Pasculotti in 2012.

Loojenga and Fontanari showed the weaker result $\operatorname{asn} M_g \leq g-2$ for $g \leq 5$.

There is a filtration on $\overline{M}_{g,n}$ given by the number of rational components of a curve. $\overline{M}^{\leqslant k}_{g,n}$ parametrizes stable curves with at most k rational components. Figure

This is clearly an increasing filtration.

$$\overline{M}^{\leqslant 0}_{g,n} \subset \cdots \subset \overline{M}^{\leqslant 2g-2+n}_{g,n} = \overline{M}_{g,n}$$

Roth and Vakil extended Looijenga's conjecture to the following

Conjecture 1. $\operatorname{asn} \overline{M}^{\leqslant k}_{g,n} \leq g-1+k$

Recently Looijenga put up a paper on the arxiv proving the the above bounds for cohomological excess of these varieties, but there seems to be a gap in that paper so we should take these as conjectures.

It is know that the bounds $\operatorname{ce}(M_g) \leq g-2$ and $\operatorname{ce}(M_{g,n} \leq g-1$ are sharp, due to a result of Harer. The sharpness of the bounds on $\overline{M}^{\leqslant k}_{g,n}$ are not known.

I investigate similar questions for the hyperelliptic locus.

4. Hyperelliptic locus

4.1. **Hyperelliptic curves.** A hyperelliptic curve of genus g is a double cover of $\mathbb{P}^1$. In fancier terms it has a linear system of rank 1 and degree 2. By Riemann-Hurwitz formula a hyerlliptic curve of genus g as a double cover of $\mathbb{P}^1$ is ramified at $2g+2$ points. Any such curve can be covered by two affine open subsets one given by

$$y^2 = f(x) \quad \text{degree of } f \text{ is 2g+1 or 2g+2}$$

and

$$w^2 = v^{2g+2} f(1/v)$$

with the gluing map being $(x, y) \mapsto (1/v, w/v^{g+1})$

Let H_g be the subvariety of M_g parametrizing hyperelliptic curves, and let $\overline{H}_g$ be the closure in $\overline{M}_g$.

$\overline{H}_2 = \overline{M}_2$ and $\overline{H}_3$ is a divisor in $\overline{M}_3$. In genral $\overline{H}_g$ is a subvariety of $\overline{M}_g$ of dimenstion $2g-1$ (codimension g-2).

We have the isomorphism $\overline{H}_g \cong \overline{M}_{0,2g+2}$ which we shall describe shortly.

4.2. **Hurwitz spaces.** Let

$$f : C \to \mathbb{P}^1$$

be a ramified d sheeted covering, where C is a smooth algebraic curve of genus g. Such a covering is called **simple** if every ramification point of f has index equal to 2 and no two of them lie over the same point of $\mathbb{P}^1$. In other words, any point in $\mathbb{P}^1$ has atleast $d-1$ points in the pre-image under f.

A simple ramified covering of type (d, g) will be a d sheeted simple cover $f : C \to \mathbb{P}^1$ branched over $2g+2d-2$ points in $\mathbb{P}^1$, along with a numbering of the branch points.

The Hurwitz space $Hur_{d,g}$ parametrizes isomorphism classes of simple ramified coverings of type (d, g). $Hur_{d,g}$ is a complex and in fact a smooth algebraic variety with a finite étale morphism to $\phi : Hur_{d,g} \to M_{0,2g+2d-2}$.

There is also a morphism $\psi : Hur_{d,g} \to M_g$.

When $d = 2$, the image of ψ is the hyperelliptic locus and $\phi : Hur_{d,g} \to \overline{M}_{0,2g+2}$ is an isomorphism. Thus there is a map $\overline{M}_{0,2g+2} \to H_g$ which is just the quotient map under the action of S_{2g+2}.

4.3. Admissible covers and a compactification of $Hur_{d,g}$.

Harris and Mumford compactified the Hurwitz space by generalizing ramified covers of $\mathbb{P}^1$. They allow the base now to be a stable genus zero curve.

Let B be a stable $2g + 2d - 2$ pointed genus 0 curve. A d sheeted genus g admissible cover is a map

$$f : C \to B$$

where C is a nodal curve such that

(1) $f^{-1}(B^{\mathrm{sm}}) = C^{\mathrm{sm}}$, and restriction of π to the open set of smooth points is simply branched over the marked points of B and otherwise unramified.
(2) $f^{-1}(B^{\mathrm{sing}}) = C^{\mathrm{sing}}$ and for every node q of B and every node r of C lying over q, the two branches over r map to the branches near q with the same ramification index.

Figure

The curve C may not be stable but it will have arithmetic genus g. Stabilization of C will be a genus g stable curve.

There is a course moduli space $\overline{Hur}_{d,g}$ for admissible covers which is a complete variety and contains $Hur_{d,g}$ as a dense open subvariety, extending the maps ϕ and ψ.

$$\begin{tikzcd}
 & Hur_{d,g} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] \arrow[dd, hook] & \\
M_{0,2g+2d-2} \arrow[dd, hook] & & M_g \arrow[dd, hook] \\
 & \overline{Hur}_{d,g} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
\overline{M}_{0,2g+2d-2} & & \overline{M}_g
\end{tikzcd}$$

Again when $d = 2$, ϕ is an isomorphism and ψ maps to $\overline{H}_g$. hence the isomorphism $\overline{M}_{0,2g+2}/S_{2g+2} \cong \overline{H}_g$. Let $\pi : \overline{M}_{0,2g+2} \to \overline{H}_g$ be the quotient map.

5. Results

Induced by the filtration on $\overline{M}_g$ we have a filtration on $\overline{H}_g$.

$$\overline{H}_g^{\leqslant k} = \overline{H}_g \cap \overline{M}_g^{\leqslant k}$$

We show Looijenga's bounds hold for these varieties.

Theorem 2 (C)**.**

$$\operatorname{asn} \overline{H}_g^{\leqslant k} \leq g - 1 + k \quad \textit{for all } g, k$$

This provides evidence towards the actual conjecture of Roth and Vakil. The effectiveness of the bound is not known, but when $k = 0$, we show

Theorem 3 (C)**.**

$$\operatorname{ce}(\overline{H}_g^{\leqslant 0}) \geq g - 1 \quad \textit{for} \quad g \geq 2$$

This proves

(1) $\operatorname{asn} \overline{H}_g^{\leqslant 0} = \operatorname{ce}(\overline{H}_g^{\leqslant 0}) = g - 1$.
(2) $\operatorname{ce}(\overline{M}_g^{\leqslant 0}) \geq g - 1$.

The second result follows since $\overline{H}_g^{\leqslant 0}$ is a closed subvariety of $\overline{M}_g^{\leqslant 0}$.

First we reduce these questions to subvarieties of $\overline{M}_{0,2g+2}$. Consider

$$\overline{M}_{0,2g+2}^{(k)} = \pi^{-1} \overline{H}_g^{\leqslant k}$$

To determine which stable genus zero curves are in $\overline{M}_{0,2g+2}^{(k)}$, we first determine the dual graph of $\pi(C)$ form the dual graph of C. For that we introduce a coloring of the stable graphs of type $(0, 2g+2)$. Figure

The ramification number of a vertex will be the number of odd edges and legs of the vertex. Now given a dual graph of a stable curve of type $(0, 2g+2)$ we can get the dual graph of the corresponding hyperelliptic curve as follows:

(1) To each vertex of rafimication number $\rho \geq 2$ or higher associate a vertex of genus $(\rho - 2)/2$.
(2) To each vertex of ramification number 0 associate two vertices of genus 0.
(3) To each odd edge associate 1 edge and to each even edge associate 2 edges connecting the appropriate vertices.

Figure

We see that genus zero vertices come from vertices with ramification number 0 or internal vertices with ramification number 2.

We call a stable graph good if it does not have vertices with ramification number 0 or internal vertices with ramification number 2. Hence we have

$$\overline{M}_{0,2g+2}^{(0)} = \bigsqcup_{G \in \Gamma(0,2g+2),\ G \text{ good}} M_G$$

We get an affine stratification of length $g-1$ of $\overline{M}^{(0)}_{0,2g+2}$, by noting that M_G is affine for the genus zero graphs and a good graph of type (g,n) may have at most $g-1$ edges.

5.1. **Lower Bound.** Consider the constant sheaf $\underline{\mathbb{C}}$ on $\overline{M}^{(0)}_{0,2g+2}$, and let $\mathcal{L} = \pi_* \underline{\mathbb{C}}$. Then $\mathcal{L}$ is a constructible sheaf on $\overline{H}^{\leqslant 0}_g$. Note that

$$H^i(\overline{H}^{\leqslant 0}_g, \mathcal{L}) \cong H^i(\overline{M}^{(0)}_{0,2g+2}, \underline{\mathbb{C}})$$

Lemma 4. *$H^{3g-2}(\overline{M}^{(0)}_{0,2g+2}, \mathbb{C})$ is non-trivial and has a pure Hodge structure of weight $4g-2$.*

5.2. **A Spectral Sequence.** Consider a compact complex manifold X with a simple normal crossings divisor $D = D_1 \cup \cdots \cup D_N$. Then there is a filtration on X as follows

$$X^0$$

Let X be a smooth projective variety over $\mathbb{C}$ of dimension n, and D, a simple normal crossings divisor. By that we mean $D = D_1 \cup \ldots \cup D_N$, where each D_i is a co-dimension 1 smooth sub-variety and all intersections of D_i are transverse. Let

$$X = X_0 \supset X_1 \supset \ldots \supset X_n \supset X_{n+1} = \emptyset$$

be the following filtration on X: $X_1 = D$, and

$$X_k = \bigcup_{|I|=k \ I \subset \{1,\ldots,N\}} \bigcap_{i \in I} D_i$$

Let $X_k^\circ = X_k \setminus X_{k+1}$. Then we have $H^\bullet(X_k, X_{k+1}) \cong H^\bullet_c(X_k^\circ)$, where $H^\bullet$ denotes cohomology and $H^\bullet_c$ compactly supported cohomology with complex coefficients.

Consider the spectral sequence associated to this filtration on X. We have

$$E_1^{p,q} = H^{p+q}(X_{-p}, X_{-p+1}) = H_c^{p+q}(X^\circ_{-p})$$

and the differential d_1 is given by the composition of maps

$$\begin{tikzcd}
E_1^{p,q} \arrow[rr, "d_1"] \arrow[d, equal] & & E_1^{p+1,q} \arrow[d, equal] \\
H^{p+q}(X_{-p}, X_{-p+1}) \arrow[r, "\imath"] & H^{p+q}(X_{-p}) \arrow[r, "\delta"] & H^{p+q+1}(X_{-p-1}, X_{-p})
\end{tikzcd}$$

where $\imath$ and δ are the maps in the long exact sequence of a pair $W \subset Z$ as follows

$$\cdots \to H^{l-1}(W) \xrightarrow{\delta} H^l(Z, W) \xrightarrow{\imath} H^l(Z) \to \cdots$$

Since the filtration is finite the spectral sequence converges to $H^{p+q}(X)$. Moreover the vector spaces $E_1^{p,q}$ carry mixed Hodge structures and the differential is a map of mixed Hodge structures. This spectral sequence is "dual" to the spectral sequence of Deligne for mixed Hodge theory of smooth quasi-projective varieties. The spectral sequence converges in the E_2 page and $E_2^{p,q} = E_\infty^{p,q}$.

Now let $X = \overline{M}_{0,m}$ and $D = \overline{M}_{0,m} \setminus M_{0,m}$; then

$$X_k = \bigcup_{[T] \in \Gamma_k(0,m)} \overline{M}_T \quad \text{and} \quad X_k^\circ = \bigsqcup_{[T] \in \Gamma_k(0,m)} M_T.$$

Hence as above we have a spectral sequence in the category of mixed Hodge structures, with

$${}_mE_1^{p,q} = \bigoplus_{[T] \in \Gamma_{-p}(0,m)} H_c^{p+q}(M_T)$$

This spectral sequence tells us how to compute $H^\bullet(\overline{M}_{0,m})$ from the knowledge of $H^\bullet(M_{0,l})$ for all $l \leq m$.

5.3. **Truncation.** Similarly we have a spectral sequence for $\overline{M}^{(0)}_{0,2g+2}$.

$$ {}_gF_1^{p,q} = \bigoplus_{[T]\in\Gamma_{-p}(0,2g+2)^0} H_c^{p+q}(M_T) $$

which we call a truncated spectral sequence for the lack of a better term.

The support of this spectral sequence is shown below. Figure

It thus turns out that

$$ H_c^g(\overline{M}^{(0)}_{0,2g+2}) \cong {}_gF_\infty^{-g+1,2g-1} \cong {}_gF_2^{-g+1,2g-1} $$

And by Poincaré duality out that $H^{3g-2}(\overline{M}^{(0)}_{0,2g+2}) \cong H_c^g(\overline{M}^{(0)}_{0,2g+2})^\vee$.

Now the proof involves showing that there is a non-zero class coming from ${}_{2g+2}E_1^{-g,2g-1}$ in the larger spectral sequence into ${}_gF_1^{-g+1,2g-1}$. This class has to e in the kernel of the differential and hence ${}_gF_2^{-g+1,2g-1} \neq 0$.