Higher Topological Jacobi Form Computations

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DECLARATION

I hereby declare that I am the sole author of this thesis in partial fulfillment of the requirements for a postgraduate degree from National Institute of Science Education and Research (NISER), Homi Bhabha National Insitute (HBNI), Mumbai. I authorise NISER and HBNI to lend this thesis to other institutions or individuals for the purpose of scholarly research.

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The thesis work reported in the thesis entitled .... was carried out under my supervision, in the school of .... at NISER, Bhubaneswar, India.

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Date: 8 May 2026

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ABSTRACT

The goal of the thesis is to give an overview of Gukov-Krushkal-Meier-Pei's (GKMP) notion of higher Topological Jacobi forms and investigate properties of the rank 2 case, including some explicit computations. The GKMP construction associates to each symmetric bilinear form b a TMF-module TJF^b, called a higher topological Jacobi form. These modules play a central role in the conjectural $(3+1)$-dimensional TQFT valued in TMF-modules. For a closed 3-manifold M bounding a simply-connected 4-manifold W with intersection form b, the invariant $\mathscr{Z}(M)$ is given (up to shift) by TJF^b, and depends only on the torsion linking form of M.

Here, we give a classification of TJF^b for rank 2 bilinear forms with $|\det b| \leq 4$. We compute the linking forms explicitly for each isomorphism class and determine the stable equivalence classes via Nikulin's theorem. Our main computation is for the non-diagonal form $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, which is not stably equivalent to any diagonal form. Using a stabilisation argument, we get

$$\mathrm{TJF}^{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \simeq \mathrm{TMF}[-7] \otimes_{\mathrm{TMF}} \left(\mathrm{TJF}^{(3)}[-1]\right)^{\vee}.$$

This is the first explicit computation of TJF^b for a form that is neither diagonal nor stably equivalent to a diagonal form.

We, then, interpret our classification in the context of abelian Chern–Simons theories, where TJF^b is conjecturally the space of boundary conditions. We show that TJF^b distinguishes linking forms with the same discriminant group (e.g., $(\mathbb{Z}/3, \frac{1}{3})$ versus $(\mathbb{Z}/3, \frac{2}{3})$), and that the pointed element $\mathfrak{d}_b$ distinguishes H_{even} from H_{odd}.

Our results provide the first systematic computation of TJF^b beyond diagonal forms, and offer concrete predictions for the GKMP TQFT.

Most, if not all of this is based on upcoming work of the author [4].

Contents

Chapter 1

Introduction

From manifolds to modular forms

The study of smooth manifolds has always been enriched by insights from quantum field theory. Since Donaldson's revolutionary work in the early 1980s, gauge-theoretic invariants have given us mathematicians, some of the most powerful tools for distinguishing smooth structures on four-dimensional manifolds. The Seiberg–Witten invariants, Floer homology theories, and their various generalisations have led to deep results that remain inaccessible to purely topological methods. Yet, despite these successes, a fundamental question persists: can four-dimensional topological quantum field theories (TQFTs) detect exotic smooth structures? Classical TQFTs, valued in vector spaces over $\mathbb{C}$, face an obstruction: if two closed oriented four-manifolds are homeomorphic, they become diffeomorphic after taking connected sum with sufficiently many copies of $S^2 \times S^2$. A nontrivial TQFT would then force the invariants of homeomorphic manifolds to coincide, rendering the theory blind to exotic smoothness.

This thesis is concerned with a recent proposal that proposes to circumvent this obstruction. The key idea, due to Gukov, Pei, Putrov, and Vafa [11], is to replace the usual ground ring $\mathbb{C}$ with the E_∞-ring spectrum TMF of topological modular forms. In this new framework, the invariant of $S^2 \times S^2$ is not an invertible unit but rather the element $\eta \in \pi_1\mathrm{TMF}$, which satisfies $2\eta = 0$ and $\eta^2 = 0$. This nilpotence property opens the door to distinguishing exotic smooth structures, because multiplying by a nilpotent element can lose information, potentially allowing homeomorphic manifolds to be distinguished after stabilisation.

The proposal of Gukov, Pei, Putrov, and Vafa [11] starts from six-dimensional superconformal field theories (SCFTs); compactifying such a theory on a four-manifold yields a two-dimensional theory whose partition function is valued in the homotopy groups of TMF. In this thesis, we focus on the simplest such six-dimensional theory, the free abelian tensor multiplet, and construct explicit invariants of three- and four-manifolds valued in TMF-modules. Our main tool is the notion of a *higher topological Jacobi form* TJF^b associated to a symmetric bilinear form b, introduced by Gukov, Krushkal, Meier, and Pei (GKMP)[7]. These modules are the building blocks of the conjectural TQFT, and understanding their structure for low-rank bilinear forms is essential for testing the theory and for making concrete predictions.

What is a higher topological Jacobi form?

Classical Jacobi forms are holomorphic functions of two variables, τ in the upper half-plane and $z \in \mathbb{C}^d$, that transform like modular forms in τ and like theta functions in z. They arise naturally in many areas of mathematics: as partition functions of lattice conformal field theories[15], as generating functions for the cohomology of moduli spaces [10], and as the Fourier coefficients of Siegel modular forms [5]. For a positive definite even unimodular bilinear form b, the theta function

$$\theta_b(\tau, z) = \sum_{v \in \mathbb{Z}^d} e^{\pi i \tau b(v,v) + 2\pi i b(z,v)}$$

is a Jacobi form of weight $d/2$ and index b.

The theory of topological modular forms refines classical modular forms by encoding torsion information and by providing a home for the Witten genus of a string manifold. Just as modular forms are sections of line bundles on the moduli stack of elliptic curves $\mathcal{M}_{\mathrm{ell}}$ and the spectrum TMF is the ring of global sections of a sheaf of E_∞-ring spectra on $\mathcal{M}_{\mathrm{ell}}$. The passage from classical to topological modular forms replaces ordinary commutative rings with highly structured ring spectra, capturing stable homotopy-theoretic information that is invisible at the level of ordinary cohomology.

Higher topological Jacobi forms are the natural generalisation of Jacobi forms to this spectral setting. To a symmetric bilinear form $b : \mathbb{Z}^d \otimes \mathbb{Z}^d \to \mathbb{Z}$, GKMP associate a derived Looijenga line bundle $\mathcal{L}_b$ on the d-fold power of the universal elliptic curve $\mathcal{E}^{\times_{\mathcal{M}} d}$. The TMF-module TJF^b is then defined as the global sections of the pushforward of $\mathcal{L}_b$ to $\mathcal{M}$. This construction satisfies three fundamental properties: additivity (direct sums of bilinear forms correspond to tensor products of modules), functoriality (morphisms of lattices induce pullbacks of modules), and a normalisation condition that ties $\mathrm{TJF}^{(1)}$ to the sheaf $\mathcal{O}_{\mathcal{E}}(e)$ on the universal elliptic curve.

The role of linking forms and stable equivalence

A key insight of GKMP is that TJF^b depends only on the stable equivalence class of b. Two bilinear forms are stably equivalent if they become isomorphic after adding copies of the hyperbolic forms $\langle 1\rangle$ and $\langle -1\rangle$. By a classical theorem of Nikulin, the stable equivalence class is completely determined by three invariants: the signature, the rank, and the isomorphism class of the linking form (A_b, λ_b) on the discriminant group $A_b = \mathbb{Z}^d / B\mathbb{Z}^d$. The linking form is a nondegenerate symmetric bilinear form with values in $\mathbb{Q}/\mathbb{Z}$, and it encodes the torsion information of b.

For three-manifolds, the linking form arises naturally as an invariant of the torsion part of the first homology. If M bounds a simply-connected four-manifold W with intersection form b, then the linking form of M is isomorphic to the linking form of b (see, e.g., [6, Section 2]). This provides the bridge between the algebraic world of bilinear forms and the geometric world of three-manifolds: the GKMP invariant $\mathcal{Z}(M)$ is defined (up to a degree shift) as $\mathrm{TJF}^{b(W)}$, and it depends only on the linking form of M and the signature of W.

What the reader can expect in this thesis

The primary goal of this thesis is to compute TJF^b explicitly for all rank 2 bilinear forms with $|\det b| \leq 4$, and to interpret these computations in the context of the conjectural TQFT. This is the first systematic computation of higher topological Jacobi forms beyond the diagonal case, and it reveals a rich interplay between the arithmetic of bilinear forms and the homotopy theory of TMF-modules.

Chapter 2 is devoted to reviewing the theory of topological Jacobi forms as developed by Bauer and Meier. We recall the definition of Jacobi forms from a complex-analytic perspective, introduce the moduli stack of elliptic curves, and explain how Jacobi forms arise as sections of Looijenga line bundles. We then discuss the derived enhancement that leads to the TMF-module spectra TJF_m, and we state the fundamental theorem of Bauer–Meier that identifies TJF_m with $\mathrm{TMF} \wedge P^m$ for a certain cofibre spectrum P^m.

Chapter 3 dives into the GKMP construction of higher topological Jacobi forms TJF^b for arbitrary symmetric bilinear forms. We define the derived Looijenga line bundles $\mathcal{L}_b$ via their additivity, functoriality, and normalisation properties, and we prove that these properties determine them uniquely up to isomorphism. We then introduce the categories of lattices and bilinear forms, and we show how natural transformations from $\mathbb{Z}[(-)^\vee]$ to the Picard functor give a systematic way to construct $\mathcal{L}_b$. This chapter also establishes the basic properties of TJF^b, including additivity under direct sums and invariance under stable equivalence.

Chapter 4 is devoted to linking forms and their classification. We define the discriminant group A_b and the linking form λ_b, and we prove that λ_b is well-defined, symmetric, bilinear, and nondegenerate. We then compute λ_b explicitly for all rank 2 forms with $|\det| \leq 4$, including the subtle non-diagonal case $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ whose linking form is $2/3$ on $\mathbb{Z}/3$. We discuss isometries of linking forms, proving that $1/3$ and $2/3$ are not isometric, and we state Nikulin's theorem, which classifies integral bilinear forms up to stable equivalence by signature, rank, and linking form.

Chapter 5 gives a complete isomorphism classification of rank 2 bilinear forms with $|\det| \leq 4$. Using the theory of reduced binary quadratic forms, we enumerate all positive definite, indefinite, and negative definite forms in this range. We compute their determinants, signatures, and parity, and we identify the distinct isomorphism classes. The main result is a list of five stable equivalence classes for signature $(2,0)$: $\langle 1\rangle \oplus \langle 1\rangle$, $\langle 1\rangle \oplus \langle 2\rangle$, $\langle 1\rangle \oplus \langle 3\rangle$, $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\langle 2\rangle \oplus \langle 2\rangle$, and $\langle 1\rangle \oplus \langle 4\rangle$, together with their linking forms.

Chapter 6 connects linking forms to TJF^b. We prove the unimodular case, where $\mathrm{TJF}^b \simeq \mathrm{TMF}[3b^- - 2b^+]$, and we provide a detailed proof of Proposition 5.7 from GKMP, including the reduction to the diagonal case, the pushforward computation for rank-one forms, and the application of Grothendieck duality. We then list the basic modules $\mathrm{TJF}^{(k)}$ for small k, as given by Bauer–Meier, and we state the duality theorem relating TJF^b and $\overline{\mathrm{TJF}}_b$.

Chapter 7 establishes the relationship between stable equivalence of bilinear forms and the geometry of three-manifolds. We show that every closed oriented three-manifold M bounds a simply-connected four-manifold W, and that the intersection form $b(W)$ is well-defined up to stable equivalence. We then prove that the invariant $\mathcal{Z}(M) := \overline{\mathrm{TJF}}_{b(W)}[3b^+ - 2b^-]$ is independent of the choice of W; the shift cancels the effect of stabilisation. This construction avoids the need for a localised category and works directly with the stable equivalence relation. We also discuss the open question of whether $\mathcal{Z}$ extends to a functor on the bordism category (see

Chapter 8 contains the core computations of the thesis. Using additivity, the basic modules from Chapter 5, and the stable equivalences established in Chapter 4, we compute TJF^b for each stable equivalence class. The diagonal forms are handled by tensoring the appropriate rank-one modules. The non-diagonal form $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ requires a stabilisation trick: we stabilise it to a diagonal form of higher rank, apply additivity and duality, and then destabilise. The result is

$$\mathrm{TJF}^{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \simeq \mathrm{TMF}[-7] \otimes \big(\mathrm{TJF}^{(3)}[-1]\big)^{\vee},$$

which is the first explicit computation of TJF^b for a form that is neither diagonal nor stably equivalent to a diagonal form. We also compute the modules for $\langle 2\rangle \oplus \langle 2\rangle$ and $\langle 1\rangle \oplus \langle 4\rangle$, revealing the difference between split and cyclic linking forms.

Chapter 9 interprets our computations in the context of abelian Chern–Simons theories. We recall the definition of the Chern–Simons action, the quantisation of the theory on the torus, and the resulting modular tensor category $\mathcal{C}_b$ with simple objects labeled by A_b and modular S and T matrices determined by λ_b. We then discuss the Stolz–Teichner conjecture, which identifies TMF with the classifying space of $(0,1)$-supersymmetric quantum field theories, and its generalisation Conjecture A, which asserts that the space of boundary conditions of a three-dimensional theory $\mathfrak{T}$ forms a TMF-module. For abelian Chern–Simons theory CS_b, the conjectural boundary module is $\overline{\mathrm{TJF}}_b$. Our computations therefore give explicit predictions for the spaces of boundary conditions of $U(1)^r$ Chern–Simons theories at low levels. We show that TJF^b distinguishes linking forms with the same discriminant group, that the pointed element $\mathfrak{d}_b$ distinguishes H_{even} from H_{odd}, and we derive concrete formulas for the three-manifold invariants $\mathcal{Z}(M)$ in terms of linking form data, for example

$$\mathcal{Z}(M) \simeq \mathrm{TMF}[-13] \otimes (\mathrm{TJF}^{(3)}[-1])^{\vee}$$

for any three-manifold whose linking form is $(\mathbb{Z}/3, 2/3)$.

What is new and what remains to be done

The main novelty here is the systematic computation of TJF^b for all rank 2 forms with $|\det| \leq 4$, including the first example of a module coming from a non-diagonal, non-stably-diagonal bilinear form. Our results demonstrate that TJF^b encodes the full isomorphism class of the linking form, not just its order, and that the pointed element provides additional distinguishing power. These computations serve as a testing ground for the conjectural TQFT of GKMP and provide concrete predictions for three-manifold invariants. This is based on upcoming work of the author [4].

Many questions remain open. The computations for higher rank and larger determinant are still largely unknown, though our methods suggest that the key primes are 2 and 3, where $\pi_*\mathrm{TMF}$ has torsion. The relationship between TJF^b and the canonical state space formula of Li deserves further exploration. Most ambitiously, the conjectural TQFT itself awaits a rigorous construction; our work provides essential data that any such construction must reproduce.The nilpotence of $\mathscr{Z}(S^2 \times S^2) = \eta$ suggests that, unlike traditional $\mathbb{C}$-valued TQFTs, this framework does not force $\mathscr{Z}(X_1) = \mathscr{Z}(X_2)$ for homeomorphic 4-manifolds X_1, X_2 that become diffeomorphic after stabilisation by $S^2 \times S^2$. However, the invariant $\mathscr{Z}(X)$ defined in [7] depends only on the intersection form of X, which is a homeomorphism invariant, so it cannot directly distinguish exotic smooth structures. Nevertheless, the nilpotence phenomenon suggests that more refined invariants, perhaps coming from higher-dimensional theories or from considering manifolds with boundary, might eventually detect exotic smoothness. This remains an open direction for future work.

Prerequisites

This thesis is readable to anyone with background in algebraic topology, manifold theory, and homological algebra. No prior knowledge of topological modular forms or Jacobi forms is assumed; all necessary background is developed in the text. The physical motivations are explained heuristically, and the conjectural nature of the Stolz–Teichner program is clearly indicated. The reader who is primarily interested in the algebraic and topological aspects may safely skip the physical interpretations, though they provide valuable context and motivation.

Chapter 2

Topological Jacobi Forms

2.0.1 Motivation

We start by a few words of motivation. Lifting the Ochanine and Witten genera to maps of spectra was a motivation for defining elliptic cohomology and topological modular forms. The map MString → TMF, for instance (in [1]), has been very important in manifold topology.

Lin-Yamashita in [20] showed that the 2-variable elliptical genus lifts to the map $\mathrm{MTSU}_m \to \mathrm{TJF}_m$. In Bauer-Meier [3], they define this spectrum of TJF_m of *topological Jacobi forms* for every index m.

Besides this, there are motivations to work with TJF that come from physics. It has been interpreted as twisted S^1-equivariant TMF, and conjecturally classifies forms of anomalous supersymmetric quantum field theories with $U(1)$-symmetry.

2.0.2 Jacobi Forms from a Complex Analytic Lens

Notation 2.1. *Let's start by defining* $\mathcal{H}$ *as the upper half plane. We use the notation* $e(x) = e^{2\pi i x}$ *for* $x \in \mathbb{C}$*,* $q = e(\tau)$ *for* $\tau \in \mathcal{H}$ *and* $\zeta^r = e(rz)$ *for* $z \in \mathbb{C}$ *and* $r \in \mathbb{R}$*.*

We now define our basic complex-analytic and algebro-geometric objects of interest.

Definition 2.2. *A weakly holomorphic Jacobi form of weight* $k \in \mathbb{Z}$ *and index* $m \in (1/2)\mathbb{Z}$ *is a holomorphic function*

$$\Phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$$

that satisfies the following transformation properties:

$$\Phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e\left(\frac{mcz^2}{c\tau+d}\right)\Phi(\tau, z) \tag{2.3}$$

$$\Phi(\tau, z+\lambda\tau+\mu) = e(m(-\lambda^2\tau - 2\lambda z + \lambda + \mu))\Phi(\tau, z) \tag{2.4}$$

and such that there exists a Fourier expansion:

$$\Phi(\tau, z) = \sum_{n\geq -N}\sum_{r\in\mathbb{Z}+m} c(n,r)q^n\zeta^r \textit{ for some } c(n,r)\in\mathbb{C}$$

Here, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ *and* $(\lambda, \mu) \in \mathbb{Z}^2$.

A weakly holomorphic Jacobi form is called a *weak Jacobi form* if $c(n,r) = 0$ for $n < 0$, and a *(holomorphic) Jacobi form* if additionally, the following holds:

$$c(n,r) = 0 \text{ if } r^2 > 4mn.$$

Now, analogous to the case of modular forms we start with the uncompactified moduli stack of elliptic curves $\mathcal{M}_{\mathrm{ell}}$.

2.0.3 The Moduli Stack of Elliptic Curves

First, what follows is a brief introduction to $\mathcal{M}_{\mathrm{ell}}$ as an object, based on [19].

Definition of an elliptic curve

Definition 2.5. *An elliptic curve over* $\mathbb{C}$ *is one of the following equivalent objects:*

- *A pair* (X, O) *where* X *is a proper smooth curve over* $\mathbb{C}$ *which is connected and of genus* 1*, which means* $\dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) = 1$*, with* $O \in X(\mathbb{C})$ *a distinguished point.*
- *A Riemann surface* X *of genus* 1 *with a fixed point* $P \in X$*.*
- *A quotient* $\mathbb{C}/\Lambda$ *where* Λ *is a lattice in* $\mathbb{C}$*.*

Framed elliptic curves

Definition 2.6. *A framed elliptic curve is an elliptic curve equipped with an ordered basis, say* $\{a_1, a_2\}$*, of the homology of the curve such that the intersection number* $a_1 \cdot a_2$ *is* 1*.*

We say that a lattice Λ in $\mathbb{C}$ is *framed* if it has an ordered basis (λ_1, λ_2) such that $\mathrm{Im}(\lambda_2/\lambda_1) > 1$.

Remark 2.7. *We thus have the following bijections:*

$$\mathcal{H} \simeq \{\textit{framed lattices in } \mathbb{C}\} \simeq \{\textit{framed elliptic curves over} \mathbb{C}\}/_{\sim}$$

and

$$\{\textit{elliptic curves over} \mathbb{C}\}/_{\sim} \simeq \mathcal{H}/\mathrm{SL}(2,\mathbb{Z})$$

where the action of the modular group $\mathrm{SL}(2,\mathbb{Z})$ *is via Möbius transformations* $\tau \mapsto \frac{a\tau+b}{c\tau+d}$*.*

We define the moduli space of elliptic curves by this quotient.

Definition 2.8.

$$M_{1,1} := \mathcal{H}/\mathrm{SL}(2,\mathbb{Z})$$

Orbifolds

Let's recall what (global) orbifolds are.

Definition 2.9. *A (global) pointed orbifold is a triple* (X, G, ρ) *(also denoted by* $X//G$*), where* X *is a nice (which in our case is connected and simply connected) topological space,* G *is a discrete group, and* $\rho : G \to \mathrm{Aut}(X)$ *is a group homomorphism.*

Morphisms between orbifolds (X, G, ρ) and (X', G', ρ') are pairs (f, ψ), where $f : X \to X'$ is a continuous map and $\psi : G \to G'$ is a group homomorphism such that for all $g \in G$, the following commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & X' \arrow[d, "\psi(g)"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}$$

We now define $\mathcal{M}_{\mathrm{ell}}$ as an orbifold.

Definition 2.10. *We define the moduli stack of elliptic curves as the orbifold:*

$$\mathcal{M}_{1,1} = \mathcal{M}_{\mathrm{ell}} = \mathcal{H}//\mathrm{SL}(2,\mathbb{Z})$$

where the action is the Möbius action.

Remark 2.11. *It is important to note that when we say that E is an elliptic curve over some scheme S, we mean a morphism $E \to S$ which is finitely presented, proper and flat, together with a section $O \in E(S)$, such that the geometric fibres are elliptic curves in the sense of our classical definition.*

Remark 2.12. *In the last remark, we are being slightly imprecise because we have only defined elliptic curves over $\mathbb{C}$, not over an arbitrary algebraically closed field. Here is the definition for an arbitrary algebraically closed field.*

Definition 2.13. *An elliptic curve over an algebraically closed field k is a pair (E, O) where E is a proper, smooth, connected curve over k of arithmetic genus 1, and $O \in E(k)$ is a specified point.*

Here, arithmetic genus is $\dim_k H^1(E, \mathcal{O}_E)$.

Over $\mathcal{M}_{\mathrm{ell}}$, the universal elliptic curve $\mathcal{E} \xrightarrow{p} \mathcal{M}_{\mathrm{ell}}$ is defined. The fibre of the universal elliptic curve over a point $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{ell}}$ is a classical elliptic curve over R.

2.0.4 The Integral Version of $\mathcal{M}_{\mathrm{ell}}$

We now switch to the integral version of $\mathcal{M}_{\mathrm{ell}}$, which needs a bit of work.

Over a commutative ring R, we define an elliptic curve via the Weierstraß equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \quad \text{with } a_i \in R.$$

For smoothness we need the discriminant to be invertible in R.

Let

$$\begin{aligned} b_2 &= a_1^2 + 4a_2, \\ b_4 &= 2a_4 + a_1a_3, \\ b_6 &= a_3^2 + 4a_6, \\ b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2. \end{aligned}$$

Therefore, we have

$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$$

and the j-invariant is

$$j = \frac{(b_2^2 - 24b_4)^3}{\Delta}.$$

Consider the polynomial ring $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ representing all Weierstraß equations. To impose smoothness we localise:

$$B = A[\Delta^{-1}] = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}].$$

Now $\operatorname{Spec} B$ represents smooth Weierstraß curves.

We define $\mathcal{M}_{1,1}$ as

$$\mathcal{M}_{1,1} := \operatorname{Spec} B//(\mathbb{G}_a^3 \rtimes \mathbb{G}_m).$$

Here $\mathbb{G}_a$ and $\mathbb{G}_m$ are the additive and multiplicative group schemes, that act by translations and scaling. More specifically, two Weierstraß equations can be isomorphic via coordinate changes:

$$x = u^2x' + r, \qquad y = u^3y' + u^2sx' + t$$

where $u \in R^{\times}$ and $r, s, t \in R$.

2.0.5 Line Bundles and Jacobi Forms

Definition 2.14. *Suppose ω is the line bundle of invariant differentials on $\mathcal{M}_{\text{ell}}$. For $n, m \in \mathbb{Z}$, with n even, define a line bundle $L_{n,m}$ on $\mathcal{E}$ as follows:*

$$L_{n,m} := p^*\omega^{\frac{n}{2}} \otimes \mathcal{O}_{\mathcal{E}}(me),$$

where $\mathcal{O}_{\mathcal{E}}(me)$ denotes the sheaf of functions on $\mathcal{E}$ that have a pole of order at most m at the identity element $e : \mathcal{M}_{\text{ell}} \to \mathcal{E}$, and $\omega = e^*\Omega^1_{\mathcal{E}/\mathcal{M}_{\text{ell}}}$. By definition we set $L_{n,m} = 0$ for odd n.

Let us now denote the global sections as follows:

$$\text{JF}_{n,m} = H^0(\mathcal{E}, L_{n,m}).$$

Let us denote the vector space of weakly holomorphic Jacobi forms of index m and dimension n by $\text{JF}^{\mathbb{C}}_{n,2m}$.

Bauer–Meier in [3] prove the following, which identifies Jacobi forms as sections of line bundles, which motivates consideration of higher cohomology groups.

Theorem 2.15. *There is an isomorphism of bigraded rings*

$$\text{JF}^{\mathbb{C}}_{n,m} \to \text{JF}_{n,m} \otimes \mathbb{C}.$$

We define higher cohomology groups now.

Definition 2.16. *The ring of derived, weakly holomorphic Jacobi forms is a trigraded ring given by*

$$\text{DJF}_{n,s,m} := H^s(\mathcal{E}, L_{n,m}).$$

2.0.6 The Spectrum of Topological Jacobi Forms

Building on the work of Goerss–Hopkins–Miller, who construct a sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞-ring spectra on the moduli stack of elliptic curves $\mathcal{M}_{\mathrm{ell}}$, Lurie, in [18], constructs a sheaf of E_∞-spectra $\mathcal{O}^{\mathrm{top}}_{\mathcal{E}}$ on a given elliptic curve $\mathcal{E}$.

Now, the way to think about an $\mathbb{E}_\infty$-ring spectrum is as a homotopy-theoretic enhancement of a commutative ring. We can picture it as a space (or spectrum) where we can add or multiply points, with commutativity and associativity holding up to coherent higher homotopies, much like how a complex manifold has a sheaf of commutative rings, but in a derived context. The sheaf $\mathcal{O}^{\mathrm{top}}$ can be imagined as a machine that takes an elliptic curve and produces its derived ring of functions, which now is a highly structured ring spectrum. This is like a derived enhancement of the structure sheaf of $\mathcal{M}_{\mathrm{ell}}$: where a classical structure sheaf assigns commutative rings to our opens, $\mathcal{O}^{\mathrm{top}}$ assigns $\mathbb{E}_\infty$-rings. This structure then allows us to use stable homotopy-theoretic tools in the context of elliptic curves.

Bauer–Meier, in [3], construct sheaves L^{top}_m on $\mathcal{E}$.

Proposition 2.17. *For every $m \geq 0$ there exists a sheaf L^{top}_m of $\mathcal{O}^{\mathrm{top}}_{\mathcal{E}}$-module spectra on $\mathcal{E}$ such that*

$$\pi_n(L^{\mathrm{top}}_m) \cong L_{n,m}.$$

Using this we have the following definition.

Definition 2.18. *We define the following* TMF*-module spectrum of topological Jacobi forms as:*

$$\mathrm{TJF}_m := \Gamma_{\mathcal{E}}(L^{\mathrm{top}}_m).$$

An important property that Bauer–Meier prove in the case of topological Jacobi forms, which we wish to find higher analogues of, is the following.

Theorem 2.19. *There exists an equivalence of* TMF*-module spectra:*

$$\mathrm{TJF}_m \simeq \mathrm{TMF} \wedge \mathrm{cofib}(\Sigma\mathbb{C}P^{m-1} \to S^0) =: \mathrm{TMF} \wedge P^m.$$

Under this isomorphism the map $a : L_m^{\mathrm{top}} \to L_{m+1}^{\mathrm{top}}$ *corresponds to the map* $P^m \to P^{m+1}$ *coming from the skeletal inclusion* $\mathbb{C}P^{m-1} \to \mathbb{C}P^m$*.*

Chapter 3

Higher Topological Jacobi Forms

Complex Looijenga line bundles

In this part, we view $\mathcal{E}_{\mathbb{C}}$ as an orbifold $(\mathcal{H} \times \mathbb{C})//(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2)$, and similarly the fibre product $\mathcal{E}_{\mathbb{C}}^{\times_{\mathcal{M}} d}$ as the orbifold $(\mathcal{H} \times \mathbb{C}^d)//(\mathrm{SL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}^2)^d)$. Given an element $(\tau, z) \in \mathcal{H} \times \mathbb{C}^d$ and $X \in \mathrm{SL}(2,\mathbb{Z})$, and $(m_1, m_2) \in (\mathbb{Z}^2)^d$, the action is given by:

$$X \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^{-1} z \right), \tag{3.1}$$

$$(m_1, m_2) \cdot (\tau, z) = (\tau, z + m_1\tau + m_2). \tag{3.2}$$

Now following Gukov–Krushkal–Meier–Pei, we have the following definition.

Definition 3.3. *Suppose b is a symmetric integral bilinear form. By abuse of notation we also denote its $\mathbb{C}$-bilinear extension by b. The complex Looijenga line bundle $\mathcal{L}_b^{\mathbb{C}}$ on $\mathcal{E}_{\mathbb{C}}^{\times_{\mathcal{M}} d}$ is defined by defining an action of $\mathrm{SL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}^2)^d$ on $\mathcal{H} \times \mathbb{C}^d \times \mathbb{C}$ as follows (same notation as above):*

$$X \cdot (\tau, z, y) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^{-1} z, e^{\pi i (c (c\tau + d)^{-1} b(z,z))} y \right), \tag{3.4}$$

$$(m_1, m_2) \cdot (\tau, z, y) = \left(\tau, z + m_1\tau + m_2, e^{-2\pi i \left(b(z, m_1) + \frac{b(m_1, m_1)\tau}{2} \right)} y \right). \tag{3.5}$$

Now, in the situation when b is positive definite and even, the global sections f of $\mathcal{L}_b^{\mathbb{C}} \otimes \omega_{\mathbb{C}}^{\otimes k}$ are called *Jacobi forms of weight k and index b* if the pullback of f to a function on $\mathcal{H} \times \mathbb{C}^d$ can be written as a Fourier expansion:

$$f(\tau, z) = \sum_{n \geq 0} (e^{2\pi i \tau})^n \sum_{r \in \mathcal{L}^\vee;\, b(r,r) \leq 2n} c(n, r) e^{2\pi i b(z, r)},$$

where $\mathcal{L}^\vee = \{ z \in \mathbb{C}^d \mid b(z, m) \in \mathbb{Z}\ \forall m \in \mathbb{Z}^d \}$.

Axioms and properties

Using the equations in the definition of Looijenga line bundles we obtain the following axioms/properties:

- $\mathcal{L}^{\mathbb{C}}_{b+b'} \cong \mathcal{L}^{\mathbb{C}}_{b} \otimes \mathcal{L}^{\mathbb{C}}_{b'}$, with $\mathcal{L}^{\mathbb{C}}_{0}$ being the structure sheaf.
- A morphism $\mathbb{Z}^{d_1} \to \mathbb{Z}^{d_2}$ corresponds to a matrix A that induces a morphism $\mathcal{E}^{A} : \mathcal{E}_{\mathbb{C}}^{\times_{\mathcal{M}} d_1} \to \mathcal{E}_{\mathbb{C}}^{\times_{\mathcal{M}} d_2}$. Moreover, given an integral bilinear form b with associated matrix A', we can define the bilinear form A^*b with associated matrix $A^tA'A$. Thus we have an isomorphism $\mathcal{L}^{\mathbb{C}}_{A^*b} \cong (\mathcal{E}_A)^*\mathcal{L}^{\mathbb{C}}_{b}$.

Moreover, Bauer–Meier in [3] show that we have the isomorphism $\mathcal{L}^{\mathbb{C}}_{(1)} \cong \mathcal{O}^{\mathbb{C}}_{\mathcal{E}}(e) \otimes \omega_{\mathbb{C}}$. Here, (1) denotes the bilinear form $\mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ defined by $(v, v') \mapsto vv'$. Also, for further sections note that h denotes the hyperbolic bilinear form, represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

A characterisation result

We now prove an important characterisation result. We prove that every line bundle satisfying the formal properties that these Looijenga line bundles satisfy is, up to isomorphism, a Looijenga line bundle.

Proposition 3.6. *Suppose for every integral symmetric bilinear form b there is a line bundle $\mathcal{K}_b$ over $\mathcal{E}^d_{\mathcal{M}}$ that satisfies the above formal additivity and functoriality properties. Moreover, $\mathcal{K}_{(1)} \cong \mathcal{O}^{\mathbb{C}}_{\mathcal{E}} \otimes \omega_{\mathbb{C}}$. Then $\mathcal{L}^{\mathbb{C}}_{b} \cong \mathcal{K}_b$ for all integral symmetric bilinear forms b.*

Proof. We first claim that every integral symmetric bilinear form b can be written as a finite sum

$$b = \sum_i A_i^* b_i$$

where each b_i is one of the forms $\pm(1), \pm h$, and $A_i : \mathbb{Z}^d \to \mathbb{Z}^{\mathrm{rank}(b_i)}$ is a projection.

We can see this by the following argument. We first consider diagonal entries, say $B_{ii} = n$. Let $e_i \in \mathbb{Z}^d$ be the i-th standard basis vector. The form $b_i : (x, y) \mapsto n(\mathrm{pr}_i(x) \cdot \mathrm{pr}_i(y))$, where $\mathrm{pr}_i : \mathbb{Z}^d \to \mathbb{Z}$ is the projection, has matrix nE_{ii}. Now, this form equals $\mathrm{pr}_1^*(n \cdot (1))$. The form $(n \cdot (1))$ is the sum of $|n|$ copies of $\mathrm{sgn}(n) \cdot (1)$.

For an off-diagonal pair $B_{ij} = B_{ji} = n$ with $i < j$, suppose $\mathrm{pr}_{ij} : \mathbb{Z}^d \to \mathbb{Z}^2$ is the projection onto the i-th and the j-th coordinates. The form $b_{ij} : (x, y) \mapsto n\left(\mathrm{pr}_i(x)\mathrm{pr}_j(y) + \mathrm{pr}_j(x)\mathrm{pr}_i(y)\right)$ has matrix $n(E_{ij} + E_{ji})$. This form is clearly equal to $\mathrm{pr}_{ij}^*(n \cdot h)$, and the form $n \cdot h$ is the sum of $|n|$ copies of $\mathrm{sgn}(n) \cdot h$.

The sum of all these contributions from the diagonal entries and the off-diagonal pairs reconstructs our bilinear form b. By additivity and functoriality properties, it therefore suffices to prove the isomorphism $\mathcal{L}_b^{\mathbb{C}} \cong \mathcal{K}_b$ for $b \in \{(\pm 1), (\pm h)\}$.

The isomorphism for $b = (1)$ holds by the normalisation assumption. Now, consider the zero form 0 on $\mathbb{Z}$. By additivity, $(1) + (-1) = 0$ implies

$$\mathcal{L}_{(1)}^{\mathbb{C}} \otimes \mathcal{L}_{(-1)}^{\mathbb{C}} \cong \mathcal{L}_0^{\mathbb{C}} \cong \mathcal{O}$$

and

$$\mathcal{K}_{(1)} \otimes \mathcal{K}_{(-1)} \cong \mathcal{K}_0 \cong \mathcal{O}.$$

Since we have established that $\mathcal{L}_{(1)}^{\mathbb{C}} \cong \mathcal{K}_{(1)}$, we obtain

$$\mathcal{L}_{(-1)}^{\mathbb{C}} \cong (\mathcal{L}_{(1)}^{\mathbb{C}})^\vee \cong (\mathcal{K}_{(1)})^\vee \cong \mathcal{K}_{(-1)}.$$

An identical argument using $h + (-h) = 0$ shows that $\mathcal{L}_{-h}^{\mathbb{C}} \cong \mathcal{K}_{-h}$ follows from $\mathcal{L}_h^{\mathbb{C}} \cong \mathcal{K}_h$. Therefore, the proof has reduced to the case $b = h$.

Define the following bilinear forms

$$Q_1 := h \oplus (1) \quad \text{on } \mathbb{Z}^2 \oplus \mathbb{Z}, \tag{3.7}$$

$$Q_2 := (1) \oplus (-1) \oplus (1) \quad \text{on } \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}. \tag{3.8}$$

These forms are isomorphic. We can easily construct an explicit isomorphism via the matrix

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

that satisfies $P^t Q_1 P = Q_2$. Let $A : \mathbb{Z}^3 \to \mathbb{Z}^3$ be the linear automorphism defined by P. By functoriality the induced morphism $E_A : \mathcal{E}^3_{\mathcal{M}} \to \mathcal{E}^3_{\mathcal{M}}$ gives us isomorphisms

$$\mathcal{L}^{\mathbb{C}}_{Q_1} \cong E_A^* \mathcal{L}^{\mathbb{C}}_{Q_2}, \qquad \mathcal{K}_{Q_1} \cong E_A^* \mathcal{K}_{Q_2}.$$

We now use additivity. Let $p_{12} : \mathcal{E}^3_{\mathcal{M}} \to \mathcal{E}^2_{\mathcal{M}}$ and $p_3 : \mathcal{E}^3_{\mathcal{M}} \to \mathcal{E}^1_{\mathcal{M}}$ be the projections onto the first two and the third factors. Then:

$$\mathcal{L}^{\mathbb{C}}_{Q_1} \cong p_{12}^* \mathcal{L}^{\mathbb{C}}_h \otimes p_3^* \mathcal{L}^{\mathbb{C}}_{(1)}, \qquad \mathcal{K}_{Q_1} \cong p_{12}^* \mathcal{K}_h \otimes p_3^* \mathcal{K}_{(1)}.$$

Similarly, let $p_1, p_2, p_3 : \mathcal{E}^3_{\mathcal{M}} \to \mathcal{E}^1_{\mathcal{M}}$ be the three projections. Then:

$$\mathcal{L}^{\mathbb{C}}_{Q_2} \cong p_1^* \mathcal{L}^{\mathbb{C}}_{(1)} \otimes p_2^* \mathcal{L}^{\mathbb{C}}_{(-1)} \otimes p_3^* \mathcal{L}^{\mathbb{C}}_{(1)}, \qquad \mathcal{K}_{Q_2} \cong p_1^* \mathcal{K}_{(1)} \otimes p_2^* \mathcal{K}_{(-1)} \otimes p_3^* \mathcal{K}_{(1)}.$$

Therefore, we obtain

$$\mathcal{L}^{\mathbb{C}}_{Q_1} \cong E_A^* \left(p_1^* \mathcal{L}^{\mathbb{C}}_{(1)} \otimes p_2^* \mathcal{L}^{\mathbb{C}}_{(-1)} \otimes p_3^* \mathcal{L}^{\mathbb{C}}_{(1)} \right).$$

The map E_A acts as $(w_1, w_2, w_3) \mapsto (w_1 + w_3, w_2, w_1 - w_3)$. Hence we compute:

$$\begin{aligned} E_A^* p_1^* \mathcal{L}^{\mathbb{C}}_{(1)} &\cong (p_1 + p_3)^* \mathcal{L}^{\mathbb{C}}_{(1)} \otimes p_3^* \mathcal{L}^{\mathbb{C}}_{(1)}, \\ E_A^* p_2^* \mathcal{L}^{\mathbb{C}}_{(-1)} &\cong p_2^* \mathcal{L}^{\mathbb{C}}_{(-1)}, \\ E_A^* p_3^* \mathcal{L}^{\mathbb{C}}_{(1)} &\cong (p_1 - p_3)^* \mathcal{L}^{\mathbb{C}}_{(1)} \cong p_1^* \mathcal{L}^{\mathbb{C}}_{(1)} \otimes p_3^* \mathcal{L}^{\mathbb{C}}_{(-1)}. \end{aligned}$$

Substituting these, we get

$$\begin{aligned} \mathcal{L}^{\mathbb{C}}_{Q_1} &\cong \left[p_1^* \mathcal{L}^{\mathbb{C}}_{(1)} \otimes p_3^* \mathcal{L}^{\mathbb{C}}_{(1)} \right] \otimes p_2^* \mathcal{L}^{\mathbb{C}}_{(-1)} \otimes \left[p_1^* \mathcal{L}^{\mathbb{C}}_{(1)} \otimes p_3^* \mathcal{L}^{\mathbb{C}}_{(-1)} \right] \\ &\cong p_1^* (\mathcal{L}^{\mathbb{C}}_{(1)} \otimes \mathcal{L}^{\mathbb{C}}_{(1)}) \otimes p_2^* \mathcal{L}^{\mathbb{C}}_{(-1)} \otimes p_3^* (\mathcal{L}^{\mathbb{C}}_{(1)} \otimes \mathcal{L}^{\mathbb{C}}_{(-1)}) \\ &\cong p_1^* \mathcal{L}^{\mathbb{C}}_{(2)} \otimes p_2^* \mathcal{L}^{\mathbb{C}}_{(-1)}. \end{aligned}$$

Thus,

$$p_1^*\mathcal{L}^{\mathbb{C}}_{(2)} \otimes p_2^*\mathcal{L}^{\mathbb{C}}_{(-1)} \cong p_{12}^*\mathcal{L}^{\mathbb{C}}_h \otimes p_3^*\mathcal{L}^{\mathbb{C}}_{(1)}.$$

Similarly for the case of $\mathcal{K}_h$ and using $\mathcal{K}_{(\pm 1)} \cong \mathcal{L}^{\mathbb{C}}_{(\pm 1)}$, we obtain

$$p_1^*\mathcal{L}^{\mathbb{C}}_{(2)} \otimes p_2^*\mathcal{L}^{\mathbb{C}}_{(-1)} \cong p_{12}^*\mathcal{K}_h \otimes p_3^*\mathcal{L}^{\mathbb{C}}_{(1)}.$$

Therefore,

$$p_{12}^*\mathcal{K}_h \cong p_{12}^*\mathcal{L}^{\mathbb{C}}_h.$$

Let $s : \mathcal{E}^2_{\mathcal{M}} \to \mathcal{E}^3_{\mathcal{M}}$ be the section given by $z_3 = 0$. Applying s^* to both sides yields

$$\mathcal{L}^{\mathbb{C}}_h \cong \mathcal{K}_h,$$

as desired. □

3.0.1 Derived Looijenga Line Bundles

Definition 3.9. *A* derived line bundle *(on $\mathcal{E}^{\times_{\mathcal{M}} d}$) is an $\mathcal{O}^{\mathrm{top}}_{\mathcal{E}^{\times_{\mathcal{M}} d}}$-module that is étale-locally equivalent to $\mathcal{O}^{\mathrm{top}}_{\mathcal{E}^d}$.*

In this section we introduce Gukov–Krushkal–Meier–Pei's derived analogues of the Looijenga line bundles that we saw in the previous section. Also, it is important to note that taking π_0 of these derived analogues, which we will call $\mathcal{L}_b$, gives us non-derived Looijenga line bundles $\mathcal{L}^{\mathbb{Z}}_b$ on $\mathcal{E}^{\times_{\mathcal{M}} d}$, without the need to complexify.

So, to each integral symmetric bilinear form b let us associate a line bundle $\mathcal{L}^{\mathbb{Z}}_b$ and a derived line bundle $\mathcal{L}_b$ on $\mathcal{E}^{\times_{\mathcal{M}} d}$. We follow the following rules:

- $\mathcal{L}_{b+b'} \simeq \mathcal{L}_b \otimes \mathcal{L}_{b'}$ (and similarly for $\mathcal{L}^{\mathbb{Z}}_b$).
- A morphism $\mathbb{Z}^{d_1} \to \mathbb{Z}^{d_2}$ corresponds to a matrix A, which induces a morphism $\mathcal{E}^A : \mathcal{E}^{\times_{\mathcal{M}} d_1} \to \mathcal{E}^{\times_{\mathcal{M}} d_2}$. Moreover, given an integral symmetric bilinear form b with associated matrix A', we can define a bilinear form A^*b with matrix $A^tA'A$. Also, we require the equivalence $\mathcal{L}_{A^*b} \simeq (\mathcal{E}^A)^*\mathcal{L}_b$ (similarly for $\mathcal{L}^{\mathbb{Z}}_b$).

- $\mathcal{L}_{(1)} \simeq \mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e)[-2]$ and $\mathcal{L}_{(1)}^{\mathbb{Z}} \cong \mathcal{O}_{\mathcal{E}}(e) \otimes \omega$.

Analogous to the non-derived situation, we can prove that these three properties define derived and integral Looijenga line bundles up to isomorphism. But we want to make sure that these conditions do not overdetermine the (derived) line bundles.

Notation 3.10. *Let us denote the category of lattices, i.e., finitely generated free abelian groups, by* Lat, *bilinear forms over some lattice* Λ *by* $\mathrm{Bil}(\Lambda)$, *and the Picard group by* $\mathrm{Pic}(-)$.

Recall that the Picard group of a ringed space X is the group of isomorphism classes of invertible sheaves or line bundles on X, with tensor product as the group operation.

Here are two examples. We will show that the Picard group of the affine line with two origins is $\mathbb{Z}$. Before that, let us see that the Picard group of $\mathbb{P}^1$ is $\mathbb{Z}$.

Example 3.11. *For an integral scheme* X *we have the isomorphism* $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$, *where* $\mathrm{Cl}(X)$ *is the divisor class group, with the forward map* $[D] \mapsto \mathcal{O}(D)$ *and the reverse map* $\mathcal{L} \mapsto [\mathrm{div}\,(\mathcal{L})]$. *We know that every divisor on* $\mathbb{P}^1$ *is of the form* $D = \sum n_P P$ *where* $P \in \mathbb{P}^1$. *So we define* $\deg(D) = \sum n_P$. *Now, if* $f \in k(x)^\times$ *is a rational function, then* $f = p(x)/q(x)$ *for polynomials* p, q *and* $\deg p = \deg q$, *thus* $\deg((f)) = 0$. *Therefore, the degree map factors* $\deg : \mathrm{Cl}(\mathbb{P}^1) \to \mathbb{Z}$.

Suppose $\deg([D]) = 0$. *Then we can write* $D = \sum n_i P_i$ *with* $\sum_i n_i = 0$. *We construct* $g \in k(x)^\times$ *such that* $D = (g)$. *In affine coordinates, assume no* P_i *is at infinity. Then*

$$g(x) := \prod_{j=1}^{r} (x - P_j)^{n_i}.$$

Thus $(g) = \sum n_i(P_i)$. *If one* P_i *is at infinity, we adjust accordingly. Therefore* $[D] = 0$ *in* $\mathrm{Cl}(\mathbb{P}^1)$, *so* deg *is injective.*

For surjectivity, for any $n \in \mathbb{Z}$*, take* $D = n \cdot (0)$*. Then we obtain a preimage* D *of* n*.*

Using the isomorphism between the Picard group and the divisor class group we define

$$\deg : \mathrm{Pic}(\mathbb{P}^1) \to \mathbb{Z}, \qquad \deg(\mathbb{L}) = \deg([\mathrm{div}\, s])$$

for any rational section s *of* $\mathcal{L}$*. This is well-defined because if* s' *is another rational section, then* $s'/s \in k(x)^\times$*, which implies*

$$\mathrm{div}\,(s') = \mathrm{div}\,(s) + \mathrm{div}\,(s'/s) \quad \Rightarrow \quad \deg[\mathrm{div}\,(s)] = \deg[\mathrm{div}\,(s')].$$

Therefore, we obtain a well-defined isomorphism $\deg : \mathrm{Pic}(\mathbb{P}^1) \xrightarrow{\sim} \mathbb{Z}$*, which gives the desired result.*

Example 3.12. *Suppose* X *is the affine line with two origins over some field* k*. Let* $U_1 = \mathrm{Spec}\, k[x]$*,* $U_2 = \mathrm{Spec}\, k[y]$*. They are glued along* $U_{12} = \mathrm{Spec}\, k[x, x^{-1}] \cong \mathrm{Spec}\, k[y, y^{-1}]$ *via* $x \mapsto y$*. Let* $p_1 = (x) \in U_1$ *and* $p_2 = (y) \in U_2$ *be the two distinct origins. We will assume that for schemes* X*, we have* $\mathrm{Pic}(X) = \mathrm{Cl}^{\mathrm{Ca}}(X)$*, where* $\mathrm{Cl}^{\mathrm{Ca}}(-)$ *is the Cartier divisor class group. Now, a line bundle on* X *is determined by a gluing of* $\mathcal{O}_{U_1}$ *and* $\mathcal{O}_{U_2}$ *along* U_{12}*, via a transition function* $g \in \mathcal{O}_{U_{12}}^\times$*. Transition functions* g, g' *define isomorphic line bundles iff* $g' = \frac{h_1}{h_2} g$*, where* $h_1 \in \mathcal{O}_{U_1}^\times = k^\times$*,* $h_2 \in \mathcal{O}_{U_2}^\times = k^\times$*. Hence the Picard group is isomorphic to*

$$\mathcal{O}_{U_{12}}^\times / (\mathcal{O}_{U_1}^\times \cdot \mathcal{O}_{U_2}^\times),$$

where the denominator is the image of $(h_1, h_2) \mapsto h_1/h_2$*. Now,* $\mathcal{O}_{U_{12}}^\times = k^\times \times x^{\mathbb{Z}}$*,* $\mathcal{O}_{U_1}^\times = k^\times = \mathcal{O}_{U_2}^\times$*. The map* $\mathcal{O}_{U_1}^\times \times \mathcal{O}_{U_2}^\times \to \mathcal{O}_{12}^\times$ *sends* (c_1, c_2) *to* c_1/c_2*. Therefore, the image is* $k^\times \times \{1\}$*. Hence*

$$\mathrm{Pic}(X) \cong \frac{k^\times \times x^{\mathbb{Z}}}{k^\times \times \{1\}} \cong x^{\mathbb{Z}} \cong \mathbb{Z}.$$

3.0.2 Deligne–Mumford Stacks and Tensor Products

Now, for a lattice Λ, define $\mathcal{E} \otimes \Lambda$ so that $\mathcal{E} \otimes \mathbb{Z}^d = \mathcal{E}^{\times_{\mathcal{M}} d}$. Before saying a few words about this definition, let us take a detour to the world of Deligne–Mumford stacks.

Let us assume for now that we know what stacks are. Then here is our definition of Deligne–Mumford stacks.

Definition 3.13. *Fix a base scheme S, and consider the category $(\mathrm{Sch}/S)_{\acute{e}t}$ of étale sites over S. Suppose F is a stack on this category. Then F is called a Deligne–Mumford stack if*

- *the diagonal $\Delta_F : F \to F \times_S F$ is representable (which means that for every scheme T and pair of points $a, b \in F(T)$, the isomorphism functor that sends a T-scheme $T' \to T$ to the set of isomorphisms between the families $a_{T'}$ and $b_{T'}$ over T' is representable by a scheme), quasi-compact, and separated;*
- *there exists a scheme S' and a representable, surjective, étale morphism $S' \to F$, called a cover of F.*

Recall that a morphism of schemes is called *étale* if it is flat (the induced map on every stalk is a flat map of rings) and unramified. A morphism of schemes $f : X \to Y$ is unramified if it is locally of finite presentation and for each $a \in X$, we have $\mathfrak{m}_{f(a)}\mathcal{O}_{X,a} = \mathfrak{m}_a$ and the induced morphism of residue fields $\mathcal{O}_{Y,f(a)}/\mathfrak{m}_{f(a)} \to \mathcal{O}_{X,a}/\mathfrak{m}_a$ is a finite and separable field extension.

Classically, we have a bifunctor

$$AbGroup_{\mathcal{C}/\mathcal{M}} \times \mathrm{AbGroup} \xrightarrow{\otimes} AbGroup_{\mathcal{C}/\mathcal{M}},$$

where $\mathcal{C}$ is the category of Deligne–Mumford stacks, $AbGroup_{\mathcal{C}}$ is the category of abelian group objects in $\mathcal{C}$, and AbGroup is the category of abelian groups. This bifunctor has the adjunction property: for $X \in \mathit{AbGroup}_{\mathcal{C}/\mathcal{M}}$, $\Lambda \in \mathrm{AbGroup}$, and $Z \in \mathit{AbGroup}_{\mathcal{C}/\mathcal{M}}$, we have the natural isomorphism

$$\hom_{\mathit{AbGroup}_{\mathcal{C}/\mathcal{M}}}(X \otimes \Lambda, Z) \cong \hom_{\mathrm{AbGroup}}(\Lambda, \hom_{\mathit{AbGroup}_{\mathcal{C}/\mathcal{M}}}(X, Z)).$$

Thus, for a free abelian group $\Lambda = \mathbb{Z}^d$, the tensor product is given by

$$\mathcal{E} \otimes \Lambda = \mathcal{E}^{\times_{\mathcal{M}} d}.$$

In the derived setting when we have $(\mathcal{E}, \mathcal{O}_{\mathcal{E}}^{\mathrm{top}})$, the tensor multiplication would work similarly for $(\mathcal{E}, \mathcal{O}_{\mathcal{E}}^{\mathrm{top}}) \otimes \Lambda$.

3.0.3 Functorial Constructions

Let us consider the functors:

$$\mathrm{Bil} : \mathrm{Lat}^{\mathrm{op}} \to \mathrm{AbGroup}, \qquad \Lambda \mapsto \mathrm{Bil}(\Lambda), \tag{3.14}$$

$$\mathrm{Pic}^{\mathbb{Z}} : \mathrm{Lat}^{\mathrm{op}} \to \mathrm{AbGroup}, \qquad \Lambda \mapsto \mathrm{Pic}(\mathcal{E} \otimes \Lambda), \tag{3.15}$$

$$\mathrm{Pic}^{\mathrm{top}} : \mathrm{Lat}^{\mathrm{op}} \to \mathrm{AbGroup}, \qquad \Lambda \mapsto \mathrm{Pic}(\mathcal{E} \otimes \Lambda, \mathcal{O}_{\mathcal{E}\otimes\Lambda}^{\mathrm{top}}). \tag{3.16}$$

The last two functors give us isomorphism classes of line bundles and derived line bundles. Now, additivity and functoriality are equivalent to asking for natural transformations $\mathcal{L}_{-}^{\mathbb{Z}} : \mathrm{Bil} \to \mathrm{Pic}^{\mathbb{Z}}$ and $\mathcal{L}_{-} : \mathrm{Bil} \to \mathrm{Pic}^{\mathrm{top}}$.

For this, consider another functor:

$$\mathbb{Z}[(-)^{\vee}] : \mathrm{Lat}^{\mathrm{op}} \to \mathrm{AbGroup}, \qquad \Lambda \mapsto \mathbb{Z}[\Lambda^{\vee}].$$

We now have the following proposition, which we will use later.

Proposition 3.17. *Suppose we have a functor* $G : \mathrm{Lat}^{\mathrm{op}} \to \mathrm{AbGroup}$*. Then specifying a natural transformation* $\eta : \mathbb{Z}[(-)^{\vee}] \to G$ *is equivalent to specifying an object* $x \in G(\mathbb{Z})$*.*

Proof. For a lattice Λ, let $\Lambda^\vee = \hom_{\mathrm{Lat}}(\Lambda, \mathbb{Z})$. The functor $\mathbb{Z}[(-)^\vee]$ sends a lattice Λ to the free abelian group on its dual lattice $\Lambda^\vee$.

Given $\eta : \mathbb{Z}[(-)^\vee] \Rightarrow G$, define

$$x_\eta := \eta_{\mathbb{Z}}([\mathrm{id}_{\mathbb{Z}}]) \in G(\mathbb{Z}),$$

where $[\mathrm{id}_{\mathbb{Z}}]$ is the basis element corresponding to $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$.

Given $x \in G(\mathbb{Z})$, construct $\eta^x : \mathbb{Z}[(-)^\vee] \Rightarrow G$ as follows. For a lattice Λ and a basis element $[\psi] \in \mathbb{Z}[\Lambda^\vee]$, regard ψ as a morphism $\psi^{\mathrm{op}} : \mathbb{Z} \to \Lambda$ in $\mathrm{Lat}^{\mathrm{op}}$. Now define

$$\eta^x_\Lambda([\psi]) := G(\psi^{\mathrm{op}})(x) \in G(\Lambda)$$

and extend linearly to all of $\mathbb{Z}[\Lambda^\vee]$.

Let $f^{\mathrm{op}} : \Lambda \to \Lambda'$ be a morphism in $\mathrm{Lat}^{\mathrm{op}}$. Then for any $[\psi] \in \mathbb{Z}[(\Lambda')^\vee]$ we have

$$\begin{aligned}
\eta^x_\Lambda\left(\mathbb{Z}[(f^{\mathrm{op}})^\vee]([\psi])\right) &= \eta^x_\Lambda([\psi \circ f]) \\
&= G((\psi \circ f)^{\mathrm{op}})(x) \\
&= G(f^{\mathrm{op}} \circ \psi^{\mathrm{op}})(x) \\
&= G(f^{\mathrm{op}})(G(\psi^{\mathrm{op}})(x)) \\
&= G(f^{\mathrm{op}})(\eta^x_{\Lambda'}([\psi])).
\end{aligned}$$

Thus the following square commutes:

$$\begin{tikzcd}
\mathbb{Z}[(\Lambda')^\vee] \arrow[r, "\eta^x_{\Lambda'}"] \arrow[d, "\mathbb{Z}[(f^{\mathrm{op}})^\vee]"'] & G(\Lambda') \arrow[d, "G(f^{\mathrm{op}})"] \\
\mathbb{Z}[\Lambda^\vee] \arrow[r, "\eta^x_\Lambda"'] & G(\Lambda)
\end{tikzcd}$$

Therefore η^x is a natural transformation.

If we start with $x \in G(\mathbb{Z})$ and form η^x, then

$$x_{\eta^x} = \eta^x_{\mathbb{Z}}([\mathrm{id}_{\mathbb{Z}}]) = G(\mathrm{id}^{\mathrm{op}}_{\mathbb{Z}})(x) = G(\mathrm{id}_{\mathbb{Z}})(x) = x.$$

If we start with η and set $x = \eta_{\mathbb{Z}}([\mathrm{id}_{\mathbb{Z}}])$, then for any Λ and any $[\psi] \in \mathbb{Z}[\Lambda^{\vee}]$, consider the naturality square of η for $\psi^{\mathrm{op}} : \mathbb{Z} \to \Lambda$:

$$\begin{tikzcd}
\mathbb{Z}[(\mathbb{Z})^{\vee}] \arrow[r, "\eta_{\mathbb{Z}}"] \arrow[d, "\mathbb{Z}[(\psi^{\mathrm{op}})^{\vee}]"'] & G(\mathbb{Z}) \arrow[d, "G(\psi^{\mathrm{op}})"] \\
\mathbb{Z}[\Lambda^{\vee}] \arrow[r, "\eta_{\Lambda}"'] & G(\Lambda)
\end{tikzcd}$$

Chasing $[\mathrm{id}_{\mathbb{Z}}]$ around the diagram gives

$$\eta_{\Lambda}([\psi]) = G(\psi^{\mathrm{op}})(x) = \eta^{x}_{\Lambda}([\psi]).$$

Since this holds for all basis elements, we obtain $\eta_{\Lambda} = \eta^{x}_{\Lambda}$ for every Λ; thus $\eta = \eta^{x}$.

Hence the assignment $\eta \mapsto \eta_{\mathbb{Z}}([\mathrm{id}_{\mathbb{Z}}])$ gives a bijection $\mathrm{Nat}_{\mathrm{AbGroup}}(\mathbb{Z}[(-)^{\vee}], G) \cong G(\mathbb{Z})$ with inverse $x \mapsto \eta^{x}$. □

Using this, the natural transformation $\mathbb{Z}[(-)^{\vee}] \to \mathrm{Pic}^{\mathbb{Z}}$ is defined by sending $1 \cdot [\mathrm{id}_{\mathbb{Z}}]$ to ω, and the natural transformation $\mathbb{Z}[(-)^{\vee}] \to \mathrm{Pic}^{\mathrm{top}}$ is defined by sending $1 \cdot [\mathrm{id}_{\mathbb{Z}}]$ to $\mathcal{O}^{\mathrm{top}}(e)[-2]$.

Similarly, we have a natural transformation $\mathbb{Z}[(-)^{\vee}] \to \mathrm{Bil}$ that is specified by sending $1 \cdot [\mathrm{id}_{\mathbb{Z}}]$ to (1). For a lattice Λ, a basis element $\lambda \in \mathbb{Z}[\Lambda^{\vee}]$ is sent to the bilinear form

$$\alpha_{\Lambda}([\lambda])(v, v') = \lambda(v) \cdot \lambda(v').$$

Using this, Meier in upcoming work [21] proves the following.

Theorem 3.18. *There exists a natural transformation* $\mathrm{Bil} \to \mathrm{Pic}^{\mathrm{top}}$ *that sends* (1) *to* $\mathcal{O}^{\mathrm{top}}_{\mathcal{E}}(e)[-2]$.

3.0.4 Definition of TJF^{b} and $\overline{\mathrm{TJF}}_{b}$

Given our setup, we are now in a position to define our TMF-modules of interest.

Definition 3.19. *For a symmetric bilinear form b on $\mathbb{Z}^d$, define*

$$\mathrm{TJF}^b := \Gamma(\mathcal{M}, p_*\mathcal{L}_b) \quad \textit{and} \quad \overline{\mathrm{TJF}}_b := \Gamma(\mathcal{M}, p_*\mathcal{L}_b^\vee)^\vee,$$

where $p : \mathcal{E} \otimes \mathbb{Z}^d \to \mathcal{M}$ is the projection and $(-)^\vee$ denotes the TMF*-linear dual.*

3.0.5 Properties of Derived Looijenga Line Bundles

In this subsection we establish some basic properties of derived Looijenga line bundles, based more or less entirely on [7].

Lemma 3.20. *If $e : \mathcal{M} \to \mathcal{E}$ denotes the unit section, we have an equivalence:*

$$e^*\mathcal{L}_{(1)} \simeq e^*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e)[-2] \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}.$$

Proof. The first equivalence comes from the normalisation axiom of the Looijenga line bundle construction. We define $\mathcal{L}_{(1)} = \mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e)[-2]$. Pulling back along the unit section gives us the first equivalence.

The second equivalence is subtler and uses $U(1)$-equivariant elliptic cohomology. Gepner–Meier in [9] prove that $\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(-e) \simeq \mathcal{O}_{\mathcal{E}}^{S^{\mathbb{C}}}$, where $\mathcal{O}_{\mathcal{E}}^{S^{\mathbb{C}}}$ is the $U(1)$-equivariant elliptic cohomology of the Riemann sphere $S^{\mathbb{C}}$ with a natural $U(1)$-action by rotation. Gepner–Meier also prove that $e^*\mathcal{O}_{\mathcal{E}}^{X} \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{res}\, X}$. This means that pulling back to the unit section essentially equates to forgetting the group action.

Now, using the first and second results we get

$$e^*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(-e) \simeq e^*\mathcal{O}_{\mathcal{E}}^{S^{\mathbb{C}}},$$
$$e^*\mathcal{O}_{\mathcal{E}}^{S^{\mathbb{C}}} \simeq \mathcal{O}_{\mathcal{M}}^{S^2}.$$

Now, since S^n is the one-point compactification of $\mathbb{R}^n$, it is the Thom space of the trivial n-bundle over a point. Therefore,

$$\mathcal{O}_{\mathcal{M}}^{S^n} \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{pt}}[-n] \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}[-n].$$

Thus we now have

$$\begin{aligned} e^*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(-e) &\simeq \mathcal{O}_{\mathcal{M}}^{S^2} \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}[-2], \\ e^*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(-e) &\simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}[-2]. \end{aligned}$$

Now we have

$$\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e) \otimes \mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(-e) \simeq \mathcal{O}_{\mathcal{E}}^{\mathrm{top}}.$$

Pulling back along e gives

$$e^*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e) \otimes e^*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(-e) \simeq e^*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}} \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}.$$

Substituting our previous result, we obtain

$$e^*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e) \otimes \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}[-2] \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}.$$

Tensoring both sides with $\mathcal{O}_{\mathcal{M}}^{\mathrm{top}}[2]$ gives

$$e^*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e) \otimes \mathcal{O}_{\mathcal{M}}^{\mathrm{top}} \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}[2],$$

hence

$$e^*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e) \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}[2],$$

which implies

$$e^*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e)[-2] \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}.$$

□

Lemma 3.21. *Suppose* b, b' *are defined on* $\mathbb{Z}^d$ *and* $\mathbb{Z}^{d'}$*. Consider the projections* π_1 *and* π_2 *from* $\mathbb{Z}^d \oplus \mathbb{Z}^{d'}$ *to* $\mathbb{Z}^d$ *and* $\mathbb{Z}^{d'}$*, and denote the corresponding projections from* $\mathcal{E}^{\times_{\mathcal{M}}(d+d')}$ *to* $\mathcal{E}^{\times_{\mathcal{M}}d}$ *and* $\mathcal{E}^{\times_{\mathcal{M}}d'}$ *by the same symbols. Then we have*

$$\mathcal{L}_{b\oplus b'} \cong \pi_1^*\mathcal{L}_b \otimes \pi_2^*\mathcal{L}_{b'}.$$

Proof. We have the following isomorphism of bilinear forms:

$$b \oplus b' \cong \pi_1^* b + \pi_2^* b'.$$

Since $\mathcal{L}_-$ is a natural transformation, we obtain

$$\mathcal{L}_{b \oplus b'} \simeq \mathcal{L}_{\pi_1^* b + \pi_2^* b'}.$$

Now, using additivity,

$$\mathcal{L}_{\pi_1^* b + \pi_2^* b'} \simeq \mathcal{L}_{\pi_1^* b} \otimes \mathcal{L}_{\pi_2^* b'}.$$

Now, using functoriality,

$$\mathcal{L}_{\pi_1^* b} \otimes \mathcal{L}_{\pi_2^* b'} \simeq \pi_1^*(\mathcal{L}_b) \otimes \pi_2^*(\mathcal{L}_{b'}).$$

Hence

$$\mathcal{L}_{b \oplus b'} \simeq \pi_1^*(\mathcal{L}_b) \otimes \pi_2^*(\mathcal{L}_{b'}).$$

□

Lemma 3.22. *Suppose for symmetric integral bilinear forms b, b' there exists another symmetric integral bilinear form c such that $b \oplus c \cong b' \oplus c$. Then $\mathcal{L}_b \simeq \mathcal{L}_{b'}$.*

Proof. Suppose b, b' are defined on $\mathbb{Z}^d$ and c on $\mathbb{Z}^{d'}$. The projections on the lattices induce

$$\pi_1 : \mathcal{E}^{\times_{\mathcal{M}} (d+d')} \to \mathcal{E}^{\times_{\mathcal{M}} d}, \qquad \pi_2 : \mathcal{E}^{\times_{\mathcal{M}} (d+d')} \to \mathcal{E}^{\times_{\mathcal{M}} d'}.$$

Since $\mathcal{L}_-$ is a natural transformation, we have

$$\mathcal{L}_{b \oplus c} \simeq \mathcal{L}_{b' \oplus c}.$$

Using the previous lemma,

$$\mathcal{L}_{b \oplus c} \simeq \pi_1^* \mathcal{L}_b \otimes \pi_2^* \mathcal{L}_c, \qquad \mathcal{L}_{b' \oplus c} \simeq \pi_1^* \mathcal{L}_{b'} \otimes \pi_2^* \mathcal{L}_c.$$

Thus,

$$\pi_1^*\mathcal{L}_b \otimes \pi_2^*\mathcal{L}_c \simeq \pi_1^*\mathcal{L}_{b'} \otimes \pi_2^*\mathcal{L}_c.$$

Now, pulling back by the unit map $e : \mathcal{M} \to \mathcal{E}^{\times_{\mathcal{M}}(d+d')}$, we obtain

$$e^*(\pi_1^*\mathcal{L}_b \otimes \pi_2^*\mathcal{L}_c) \simeq e^*(\pi_1^*\mathcal{L}_{b'} \otimes \pi_2^*\mathcal{L}_c),$$

so

$$e^*\pi_1^*\mathcal{L}_b \otimes e^*\pi_2^*\mathcal{L}_c \simeq e^*\pi_1^*\mathcal{L}_{b'} \otimes e^*\pi_2^*\mathcal{L}_c.$$

Now,

$$e^*\pi_1^*\mathcal{L}_b \simeq (\pi_1 \circ e)^*\mathcal{L}_b \simeq \mathrm{id}^*\mathcal{L}_b \simeq \mathcal{L}_b,$$

and similarly $e^*\pi_1^*\mathcal{L}_{b'} \simeq \mathcal{L}_{b'}$. Also,

$$e^*\pi_2^*\mathcal{L}_c \simeq (\pi_2 \circ e)^*\mathcal{L}_c \simeq c_e^*\mathcal{L}_c,$$

where c_e is the constant map from $\mathcal{E}^{\times_{\mathcal{M}} d}$ to the identity section $e_{d'} \in \mathcal{E}^{\times_{\mathcal{M}} d'}$. The pullback of any bundle along a constant map is a trivialisable bundle; call it T. Then $e^*\pi_2^*\mathcal{L}_c \simeq T$.

Using all this, we obtain

$$\mathcal{L}_b \otimes T \simeq \mathcal{L}_{b'} \otimes T.$$

Tensoring both sides with $T^\vee$ yields

$$(\mathcal{L}_b \otimes T) \otimes T^\vee \simeq (\mathcal{L}_{b'} \otimes T) \otimes T^\vee,$$

so by associativity,

$$\mathcal{L}_b \otimes (T \otimes T^\vee) \simeq \mathcal{L}_{b'} \otimes (T \otimes T^\vee).$$

Since $T \otimes T^\vee \simeq \mathcal{O}$, we conclude $\mathcal{L}_b \simeq \mathcal{L}_{b'}$. $\square$

Proposition 3.23. *Suppose $b \oplus b'$ is the direct sum of symmetric integral bilinear forms b, b'. Then*

$$\mathrm{TJF}^{b \oplus b'} \simeq \mathrm{TJF}^b \otimes_{\mathrm{TMF}} \mathrm{TJF}^{b'}.$$

Proof. Let $\mathcal{E}^b$ and $\mathcal{E}^{b'}$ be the elliptic curve products where $\mathcal{L}_b$ and $\mathcal{L}_{b'}$ live. Define the structure maps $f : \mathcal{E}^b \to \mathcal{M}$, $g : \mathcal{E}^{b'} \to \mathcal{M}$, and $h : \mathcal{E}^{b\oplus b'} \to \mathcal{M}$. Also, define the projections $\pi^b : \mathcal{E}^{b\oplus b'} \to \mathcal{E}^b$ and $\pi^{b'} : \mathcal{E}^{b\oplus b'} \to \mathcal{E}^{b'}$.

We have a natural isomorphism $\mathcal{E}^{b\oplus b'} \cong \mathcal{E}^b \times_{\mathcal{M}} \mathcal{E}^{b'}$ that makes the following square Cartesian:

$$\begin{tikzcd}
\mathcal{E}^{b\oplus b'} \arrow[r, "\pi^{b'}"] \arrow[d, "\pi^b"'] & \mathcal{E}^{b'} \arrow[d, "g"] \\
\mathcal{E}^b \arrow[r, "f"'] & \mathcal{M}
\end{tikzcd}$$

Using a previous result, we have

$$\mathcal{L}_{b\oplus b'} \simeq (\pi^b)^*\mathcal{L}_b \otimes (\pi^{b'})^*\mathcal{L}_{b'}.$$

We want $\mathrm{TJF}^{b\oplus b'} = h_*\mathcal{L}_{b\oplus b'}$. Using $h = f \circ \pi^b$, we obtain

$$\mathrm{TJF}^{b\oplus b'} = h_*\mathcal{L}_{b\oplus b'} = f_*(\pi^b_*\mathcal{L}_{b\oplus b'}).$$

Now, using the projection formula for π^b,

$$\pi^b_*\mathcal{L}_{b\oplus b'} \simeq \pi^b_*\big((\pi^b)^*\mathcal{L}_b \otimes (\pi^{b'})^*\mathcal{L}_{b'}\big) \simeq \mathcal{L}_b \otimes \pi^b_*(\pi^{b'})^*\mathcal{L}_{b'}.$$

Using the Cartesian square, the base change isomorphism gives

$$\pi^b_*(\pi^{b'})^*\mathcal{L}_{b'} \simeq f^*g_*\mathcal{L}_{b'}.$$

Thus,

$$\pi^b_*\mathcal{L}_{b\oplus b'} \simeq \mathcal{L}_b \otimes f^*g_*\mathcal{L}_{b'}.$$

Now, applying f_* to both sides,

$$f_*(\pi^b_*\mathcal{L}_{b\oplus b'}) \simeq f_*(\mathcal{L}_b \otimes f^*g_*\mathcal{L}_{b'}) \simeq f_*\mathcal{L}_b \otimes g_*\mathcal{L}_{b'}.$$

Therefore,

$$\mathrm{TJF}^{b\oplus b'} = h_*\mathcal{L}_{b\oplus b'} \simeq f_*\mathcal{L}_b \otimes g_*\mathcal{L}_{b'} = \mathrm{TJF}^b \otimes \mathrm{TJF}^{b'}.$$

□

Chapter 4

Linking Forms and Stable Equivalence

The algebraic data that ultimately determines the stable equivalence class of a bilinear form (and hence the TMF-module TJF^b) is encoded by its *linking form.* The goal of this chapter is to make this relationship precise, to define the linking form, and to prove the key properties that will be used throughout the classification.

4.0.1 The discriminant group

Let $b : \mathbb{Z}^d \otimes \mathbb{Z}^d \to \mathbb{Z}$ be a symmetric bilinear form. Let's choose a basis for $\mathbb{Z}^d$ and let B be the $d \times d$ symmetric integer matrix representing b, i.e., $b(x, y) = x^T By$ for column vectors $x, y \in \mathbb{Z}^d$.

Definition 4.1. *The* adjoint map *of b is the homomorphism*

$$b_{\mathrm{adj}} : \mathbb{Z}^d \longrightarrow (\mathbb{Z}^d)^* \cong \mathbb{Z}^d, \qquad x \longmapsto \big(y \mapsto b(x, y)\big).$$

Under the identification of $(\mathbb{Z}^d)^*$ with $\mathbb{Z}^d$ via the standard basis, b_{adj} corresponds to multiplication by the matrix B.

Definition 4.2. *The* discriminant group *(or* cokernel*) of b is the abelian group*

$$A_b := \mathrm{coker}(b_{\mathrm{adj}}) = \mathbb{Z}^d / B\mathbb{Z}^d.$$

If $\det B \neq 0$, the order of A_b is $|\det B|$, and A_b is finite.

Remark 4.3. *The discriminant group is a fundamental invariant of the bilinear form. It vanishes if and only if b is unimodular (i.e., $\det B = \pm 1$). In general, it measures the failure of b to be unimodular.*

Example 4.4 (Rank one). *For* $b = \langle n \rangle$*, i.e.,* $b(x, y) = nxy$ *with* $n \neq 0$*, we have* $B = (n)$*, so* $A_b = \mathbb{Z}/n\mathbb{Z}$*.*

Example 4.5 (Diagonal rank two). *For* $b = \langle a \rangle \oplus \langle b \rangle = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ *with* $a, b \neq 0$*, we have*

$$B\mathbb{Z}^2 = \{(ax, by) : x, y \in \mathbb{Z}\},$$

so $A_b \cong \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$*.*

Example 4.6 (Non-diagonal rank two). *For* $b = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$*, we have* $\det B = 3$*. The matrix* B *is invertible over* $\mathbb{Q}$ *but not over* $\mathbb{Z}$*. One computes*

$$B\mathbb{Z}^2 = \{(2x + y, x + 2y) : x, y \in \mathbb{Z}\}.$$

A direct check shows that $(1, 0) \notin B\mathbb{Z}^2$ *and* $(0, 1) \notin B\mathbb{Z}^2$*, but* $2(1, 0) = (2, 0) \equiv (-1, 1) \pmod{B\mathbb{Z}^2}$ *and* $3(1, 0) = (3, 0) \equiv (0, 0)$ *after some manipulation. In fact,* $A_b \cong \mathbb{Z}/3\mathbb{Z}$ *generated by the class of* $(1, 0)$*. We will verify this systematically later.*

4.0.2 The linking form

We now define a bilinear form on the discriminant group A_b with values in $\mathbb{Q}/\mathbb{Z}$. This is the *linking form.* It is a classical invariant in the theory of quadratic forms ([22]) and appears naturally in the study of the torsion linking pairings of three-manifolds [8], [14].

Definition 4.7. *Let* b *be a nondegenerate symmetric bilinear form (i.e.,* $\det B \neq 0$*). The* linking form *associated to* b *is the map* $\lambda_b : A_b \times A_b \to \mathbb{Q}/\mathbb{Z}$ *defined by*

$$\lambda_b([x], [y]) := x^T B^{-1} y \pmod{\mathbb{Z}},$$

where $x, y \in \mathbb{Z}^d$ *are lifts of the classes* $[x], [y] \in A_b$*, and* B^{-1} *is the inverse matrix over* $\mathbb{Q}$*.*

Lemma 4.8. *The linking form λ_b is well-defined, i.e., independent of the choice of lifts x and y.*

Proof. Suppose $x' = x + Bz$ and $y' = y + Bw$ be different lifts, with $z, w \in \mathbb{Z}^d$. Then

$$\begin{aligned}(x')^T B^{-1} y' &= (x + Bz)^T B^{-1}(y + Bw)\\ &= x^T B^{-1} y + x^T B^{-1} Bw + (Bz)^T B^{-1} y + (Bz)^T B^{-1} Bw.\end{aligned}$$

Now $B^{-1}B = I_d$, and $(Bz)^T = z^T B^T = z^T B$ because B is symmetric. Therefore:

$$x^T B^{-1} Bw = x^T w, \qquad (Bz)^T B^{-1} y = z^T B B^{-1} y = z^T y, \qquad (Bz)^T B^{-1} Bw = z^T Bw.$$

Thus,

$$(x')^T B^{-1} y' = x^T B^{-1} y + x^T w + z^T y + z^T Bw.$$

The terms $x^T w$, $z^T y$, and $z^T Bw$ are all integers because x, y, z, w are integer vectors and B has integer entries. Hence

$$(x')^T B^{-1} y' \equiv x^T B^{-1} y \pmod{\mathbb{Z}},$$

so the class in $\mathbb{Q}/\mathbb{Z}$ is unchanged. □

Lemma 4.9. *The linking form λ_b is:*

1. ***Symmetric:*** $\lambda_b([x],[y]) = \lambda_b([y],[x])$ *for all* $[x],[y] \in A_b$.
2. ***Bilinear:*** $\lambda_b([x]+[x'],[y]) = \lambda_b([x],[y]) + \lambda_b([x'],[y])$ *and similarly in the second variable.*
3. ***Nondegenerate:*** *If* $\lambda_b([x],[y]) = 0$ *for all* $[y] \in A_b$, *then* $[x] = 0$ *in* A_b.

Proof. 1. Symmetry follows from the symmetry of B: $x^T B^{-1} y = (x^T B^{-1} y)^T = y^T (B^{-1})^T x = y^T B^{-1} x$ because B^{-1} is symmetric.

2. Bilinearity is immediate from the definition.
3. For nondegeneracy, suppose $[x] \in A_b$ satisfies $\lambda_b([x],[y]) = 0$ for all $[y]$. Then $x^T B^{-1} y \in \mathbb{Z}$ for all $y \in \mathbb{Z}^d$. This means that $x^T B^{-1} \in (\mathbb{Z}^d)^*$, i.e., $x^T B^{-1}$ is an integer row vector. Hence $x^T B^{-1} \in \mathbb{Z}^d$, so $x^T \in \mathbb{Z}^d B$, i.e., $x \in B\mathbb{Z}^d$. Therefore $[x] = 0$ in A_b.

□

4.0.3 Examples of linking forms for rank two forms

We now compute the linking forms for the rank two bilinear forms that will appear in our classification. These computations are essential for applying Nikulin's theorem.

Determinant 2: $\langle 1\rangle \oplus \langle 2\rangle$

Let $b = \langle 1\rangle \oplus \langle 2\rangle$ with matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

The discriminant group is $A_b = \mathbb{Z}^2/B\mathbb{Z}^2$. Since $B\mathbb{Z}^2 = \{(x, 2y) : x, y \in \mathbb{Z}\}$, a set of coset representatives is $\{(0,0),(0,1)\}$. Thus $A_b \cong \mathbb{Z}/2\mathbb{Z}$ generated by $g = [0,1]$. The linking form is

$$\lambda_b(g,g) = (0,1) \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \pmod{\mathbb{Z}}.$$

Hence $\lambda_b(x,y) = \frac{1}{2}xy$ on $\mathbb{Z}/2 \cong \{0,1\}$.

Determinant 3, diagonal: $\langle 1\rangle \oplus \langle 3\rangle$

Let $b = \langle 1\rangle \oplus \langle 3\rangle$ with

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}.$$

Here $B\mathbb{Z}^2 = \{(x, 3y)\}$, so $A_b \cong \mathbb{Z}/3\mathbb{Z}$ generated by $g = [0, 1]$. Then

$$\lambda_b(g, g) = \frac{1}{3} \pmod{\mathbb{Z}}.$$

Thus $\lambda_b(x, y) = \frac{1}{3}xy$ on $\mathbb{Z}/3$.

Determinant 3, non-diagonal: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Let us consider the bilinear form b, which is represented by the symmetric matrix

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We have $\det B = 4 - 1 = 3$, and the inverse over $\mathbb{Q}$ is

$$B^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Step 1: Computing the discriminant group $A_b = \mathbb{Z}^2/B\mathbb{Z}^2$. The lattice $B\mathbb{Z}^2$ is the set of all integer vectors of the form $B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}$ with $x, y \in \mathbb{Z}$. We need to determine the structure of the quotient $\mathbb{Z}^2/B\mathbb{Z}^2$.

A classical approach is to compute the *Smith normal form* of B. Recall that for any integer matrix B, there exist unimodular matrices $P, Q \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$PBQ = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

where d_1, d_2 are positive integers with $d_1 \mid d_2$. The numbers d_1, d_2 are called the *invariant factors* of B. Moreover, $\mathbb{Z}^2/B\mathbb{Z}^2 \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z}$.

Let us compute the Smith normal form of B. Perform the following elementary row and column operations (all matrices P and Q will be products of elementary matrices, hence unimodular):

$$\begin{aligned}
B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} &\xrightarrow{\text{swap rows}} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{(row operation, determinant } -1) \\
&\xrightarrow{\text{replace row2 by row2 - 2 row1}} \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \quad \text{(row operation, determinant 1)} \\
&\xrightarrow{\text{replace col2 by col2 - 2 col1}} \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \quad \text{(column operation, determinant 1)} \\
&\xrightarrow{\text{multiply col2 by } -1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{(column operation, determinant } -1).
\end{aligned}$$

Thus the Smith normal form of B is $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. The invariant factors are $d_1 = 1$ and $d_2 = 3$. Consequently,

$$A_b = \mathbb{Z}^2/B\mathbb{Z}^2 \cong \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}.$$

Step 2: Identifying a generator. We now find an explicit generator of A_b. From the Smith normal form computation, we can track the change of basis. The operations above give unimodular matrices P and Q such that

$$PBQ = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Explicitly, we have:

$$\begin{aligned}
P &= \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \quad \text{(since we swapped rows, then subtracted 2 times row1 from row2)}, \\
Q &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad \text{(since we subtracted 2 times col1 from col2, then multiplied col2 by -1)}.
\end{aligned}$$

One can verify that $PBQ = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Indeed,

$$PB = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix},$$

and then

$$(PB)Q = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \xrightarrow{\text{multiply col2 by } -1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Now, under the isomorphism $\mathbb{Z}^2/B\mathbb{Z}^2 \cong \mathbb{Z}^2/\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\mathbb{Z}^2$ given by $[v] \mapsto [Pv]$ (since Q is an automorphism of $\mathbb{Z}^2$), the generator of the $\mathbb{Z}/3\mathbb{Z}$ factor corresponds to the vector $(0,1)$ in the target. Therefore, a generator of A_b is given by $g = P^{-1}(0,1)^T$.

Compute P^{-1}. Since $P = \begin{pmatrix}0 & 1\\ 1 & -2\end{pmatrix}$, its determinant is $(0)(-2)-(1)(1) = -1$, so it is unimodular. The inverse is

$$P^{-1} = \frac{1}{-1}\begin{pmatrix}-2 & -1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}.$$

Thus $g = P^{-1}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$.

Hence the class of $g = (1,0)$ generates A_b. We check directly that $3(1,0) \in B\mathbb{Z}^2$:

$$3(1,0) = (3,0) = (2,1) + (1,-1) = B\begin{pmatrix}1\\ 0\end{pmatrix} + B\begin{pmatrix}0\\ -1\end{pmatrix} \in B\mathbb{Z}^2.$$

Also $(1,0) \notin B\mathbb{Z}^2$ because if $(1,0) = B(x,y) = (2x+y, x+2y)$, then $x+2y = 0$ implies $x = -2y$, so the first coordinate becomes $2(-2y)+y = -4y+y = -3y = 1$, which has no integer solution. Therefore $[(1,0)]$ has order exactly 3 in A_b.

Step 3: Expressing $(0,1)$ in terms of the generator. We compute the class of $(0,1)$ in A_b. Observe that

$$(0,1) = (1,0) - (1,-1).$$

Now $(1,-1) = B(1,-1)$ because

$$B\begin{pmatrix}1\\ -1\end{pmatrix} = \begin{pmatrix}2(1)+(-1)\\ 1+2(-1)\end{pmatrix} = \begin{pmatrix}2-1\\ 1-2\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}.$$

Thus $(1,-1) \in B\mathbb{Z}^2$, so its class in A_b is zero. Therefore,

$$[(0,1)] = [(1,0)] - [(1,-1)] = [(1,0)] - 0 = [(1,0)].$$

Hence $[(0,1)] = [(1,0)]$ in A_b. This confirms that A_b is cyclic of order 3 generated by the class of $(1,0)$, and $(0,1)$ represents the same class.

Step 4: Computing the linking form. Let $g = [(1,0)]$ be the generator. Using the definition of the linking form,

$$\lambda_b(g,g) = (1,0)^T B^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pmod{\mathbb{Z}}.$$

We compute:

$$B^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Then

$$(1,0)\cdot\frac{1}{3}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{2}{3}.$$

Thus

$$\lambda_b(g,g) = \frac{2}{3} \pmod{\mathbb{Z}}.$$

Since $A_b \cong \mathbb{Z}/3\mathbb{Z}$ is cyclic, the linking form is completely determined by the value on the generator. For any $x, y \in \mathbb{Z}/3\mathbb{Z}$, writing $x = ag$, $y = bg$ with $a, b \in \{0,1,2\}$, we have

$$\lambda_b(x,y) = ab\cdot\lambda_b(g,g) = ab\cdot\frac{2}{3} \pmod{\mathbb{Z}}.$$

In particular, $\lambda_b(1,1) = 2/3$ (where 1 denotes the generator).

So to summarise, for the non-diagonal form $b = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, we have:

- Discriminant group: $A_b \cong \mathbb{Z}/3\mathbb{Z}$, generated by $[(1,0)]$.
- Linking form: $\lambda_b(x,y) = \frac{2}{3}xy$ for $x, y \in \mathbb{Z}/3\mathbb{Z}$.

Remark 4.10. *Note that this linking form is* not *isometric to the linking form of the diagonal form* $\langle 1\rangle \oplus \langle 3\rangle$*, which gives* $\lambda(1,1) = 1/3$*. Indeed, an isometry would require an automorphism* ϕ *of* $\mathbb{Z}/3\mathbb{Z}$ *such that* $\phi(1)^2 \cdot (2/3) = 1/3$*. The automorphisms of* $\mathbb{Z}/3\mathbb{Z}$ *are multiplication by* ± 1*, and* $(\pm 1)^2 = 1$*, so this would require* $2/3 = 1/3$*, which is false. Therefore the two forms are not stably equivalent.*

Determinant 4, even: $\langle 2\rangle \oplus \langle 2\rangle$

For $b = \langle 2\rangle \oplus \langle 2\rangle$,

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Then $B\mathbb{Z}^2 = 2\mathbb{Z} \times 2\mathbb{Z}$, so $A_b \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ with generators $e_1 = [1, 0]$, $e_2 = [0, 1]$. The linking form is

$$\lambda(e_1, e_1) = \frac{1}{2}, \quad \lambda(e_1, e_2) = 0, \quad \lambda(e_2, e_2) = \frac{1}{2} \pmod{\mathbb{Z}}.$$

Thus $\lambda((x_1, x_2), (y_1, y_2)) = \frac{1}{2}(x_1y_1 + x_2y_2)$.

Determinant 4, odd: $\langle 1\rangle \oplus \langle 4\rangle$

For $b = \langle 1\rangle \oplus \langle 4\rangle$,

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix}.$$

Here $B\mathbb{Z}^2 = \{(x, 4y)\}$, so $A_b \cong \mathbb{Z}/4\mathbb{Z}$ generated by $g = [0, 1]$. Then

$$\lambda(g, g) = \frac{1}{4} \pmod{\mathbb{Z}}.$$

Thus $\lambda(x, y) = \frac{1}{4}xy$ on $\mathbb{Z}/4$.

4.0.4 Isometry of linking forms

We need to understand when two linking forms are isomorphic. This is a subtle point, especially for cyclic groups.

Definition 4.11. *Two linking forms (A, λ) and (A', λ') are* isometric *if there exists a group isomorphism $\phi : A \to A'$ such that*

$$\lambda'(\phi(x), \phi(y)) = \lambda(x, y) \quad \textit{for all } x, y \in A.$$

Lemma 4.12. *Let $A = \mathbb{Z}/n\mathbb{Z}$ with n odd. A linking form on A is determined by the value $\lambda(1, 1) = k/n$ where $\gcd(k, n) = 1$. Two such forms with parameters k/n and*

k'/n are isometric if and only if $k \equiv k' \cdot u^2 \pmod{n}$ for some unit $u \in (\mathbb{Z}/n\mathbb{Z})^\times$ (i.e., u coprime to n).

Proof. Any automorphism of $\mathbb{Z}/n\mathbb{Z}$ is multiplication by some $u \in (\mathbb{Z}/n\mathbb{Z})^\times$. Under such an automorphism, $\phi(1) = u$, so

$$\lambda'(\phi(1), \phi(1)) = \lambda'(u, u) = u^2 \cdot \lambda'(1, 1) = u^2 \cdot \frac{k'}{n}.$$

We require this to equal $\lambda(1, 1) = k/n$. Hence $k \equiv k'u^2 \pmod{n}$. The condition $\gcd(k, n) = 1$ ensures that k is a unit modulo n, and the set of squares in $(\mathbb{Z}/n\mathbb{Z})^\times$ is the subgroup of quadratic residues. □

Example 4.13 ($n = 3$)**.** *The group $(\mathbb{Z}/3\mathbb{Z})^\times = \{1, 2\}$ has squares $\{1^2 = 1, 2^2 = 4 \equiv 1\}$. Thus the only square is* 1*. Therefore, the linking forms* $1/3$ *and* $2/3$ *are* not *isometric because $2 \not\equiv 1 \cdot u^2 \pmod{3}$ for any u (since $u^2 \equiv 1$). This corrects the mistaken claim in the original text that they might be isomorphic via $\phi(x) = -x$; that would send $1/3$ to $(-1)^2 \cdot 1/3 = 1/3$, not $2/3$.*

Example 4.14 ($n = 4$)**.** *For $A = \mathbb{Z}/4\mathbb{Z}$, the automorphism group is $\{1, 3\}$ (since $3 \equiv -1$). The squares are $1^2 = 1$ and $3^2 = 9 \equiv 1$. So again only 1. The linking form $1/4$ is not isometric to $3/4$ (since $3 \not\equiv 1$). However, note that $3/4 \equiv -1/4$, so the negative of the linking form is a different isometry class when -1 is not a square modulo n. For $n = 4$, the units are $\{1, 3\}$ and $3^2 = 9 \equiv 1$, so $-1 \equiv 3$ is not a square; hence the linking forms $1/4$ and $3/4$ are not isometric. In general, for n odd, -1 is a square modulo n if and only if every prime factor of n is congruent to $1 \pmod{4}$; for n even, the situation is more subtle (see [22, Chapter 5]).*

4.0.5 Nikulin's classification theorem

We now state the fundamental theorem that classifies integral symmetric bilinear forms up to stable equivalence.

Theorem 4.15 ([23])**.** *Let b and b' be two nondegenerate integral symmetric bilinear forms. They are stably equivalent if and only if they have:*

1. *The same signature (b^+, b^-) (the number of positive and negative eigenvalues over $\mathbb{R}$);*
2. *The same rank (or equivalently, the same type: definite or indefinite);*
3. *Isomorphic linking forms $(A_b, \lambda_b) \cong (A_{b'}, \lambda_{b'})$.*

Remark 4.16. *Recall that two forms b and b' are* stably equivalent *if there exist nonnegative integers r, s such that*

$$b \oplus \langle 1 \rangle^{\oplus r} \oplus \langle -1 \rangle^{\oplus s} \;\cong\; b' \oplus \langle 1 \rangle^{\oplus r} \oplus \langle -1 \rangle^{\oplus s}.$$

(Note that the same numbers r, s appear on both sides. This ensures that the signature and rank are invariant under stable equivalence.)

Remark 4.17. *For unimodular forms ($A_b = 0$), the linking form is trivial, and the classification reduces to signature and rank. For non-unimodular forms, the linking form captures the essential information about the torsion part of the cokernel.*

4.0.6 Application to the determinant-3 rank-two forms

We now apply Nikulin's theorem to the two positive definite rank-two forms with determinant 3:

$$b_{\text{diag}} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad b_{\text{nd}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Both have:

- Signature $(2, 0)$ (positive definite);
- Rank 2;

- Determinant 3, so $|A_b| = 3$.

Their linking forms are:

- For b_{diag}: $\lambda(1,1) = 1/3$ on $\mathbb{Z}/3$.
- For b_{nd}: $\lambda(1,1) = 2/3$ on $\mathbb{Z}/3$.

As shown in Example 3.6, 1/3 and 2/3 are not related by multiplication by a square in $(\mathbb{Z}/3)^\times$ (since the only square is 1). Therefore, the linking forms are *not isometric.*

By Nikulin's theorem, b_{diag} and b_{nd} are *not stably equivalent.* Consequently, the TMF-modules $\mathrm{TJF}^{b_{\mathrm{diag}}}$ and $\mathrm{TJF}^{b_{\mathrm{nd}}}$ are different (though they may become isomorphic after localising at certain primes, as we will see in later chapters).

Remark 4.18. *This example shows that the linking form is a sensitive invariant. Even though both forms have the same determinant and signature, the different linking forms lead to different stable equivalence classes and hence different* TMF*-modules. This will have consequences for the invariants of three-manifolds: two three-manifolds with linking forms* 1/3 *and* 2/3 *will have distinct GKMP invariants.*

Now, Nikulin's theorem tells us that for nondegenerate forms, the triple

$$(\text{rank}, \text{signature}, (A_b, \lambda_b))$$

is a complete set of invariants for stable equivalence. In particular, the linking form alone determines the stable equivalence class up to the signature and rank.

For our purposes, we will primarily be concerned with the linking form because:

1. The signature and rank are easily read off from the matrix.
2. The linking form encodes the subtle torsion information that distinguishes forms with the same determinant.

3. The TMF-module TJF^b depends only on the stable equivalence class, hence only on these invariants.

In the following chapters, we will compute TJF^b for all rank-two forms with $|\det b| \leq 4$ and show how the linking form manifests itself in the module structure.

Chapter 5

Isomorphism Classification of Rank 2 Bilinear Forms with $|\det| \leq 4$

We now apply the general theory to the specific case of rank 2 bilinear forms with small determinant. Our goal is to list all isomorphism classes and compute their linking forms.

Recall that a rank 2 symmetric bilinear form is represented by a 2×2 symmetric integer matrix

$$B = \begin{pmatrix} a & c \\ c & d \end{pmatrix}, \qquad a, b, c \in \mathbb{Z}.$$

Definition 5.1. *For a bilinear form b represented by B, define:*

- Determinant*: $\Delta(b) = \det B = ad - c^2$.*
- Signature*: $\sigma(b) = (b^+, b^-)$, the number of positive and negative eigenvalues of B over $\mathbb{R}$.*
- Parity*: b is* even *if $b(v, v) \in 2\mathbb{Z}$ for all $v \in \mathbb{Z}^2$; otherwise it is* odd.

For positive definite forms ($\Delta > 0$), we can use the theory of reduced binary quadratic forms.

Definition 5.2. *A positive definite form with matrix $\begin{pmatrix} a & c \\ c & d \end{pmatrix}$ is called* reduced *if it satisfies*

$$0 < a \leq d, \qquad -a < 2c \leq a,$$

with the additional condition $c \geq 0$ when $a = d$.

The importance of reduced forms is that every positive definite binary quadratic form is $\mathrm{GL}_2(\mathbb{Z})$-equivalent to a unique reduced form. This allows us to enumerate all forms with $|\det| \leq 4$ by solving the equation $ad - c^2 = \Delta$ under the reducedness conditions.

We now enumerate reduced forms. For that, let's go through each possible determinant.

Determinant 1

We need $ad - c^2 = 1$, $0 < a \leq d$, and $-a < 2c \leq a$.

We first derive a lower bound for a. Since $ad \geq a^2$ (because $d \geq a$) and $c^2 \leq (a/2)^2$ (from $|2c| \leq a$), we have

$$1 = ad - c^2 \geq a^2 - \frac{a^2}{4} = \frac{3a^2}{4}.$$

Thus $a^2 \leq 4/3$, so $a = 1$ is forced.

Now with $a = 1$, the condition $d - c^2 = 1$ gives $d = 1 + c^2$. The inequality $a \leq d$ gives $1 \leq 1 + c^2$, which is always true. The condition $-1 < 2c \leq 1$ forces $c = 0$. Then $d = 1$. So the unique reduced form is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e., $\langle 1 \rangle \oplus \langle 1 \rangle$.

Determinant 2

We need $ad - c^2 = 2$. Try $a = 1$: then $d - c^2 = 2$, so $d = 2 + c^2$. The condition $-1 < 2c \leq 1$ gives $c = 0$. Then $d = 2$. This yields $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, i.e., $\langle 1 \rangle \oplus \langle 2 \rangle$.

From the reducedness conditions, we have $a^2 \leq ad = \Delta + c^2$. Since $c^2 \leq a^2/4$, we obtain $a^2 \leq \Delta + a^2/4$, i.e., $3a^2/4 \leq \Delta$. For $\Delta = 2$, this gives $a^2 \leq 8/3$, so $a \leq 1$. Hence only $a = 1$ is possible. This confirms that our enumeration is complete.

Determinant 3

We need $ad - c^2 = 3$.

Try $a = 1$: then $d - c^2 = 3$, so $d = 3 + c^2$. The condition $-1 < 2c \leq 1$ gives $c = 0$. Then $d = 3$. This yields $\langle 1\rangle \oplus \langle 3\rangle$.

Try $a = 2$: then $2d - c^2 = 3$, so $d = (3 + c^2)/2$. The condition $a \leq d$ gives $2 \leq (3 + c^2)/2$, i.e., $c^2 \geq 1$. The condition $-2 < 2c \leq 2$ gives $|c| \leq 1$. So $c = \pm 1$ are possible. For $c = 1$, $d = (3 + 1)/2 = 2$. This yields $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. For $c = -1$, we get $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, which is $\mathrm{GL}_2(\mathbb{Z})$-equivalent to the $c = 1$ case via the transformation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

From the reducedness conditions, we have $a^2 \leq ad = \Delta + c^2$. Since $c^2 \leq a^2/4$, we obtain $a^2 \leq \Delta + a^2/4$, i.e., $3a^2/4 \leq \Delta$. For $\Delta = 3$, this gives $a^2 \leq 4$, so $a \leq 2$. Thus we only need to check $a = 1$ and $a = 2$.

Determinant 4

We need $ad - c^2 = 4$.

Try $a = 1$: then $d - c^2 = 4$, so $d = 4 + c^2$. The condition $-1 < 2c \leq 1$ gives $c = 0$, $d = 4$. This yields $\langle 1\rangle \oplus \langle 4\rangle$.

Try $a = 2$: then $2d - c^2 = 4$, so $d = (4 + c^2)/2$. The condition $a \leq d$ gives $2 \leq (4 + c^2)/2$, i.e., $c^2 \geq 0$ (automatic). The condition $-2 < 2c \leq 2$ gives $|c| \leq 1$. For $c = 0$, $d = 2$, yielding $\langle 2\rangle \oplus \langle 2\rangle$. For $c = \pm 1$, $d = (4 + 1)/2 = 2.5$, not an integer.

Try $a = 3$: then $3d - c^2 = 4$, so $d = (4 + c^2)/3$. The condition $a \leq d$ gives $3 \leq (4 + c^2)/3$, i.e., $c^2 \geq 5$. But the condition $-3 < 2c \leq 3$ gives $|c| \leq 1$. No solution.

Try $a = 4$: then $4d - c^2 = 4$, so $d = (4 + c^2)/4 = 1 + c^2/4$. The condition $a \leq d$ gives $4 \leq 1 + c^2/4$, i.e., $c^2 \geq 12$. The condition $-4 < 2c \leq 4$ gives $|c| \leq 2$. No solution.

Thus we have two isomorphism classes for $\det = 4$: $\langle 1\rangle \oplus \langle 4\rangle$ and $\langle 2\rangle \oplus \langle 2\rangle$.

5.0.1 Negative definite and indefinite forms

For negative definite forms ($\Delta > 0$, signature $(0, 2)$), we simply take the negatives of the positive definite forms. For example, $-\langle 1\rangle \oplus \langle 1\rangle = \langle -1\rangle \oplus \langle -1\rangle$.

For indefinite forms ($\Delta < 0$), the classification is more subtle. The standard indefinite forms with small determinant include:

$$H_{\text{even}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H_{\text{odd}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \langle 2\rangle \oplus \langle -2\rangle = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Other forms, such as $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ (determinant -2) and $\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$ (determinant -3), also exist. However, these are stably equivalent to the forms listed above: adding a copy of $\langle 1\rangle \oplus \langle -1\rangle$ (which is stably trivial) can transform them into the diagonal forms. For example,

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \oplus \langle 1\rangle \oplus \langle -1\rangle \cong H_{\text{even}} \oplus H_{\text{even}}.$$

Thus, up to stable equivalence, the three forms listed above represent all indefinite forms with $|\det| \leq 4$.

Theorem 5.3. *Up to $\mathrm{GL}_2(\mathbb{Z})$-isomorphism, the nondegenerate symmetric bilinear forms on $\mathbb{Z}^2$ with $|\det| \leq 4$ are as follows:*

Indefinite forms *($\Delta < 0$):*

$$H_{\text{even}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H_{\text{odd}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \langle 2\rangle \oplus \langle -2\rangle = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Positive definite forms *($\Delta > 0$):*

$$\begin{aligned}
\det = 1 :&\quad \langle 1\rangle \oplus \langle 1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\det = 2 :&\quad \langle 1\rangle \oplus \langle 2\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\
\det = 3 :&\quad \langle 1\rangle \oplus \langle 3\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\
\det = 4 :&\quad \langle 1\rangle \oplus \langle 4\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad \langle 2\rangle \oplus \langle 2\rangle = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}
\end{aligned}$$

Negative definite forms *are obtained by multiplying any positive definite form by* -1.

5.0.2 Computation of linking forms for all rank 2 forms

We now compute the discriminant group A_b and the linking form λ_b for each form in the classification.

Unimodular forms (det $= \pm 1$)

For H_{even}, H_{odd}, and $\langle 1\rangle \oplus \langle 1\rangle$, we have $\det B = \pm 1$, so B is invertible over $\mathbb{Z}$. Hence $B\mathbb{Z}^2 = \mathbb{Z}^2$ and $A_b = 0$. The linking form is trivial.

Determinant 2: $\langle 1\rangle \oplus \langle 2\rangle$

We already computed: $A_b \cong \mathbb{Z}/2$, $\lambda(x, y) = \frac{1}{2}xy$.

Determinant 3, diagonal: $\langle 1\rangle \oplus \langle 3\rangle$

We computed: $A_b \cong \mathbb{Z}/3$, $\lambda(x, y) = \frac{1}{3}xy$.

Determinant 3, non-diagonal: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

We computed: $A_b \cong \mathbb{Z}/3$, $\lambda(x, y) = \frac{2}{3}xy$.

Determinant 4, even: $\langle 2\rangle \oplus \langle 2\rangle$

For $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, we have $B\mathbb{Z}^2 = 2\mathbb{Z} \times 2\mathbb{Z}$, so $A_b \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Let $e_1 = [1, 0]$, $e_2 = [0, 1]$. Then $B^{-1} = \frac{1}{2}I_2$, so

$$\lambda(e_1, e_1) = \frac{1}{2}, \quad \lambda(e_1, e_2) = 0, \quad \lambda(e_2, e_2) = \frac{1}{2} \pmod{\mathbb{Z}}.$$

Thus $\lambda((x_1, x_2), (y_1, y_2)) = \frac{1}{2}(x_1y_1 + x_2y_2)$.

Determinant 4, odd: $\langle 1\rangle \oplus \langle 4\rangle$

For $B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, we have $B\mathbb{Z}^2 = \{(x, 4y)\}$, so $A_b \cong \mathbb{Z}/4$ generated by $g = [0, 1]$. Then $B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix}$, so

$$\lambda(g, g) = \frac{1}{4} \pmod{\mathbb{Z}}.$$

Thus $\lambda(x, y) = \frac{1}{4}xy$ on $\mathbb{Z}/4$.

Determinant -4**:** $\langle 2\rangle \oplus \langle -2\rangle$

For $B = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, we have $B\mathbb{Z}^2 = 2\mathbb{Z} \times 2\mathbb{Z}$, so $A_b \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Then $B^{-1} = \frac{1}{-4}\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$. Thus

$$\lambda(e_1, e_1) = \frac{1}{2}, \quad \lambda(e_2, e_2) = -\frac{1}{2} \equiv \frac{1}{2} \pmod{\mathbb{Z}}, \quad \lambda(e_1, e_2) = 0.$$

This is isomorphic to the linking form of $\langle 2\rangle \oplus \langle 2\rangle$.

5.0.3 Stable equivalence classes

Now we apply Nikulin's theorem to determine the stable equivalence classes.

Theorem 5.4. *The stable equivalence classes of nondegenerate rank 2 bilinear forms with* $|\det| \leq 4$ *are:*

1. ***Signature*** $(1, 1)$***:*** *A single class containing* H_{even}*,* H_{odd}*, and* $\langle 2\rangle \oplus \langle -2\rangle$*. (Note that these forms have determinants* -1*,* -1*, and* -4 *respectively. The determinant is not a stable invariant: adding* $\langle 1\rangle \oplus \langle -1\rangle$ *multiplies the determinant by* -1*, so the stable equivalence class can contain forms with different determinants. Indeed,* $\langle 2\rangle \oplus \langle -2\rangle$ *is stably equivalent to* H_{even} *because*

$$\langle 2\rangle \oplus \langle -2\rangle \oplus \langle 1\rangle \oplus \langle -1\rangle \cong H_{\text{even}} \oplus H_{\text{even}}.$$

Thus the stable equivalence class is determined by the signature and the linking form, not by the determinant alone.)

2. ***Signature*** $(2,0)$***:*** *Five distinct classes:*

$$\begin{aligned}
&C_{1,1} : \langle 1\rangle \oplus \langle 1\rangle \quad (A = 0),\\
&C_{1,2} : \langle 1\rangle \oplus \langle 2\rangle \quad (A = \mathbb{Z}/2,\ \lambda = 1/2),\\
&C_{2,1,2} : \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad (A = \mathbb{Z}/3,\ \lambda = 2/3),\\
&C_{2,2} : \langle 2\rangle \oplus \langle 2\rangle \quad (A = (\mathbb{Z}/2)^2,\ \lambda = \tfrac{1}{2}(x_1y_1 + x_2y_2)),\\
&C_{1,4} : \langle 1\rangle \oplus \langle 4\rangle \quad (A = \mathbb{Z}/4,\ \lambda = 1/4).
\end{aligned}$$

3. ***Signature*** $(0,2)$***:*** *Five distinct classes obtained by negating the positive definite forms.*

Proof. The linking forms computed above are pairwise non-isomorphic: different groups or different values on generators that are not related by automorphisms. By Nikulin's theorem, forms with different linking forms cannot be stably equivalent. The signature distinguishes the three families. □

Chapter 6

Linking Forms and Higher Topological Jacobi Forms

With the classification in hand, we now connect the algebraic data of the linking form to the homotopy-theoretic objects TJF^b.

When b is unimodular, $A_b = 0$ and the linking form is trivial. In this case, the module TJF^b is particularly simple.

Proposition 6.1 (Equation 5.7 [7])**.** *If b is unimodular with signature (b^+, b^-), then*

$$\mathrm{TJF}^b \simeq \mathrm{TMF}[3b^- - 2b^+].$$

Proof.

Remark 6.2. *Recall that $\mathrm{TJF}^b = \Gamma(\mathcal{M}, p_*\mathcal{L}_b)$, where $p : \mathcal{E}^{\times_{\mathcal{M}} d} \to \mathcal{M}$ is the projection and $\mathcal{L}_b$ is the derived Looijenga line bundle associated to b. The proposition asserts that, up to a degree shift, TJF^b is the free TMF-module of rank 1. The shift is determined entirely by the signature of b.*

Step 1: Reduction to the diagonal case via stable equivalence

A fundamental fact in the arithmetic of quadratic forms is that every unimodular bilinear form becomes diagonal after adding hyperbolic planes.

Lemma 6.3. *Let b be a unimodular symmetric bilinear form on $\mathbb{Z}^d$ with signature (b^+, b^-). Then b is stably equivalent to the diagonal form*

$$b_{\mathrm{diag}} := \langle 1\rangle^{\oplus b^+} \oplus \langle -1\rangle^{\oplus b^-}.$$

That is, there exist nonnegative integers r, s such that

$$b \oplus \langle 1 \rangle^{\oplus r} \oplus \langle -1 \rangle^{\oplus s} \;\cong\; \langle 1 \rangle^{\oplus (b^+ + r)} \oplus \langle -1 \rangle^{\oplus (b^- + s)}.$$

Proof Sketch. This is a classical result in the theory of integral quadratic forms.

- If b is *indefinite* and *odd* (i.e., there exists a vector with odd norm), then the classification of indefinite unimodular forms over $\mathbb{Z}$ (see [22]) implies that b is isomorphic to the diagonal form $\langle 1 \rangle^{\oplus b^+} \oplus \langle -1 \rangle^{\oplus b^-}$. If b is indefinite and *even*, then b is isomorphic to a form of the form $H^{\oplus k} \oplus E_8^{\oplus \ell}$ (or $H^{\oplus k} \oplus E_8^{\oplus \ell} \oplus \langle 1 \rangle \oplus \langle -1 \rangle$ if the signature is not divisible by 8). However, such forms become diagonal after adding a hyperbolic plane $H = \langle 1 \rangle \oplus \langle -1 \rangle$, which is stably trivial. Hence, up to stable equivalence, every indefinite unimodular form is diagonal.

- If b is *definite* (i.e., $b^- = 0$ or $b^+ = 0$), there exist many non-isomorphic definite unimodular forms (e.g., the E_8 lattice). However, after adding a sufficient number of hyperbolic planes $H = \langle 1 \rangle \oplus \langle -1 \rangle$, the form becomes indefinite (since H has signature $(1, 1)$). The resulting indefinite form can then be diagonalised by the indefinite case (after adding further hyperbolic planes if necessary). Therefore, any two definite unimodular forms become stably equivalent after adding enough copies of H. For a detailed treatment, see [22].

In both cases, b is stably equivalent to the diagonal form with the same signature.

□

By Lemma 3.22, stably equivalent forms give isomorphic Looijenga line bundles. Therefore,

$$\mathcal{L}_b \simeq \mathcal{L}_{b_{\mathrm{diag}}}.$$

Step 2: Additivity and reduction to rank one

The diagonal form b_{diag} is an orthogonal direct sum of b^+ copies of $\langle 1\rangle$ and b^- copies of $\langle -1\rangle$. By the additivity property of Looijenga line bundles,

$$\mathcal{L}_{b_{\mathrm{diag}}} \simeq \left(\bigotimes_{i=1}^{b^+} \mathrm{pr}_i^* \mathcal{L}_{\langle 1\rangle}\right) \otimes \left(\bigotimes_{j=1}^{b^-} \mathrm{pr}_j^* \mathcal{L}_{\langle -1\rangle}\right),$$

where $\mathrm{pr}_i : \mathcal{E}^{\times_{\mathcal{M}}(b^+ + b^-)} \to \mathcal{E}$ is the projection onto the i-th factor.

Step 3: Pushforward from a product

We need to compute $p_*\mathcal{L}_{b_{\mathrm{diag}}}$, where $p : \mathcal{E}^{\times_{\mathcal{M}} D} \to \mathcal{M}$ is the projection and $D = b^+ + b^-$.

A key technical fact is that pushforward along p respects external tensor products:

Lemma 6.4. *Let $p_i : \mathcal{E}^{\times_{\mathcal{M}} D} \to \mathcal{E}$ be the projection onto the i-th factor, and let $\mathcal{F}_i$ be classical quasi-coherent sheaves on $\mathcal{E}$ (i.e., $\mathcal{O}_{\mathcal{E}}$-modules). In the derived setting, we work with $\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}$-module spectra, but the pushforward formula holds by the same reasoning using the projection formula in derived algebraic geometry (see [7, Section 6.2]). Then*

$$p_*\left(\bigotimes_{i=1}^{D} p_i^*\mathcal{F}_i\right) \simeq \bigotimes_{i=1}^{D}(p_{1*}\mathcal{F}_i),$$

where $p_1 : \mathcal{E} \to \mathcal{M}$ is the projection for a single factor.

Proof. This follows by induction on D using the projection formula for the fibre product. For $D = 2$, consider the diagram

$$\mathcal{E} \times_{\mathcal{M}} \mathcal{E} \xrightarrow{\pi_2} \mathcal{E} \xrightarrow{p_1} \mathcal{M}.$$

Then $p_* = p_{1*} \circ \pi_{2*}$. By the projection formula,

$$\pi_{2*}(p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2) \simeq \mathcal{F}_1 \otimes \pi_{2*}p_2^*\mathcal{F}_2.$$

But $\pi_{2*}p_2^*\mathcal{F}_2 \simeq p_{1*}\mathcal{F}_2$ by base change. Applying p_{1*} gives the result. The general case is similar. □

Applying Lemma 6.4 to our situation, we obtain

$$p_*\mathcal{L}_{b_{\mathrm{diag}}} \simeq \bigotimes_{i=1}^{b^+}(p_*\mathcal{L}_{\langle 1\rangle}) \otimes \bigotimes_{j=1}^{b^-}(p_*\mathcal{L}_{\langle -1\rangle}),$$

where now $p : \mathcal{E} \to \mathcal{M}$ is the projection for a single elliptic curve.

Step 4: The rank-one pushforwards

The core of the proof is the computation of $p_*\mathcal{L}_{(1)}$ and $p_*\mathcal{L}_{(-1)}$. These are given in [7] and derived using $U(1)$-equivariant elliptic cohomology.

Lemma 6.5. *We have*

$$p_*\mathcal{L}_{(1)} \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}[-2].$$

Proof sketch. By the normalisation property, $\mathcal{L}_{(1)} \simeq \mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e)[-2]$. The result $p_*\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e) \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}$ is established in [9] using the equivalence

$$\mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(e) \simeq \mathcal{O}_{\mathcal{E}}^{\mathrm{top}}(U(1)_+)[1],$$

where $U(1)_+$ is $U(1)$ with a disjoint basepoint, and the fact that the $U(1)$-equivariant cohomology of a point is $\mathrm{TMF} \oplus \mathrm{TMF}[1]$. The pushforward then corresponds to taking $U(1)$-invariants, which picks out the first summand. The shift $[-2]$ in $\mathcal{L}_{(1)}$ remains, giving a total shift of $[-2]$. □

Lemma 6.6. *We have*

$$p_*\mathcal{L}_{(-1)} \simeq \mathcal{O}_{\mathcal{M}}^{\mathrm{top}}[3].$$

Proof. This result is proved in [7, Section 6.2] using $U(1)$-equivariant elliptic cohomology. The key steps are: (1) $\mathcal{L}_{(-1)} \simeq \mathcal{L}_{(1)}^{\vee}$ by additivity; (2) the duality theorem for $p : \mathcal{E} \to \mathcal{M}$ gives a relation between $p_*\mathcal{L}_{(1)}^{\vee}$ and $(p_*\mathcal{L}_{(1)})^{\vee}$; (3) a computation using the equivariant transfer shows that the total shift is [3]. For a detailed derivation, see [7, Lemma 6.5 and Equation 6.12]. □

Step 5: Combine the pushforwards

Now substitute Lemma 6.5 and Lemma 6.6 into the expression for $p_*\mathcal{L}_{b_{\mathrm{diag}}}$:

$$p_*\mathcal{L}_{b_{\mathrm{diag}}} \;\simeq\; \bigotimes_{i=1}^{b^+} \mathcal{O}^{\mathrm{top}}_{\mathcal{M}}[-2] \;\otimes\; \bigotimes_{j=1}^{b^-} \mathcal{O}^{\mathrm{top}}_{\mathcal{M}}[3].$$

Since $\mathcal{O}^{\mathrm{top}}_{\mathcal{M}}$ is the unit for the tensor product, this simplifies to

$$p_*\mathcal{L}_{b_{\mathrm{diag}}} \;\simeq\; \mathcal{O}^{\mathrm{top}}_{\mathcal{M}}[-2b^+ + 3b^-].$$

Step 6: Take global sections

The global sections functor $\Gamma(\mathcal{M}, -)$ sends $\mathcal{O}^{\mathrm{top}}_{\mathcal{M}}[k]$ to $\mathrm{TMF}[k]$, because $\mathrm{TMF} = \Gamma(\mathcal{M}, \mathcal{O}^{\mathrm{top}}_{\mathcal{M}})$ by definition and the shift $[k]$ passes through. Therefore,

$$\mathrm{TJF}^{b_{\mathrm{diag}}} = \Gamma(\mathcal{M}, p_*\mathcal{L}_{b_{\mathrm{diag}}}) \;\simeq\; \Gamma(\mathcal{M}, \mathcal{O}^{\mathrm{top}}_{\mathcal{M}}[-2b^+ + 3b^-]) \;\simeq\; \mathrm{TMF}[-2b^+ + 3b^-].$$

Since $\mathcal{L}_b \simeq \mathcal{L}_{b_{\mathrm{diag}}}$ by stable equivalence, we conclude

$$\mathrm{TJF}^b \simeq \mathrm{TMF}[-2b^+ + 3b^-].$$

This completes the proof of Proposition 6.1.

Step 7: Remarks on canonicity and the sign ambiguity

The isomorphism $\mathrm{TJF}^b \simeq \mathrm{TMF}[-2b^+ + 3b^-]$ is not canonical; it depends on choices made in the stable diagonalisation and in the isomorphisms of Lemma 6.5 and Lemma 6.6. The group of units of $\pi_0\mathrm{TMF}$ is $\{\pm 1\}$ (since $\pi_0\mathrm{TMF} \cong \mathbb{Z}[j]$ and j is not a unit). Therefore, the isomorphism is well-defined only up to multiplication by ± 1.

This sign ambiguity is important when defining the pointed element $\mathfrak{d}_b$ in Construction 6.3 of [7]. The element $\mathfrak{d}_b$ is the image of $1 \in \pi_0\mathrm{TMF}$ under the map

$$\mathrm{TMF} \xrightarrow{\cong} \mathrm{TJF}^b[-2b^+ + 3b^-] \xrightarrow{\text{restriction}} \mathrm{TMF}[-2b^+ + 3b^-],$$

where the restriction is induced by the inclusion $0 \to \mathbb{Z}^d$. Changing the isomorphism $\mathrm{TJF}^b \cong \mathrm{TMF}[-2b^+ + 3b^-]$ by a sign changes $\mathfrak{d}_b$ by that sign. In examples, the sign can be fixed by choosing a specific identification of TJF^b with $\mathrm{TMF}[-2b^+ + 3b^-]$. For instance, a choice of orientation for the underlying lattice or a choice of spin structure on the associated three-manifold can pick a canonical sign. For the purposes of defining the invariant $\mathcal{Z}(M)$, the sign ambiguity is inconsequential because $\mathcal{Z}(M)$ is defined up to equivalence, and multiplication by -1 is an automorphism of TMF that does not change the isomorphism class of the module. For the pointed element $\mathfrak{d}_b$, the sign matters, but it can be fixed by a convention (e.g., requiring $\mathfrak{d}_{H_{\text{even}}} = \eta$ rather than $-\eta$).

This completes the proof. □

6.0.1 Basic modules $\mathrm{TJF}^{(k)}$ for small k

The building blocks are the rank-1 modules $\mathrm{TJF}^{(k)}$. Their structure is described in [7].

Remark 6.7 (Stable transfer map)**.** *For $k \geq 1$, there exists a stable transfer map $\mathrm{tr} : \mathbb{CP}^{k-1}[1] \to S^0$ in the stable homotopy category. This map is constructed via the Pontryagin–Thom collapse map associated to the embedding $\mathbb{CP}^{k-1} \hookrightarrow \mathbb{CP}^{\infty}$ and the fact that the normal bundle of $\mathbb{CP}^{k-1}$ in $\mathbb{CP}^{\infty}$ is the tautological line bundle. The shift [1] accounts for the fact that the transfer raises the degree by 1. For a construction sketch, see [3, Section 2] .*

Proposition 6.8 (Equations 5.5, 5.6 [7])**.** *For $k \in \mathbb{Z}$, we have the following equiva-*

lences:

$$\begin{aligned}
\mathrm{TJF}^{(0)} &\simeq \mathrm{TMF} \oplus \mathrm{TMF}[1], \\
\mathrm{TJF}^{(1)} &\simeq \mathrm{TMF}[-2], \\
\mathrm{TJF}^{(-1)} &\simeq \mathrm{TMF}[3], \\
\mathrm{TJF}^{(2)} &\simeq (\nu : \mathrm{TMF}[3] \to \mathrm{TMF})[-4], \\
\mathrm{TJF}^{(-2)} &\simeq (\nu : \mathrm{TMF} \to \mathrm{TMF}[-3])[-1], \\
\mathrm{TJF}^{(3)} &\simeq (\mathrm{TMF} \wedge \mathbb{CP}^2[1] \xrightarrow{\mathrm{tr}} \mathrm{TMF})[-6], \\
\mathrm{TJF}^{(4)} &\simeq (\mathrm{TMF} \wedge \mathbb{CP}^3[1] \xrightarrow{\mathrm{tr}} \mathrm{TMF})[-8].
\end{aligned}$$

For general $k \geq 1$*,*

$$\mathrm{TJF}^{(k)} \simeq (\mathrm{TMF} \wedge \mathbb{CP}^{k-1}[1] \xrightarrow{\mathrm{tr}} \mathrm{TMF})[-2k].$$

Here $\nu \in \pi_3\mathrm{TMF}$ is the image of the Hopf map $S^7 \to S^4$. The cone of a map $f : X \to Y$ is the homotopy cofiber, denoted (f). It fits into a cofiber sequence

$$X \xrightarrow{f} Y \to (f) \to X[1].$$

Remark 6.9. *The map* $\nu : S^3 \to S^0$ *is the stable Hopf map. Its cone is a 2-cell complex that represents the element* ν *in the stable homotopy groups of spheres. Over* TMF*, the map* $\nu : \mathrm{TMF}[3] \to \mathrm{TMF}$ *is the induced map. The cone* (ν) *has homotopy groups that are extensions of* (ν) *and* $\ker(\nu)$ *in various degrees.*

6.0.2 The duality theorem

There is an important relationship between TJF^b and $\overline{\mathrm{TJF}}_b$.

Theorem 6.10 (Theorem 6.10 [7])**.** *For a bilinear form b on* $\mathbb{Z}^d$*, there is an equivalence*

$$\overline{\mathrm{TJF}}_b \simeq \mathrm{TJF}^b[-d]$$

up to a sign. In particular, $\overline{\mathrm{TJF}}_b$ *and* TJF^b *are dual up to a shift by the rank.*

Sketch. This follows from spectral Grothendieck duality for the projection $\mathcal{E}\otimes\mathbb{Z}^d \to \mathcal{M}$. The relative dualizing complex is $\mathcal{O}^{\mathrm{top}}_{\mathcal{E}\otimes\mathbb{Z}^d}[d]$, and the duality theorem gives $(p_*\mathcal{L}_b)^\vee \simeq p_*(\mathcal{L}_b^\vee)[d]$. Taking global sections and using $\mathcal{L}_b^\vee \simeq \mathcal{L}_{-b}$ yields the result. □

Chapter 7

Stable Equivalence and The Homological Cobordism Category

The invariants TJF^b associated to bilinear forms b are the building blocks of the GKMP TQFT. For a three-manifold M bounding a simply-connected four-manifold W with intersection form $b(W)$, the invariant $\mathscr{Z}(M)$ is defined (up to a degree shift) as $\overline{\mathrm{TJF}}_{b(W)}$. However, different choices of the bounding four-manifold W lead to different bilinear forms. The key observation is that these different bilinear forms are *stably equivalent*, and that TJF^b changes in a controlled way under stabilisation. This chapter makes this relationship precise.

7.0.1 Stable equivalence of bilinear forms

Recall the following standard notion from the theory of integral quadratic forms.

Definition 7.1. *Two symmetric bilinear forms b on $\mathbb{Z}^d$ and b' on $\mathbb{Z}^{d'}$ are* stably equivalent *if there exist nonnegative integers r, s, r', s' such that*

$$b \oplus \langle 1\rangle^{\oplus r} \oplus \langle -1\rangle^{\oplus s} \;\cong\; b' \oplus \langle 1\rangle^{\oplus r'} \oplus \langle -1\rangle^{\oplus s'}.$$

Here $\langle 1\rangle$ denotes the rank-1 form $(x, y) \mapsto xy$, and $\langle -1\rangle$ denotes its negative. The direct sum $\oplus$ denotes orthogonal direct sum.

Stable equivalence is an equivalence relation on bilinear forms. Its significance for us is that it relates the intersection forms of different bounding four-manifolds for the same three-manifold M.

Lemma 7.2 (Effect of stabilisation on TJF^b)**.** *For any bilinear form b, we have*

$$\mathrm{TJF}^{b\oplus\langle 1\rangle} \simeq \mathrm{TJF}^b[-2], \qquad \mathrm{TJF}^{b\oplus\langle -1\rangle} \simeq \mathrm{TJF}^b[3].$$

Proof. By additivity (Proposition 3.23), $\mathrm{TJF}^{b\oplus\langle 1\rangle} \simeq \mathrm{TJF}^b \otimes \mathrm{TJF}^{(1)}$. From Proposition 6.8, $\mathrm{TJF}^{(1)} \simeq \mathrm{TMF}[-2]$. Hence $\mathrm{TJF}^{b\oplus\langle 1\rangle} \simeq \mathrm{TJF}^b[-2]$. The case for $\langle -1\rangle$ follows similarly from $\mathrm{TJF}^{(-1)} \simeq \mathrm{TMF}[3]$. □

Thus, adding $\langle 1\rangle$ shifts the module by $[-2]$, and adding $\langle -1\rangle$ shifts it by $[3]$.

Corollary 7.3. *If b and b' are stably equivalent, then there exist integers k, ℓ such that*

$$\mathrm{TJF}^b[k] \simeq \mathrm{TJF}^{b'}[\ell]$$

in $\mathrm{Ho}(\mathrm{Mod}_{\mathrm{TMF}})$*. In particular,* TJF^b *and* $\mathrm{TJF}^{b'}$ *become isomorphic after appropriate shifts.*

7.0.2 Bounding four-manifolds and their intersection forms

A classical result in low-dimensional topology, due to Lickorish and Wallace, states that every closed oriented three-manifold bounds a simply-connected oriented four-manifold.

Theorem 7.4 (Lickorish–Wallace)**.** *Let M be a closed oriented three-manifold. Then there exists a simply-connected oriented four-manifold W with $\partial W = M$. Moreover, W can be chosen to have a handle decomposition consisting only of* 0-, 2-, *and* 4-*handles.*

Proof. The Lickorish–Wallace theorem [16, 26] states that every closed oriented three-manifold can be obtained by surgery on a framed link $L \subset S^3$. Let L have k components, each with an integer framing $n_i \in \mathbb{Z}$. Attach k two-handles to D^4 along L with the specified framings. The resulting four-manifold W has boundary equal to the result of the surgery, which is M. The fundamental group of W is generated by the meridians of the link components, but each meridian is killed by the corresponding two-handle because the attaching circle is null-homotopic in D^4 and the two-handle adds a relation. Hence $\pi_1(W) = 0$. The handle decomposition consists of one 0-handle (D^4), k two-handles, and one 4-handle to close off the boundary. For a really nice exposition, see [12, Chapter 5]. □

Let W be such a bounding four-manifold. Its intersection form

$$b(W) : H_2(W;\mathbb{Z}) \otimes H_2(W;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

is a symmetric bilinear form on a finitely generated free abelian group. Indeed, since W is simply-connected, $H_1(W) = 0$, and by the universal coefficient theorem, $\mathrm{Tor}(H_2(W;\mathbb{Z}),\mathbb{Z}) \cong H_1(W) = 0$, so $H_2(W;\mathbb{Z})$ is torsion-free, hence free. Thus $b(W)$ is an object of the category Bil defined in Chapter 5.

7.0.3 The map from three-manifolds to stable equivalence classes

We now construct a map from the set of closed oriented three-manifolds to the set of stable equivalence classes of bilinear forms.

Definition 7.5 (Stable intersection form of a three-manifold)**.** *Let M be a closed oriented three-manifold. Choose any simply-connected oriented four-manifold W with $\partial W = M$ (such a W exists by Theorem 7.4). Let $b(W)$ be its intersection form. Define*

$$\sigma(M) := [b(W)],$$

the stable equivalence class of $b(W)$.

For this to be well-defined, we must show that $\sigma(M)$ does not depend on the choice of W.

Lemma 7.6. *Let M be a closed oriented three-manifold, and let W and W' be two simply-connected oriented four-manifolds with $\partial W = \partial W' = M$. Then the intersection forms $b(W)$ and $b(W')$ are stably equivalent.*

Proof. We give a proof using Kirby calculus, as this is the most transparent for the purposes of constructing a TQFT.

Step 1: Framed link presentations. By Theorem 7.4, W and W' can be represented by framed links L and L' in S^3. Specifically, W is obtained from D^4 by attaching two-handles along L, and similarly W' from L'. The boundaries $\partial W = M$ and $\partial W' = M$ are the three-manifolds obtained by surgery on L and L', respectively.

Step 2: Kirby moves. Since L and L' give the same surgery result M, they are related by a finite sequence of *Kirby moves* [12]. There are two types of Kirby moves:

- **Handle slides**: A handle slide replaces a framed link component by its band sum with another component. This move corresponds to an isomorphism of the intersection forms of the corresponding four-manifolds (it changes the basis of H_2 by an element of $\mathrm{GL}_n(\mathbb{Z})$).
- **Stabilisation/destabilisation**: Adding or removing a ±1-framed unknot that is split from the rest of the link. Adding a $+1$-framed unknot corresponds to taking the connected sum with $\mathbb{CP}^2$, which adds a $\langle 1\rangle$ summand to the intersection form. Adding a -1-framed unknot corresponds to taking the connected sum with $\overline{\mathbb{CP}}^2$, which adds a $\langle -1\rangle$ summand.

Step 3: Effect on intersection forms. Let b and b' be the intersection forms of W and W', respectively. A sequence of Kirby moves relating L and L' gives a sequence of operations on b and b':

- Each handle slide corresponds to an isomorphism $b \cong b$ (change of basis), which does not change the isomorphism class of b.
- Each stabilisation (adding a ±1-framed unknot) corresponds to replacing b by $b \oplus \langle \pm 1 \rangle$.
- Each destabilisation (removing a ±1-framed unknot) corresponds to the inverse operation.

Therefore, after a finite sequence of such operations, b can be transformed into b'. This means that there exist nonnegative integers r, s, r', s' such that

$$b \oplus \langle 1 \rangle^{\oplus r} \oplus \langle -1 \rangle^{\oplus s} \cong b' \oplus \langle 1 \rangle^{\oplus r'} \oplus \langle -1 \rangle^{\oplus s'}.$$

Hence b and b' are stably equivalent. For a detailed treatment of Kirby calculus and its relation to intersection forms, see [12, Chapter 5] and [7, Appendix A]. □

Thus $\sigma(M)$ is well-defined.

7.0.4 Definition of the three-manifold invariant

We can now define the GKMP invariant $\mathcal{Z}(M)$. The following definition matches [7, Equation 7.1].

Definition 7.7. *Let M be a closed oriented three-manifold. Choose a simply-connected oriented four-manifold W with $\partial W = M$ (by Theorem 7.4). Let $b = b(W)$ be its intersection form, with signature (b^+, b^-). Define*

$$\mathcal{Z}(M) := \overline{\mathrm{TJF}}_b[3b^+ - 2b^-] \quad \in \mathrm{Ho}(\mathrm{Mod}_{\mathrm{TMF}}),$$

where $\overline{\mathrm{TJF}}_b = (\mathrm{TJF}^{-b})^\vee$ is the homology version of the higher topological Jacobi form.

Using the duality theorem (Theorem 6.10), $\overline{\mathrm{TJF}}_b \simeq \mathrm{TJF}^b[-\operatorname{rank}(b)]$, we obtain an equivalent formula:

$$\mathcal{Z}(M) \simeq \mathrm{TJF}^b[3b^+ - 2b^- - \operatorname{rank}(b)].$$

7.0.5 Well-definedness: independence of the choice of W

We now prove that $\mathscr{Z}(M)$ does not depend on the choice of bounding four-manifold W.

Theorem 7.8 (Well-definedness of $\mathscr{Z}(M)$)**.** *Let M be a closed oriented three-manifold. For any two simply-connected oriented four-manifolds W and W' with $\partial W = \partial W' = M$, we have*

$$\overline{\mathrm{TJF}}_{b(W)}[3b(W)^+ - 2b(W)^-] \;\simeq\; \overline{\mathrm{TJF}}_{b(W')}[3b(W')^+ - 2b(W')^-]$$

in $\mathrm{Ho}(\mathrm{Mod}_{\mathrm{TMF}})$*. Hence $\mathscr{Z}(M)$ is well-defined.*

Proof. Let $b = b(W)$ and $b' = b(W')$. By Lemma 7.6, b and b' are stably equivalent. Hence there exist nonnegative integers r, s, r', s' such that

$$b \oplus \langle 1\rangle^{\oplus r} \oplus \langle -1\rangle^{\oplus s} \;\cong\; b' \oplus \langle 1\rangle^{\oplus r'} \oplus \langle -1\rangle^{\oplus s'}. \tag{6.1}$$

Step 1: Relate the signatures and ranks. From (6.1), we have:

$$\mathrm{rank}(b') = \mathrm{rank}(b) + r + s - r' - s', \tag{6.2}$$

$$(b')^+ = b^+ + r - r', \tag{6.3}$$

$$(b')^- = b^- + s - s'. \tag{6.4}$$

Step 2: Apply $\mathrm{TJF}^{(-)}$ to both sides of (6.1). Using additivity (Proposition 3.23) and Lemma 7.2, we obtain

$$\mathrm{TJF}^b[-2r + 3s] \;\simeq\; \mathrm{TJF}^{b'}[-2r' + 3s']. \tag{6.5}$$

Step 3: Express $\mathscr{Z}(M)$ in terms of TJF^b. Using the duality theorem, $\overline{\mathrm{TJF}}_b \simeq \mathrm{TJF}^b[-\mathrm{rank}(b)]$. Therefore,

$$\mathscr{Z}(M)_W \simeq \mathrm{TJF}^b[3b^+ - 2b^- - \mathrm{rank}(b)]. \tag{6.6}$$

Similarly, using W':

$$\mathcal{Z}(M)_{W'} \simeq \mathrm{TJF}^{b'}[3(b')^+ - 2(b')^- - \mathrm{rank}(b')]. \tag{6.7}$$

Step 4: Substitute the relations (6.2)-(6.4) into (6.7).

$$\begin{aligned}\mathcal{Z}(M)_{W'} &\simeq \mathrm{TJF}^{b'}[3(b^+ + r - r') - 2(b^- + s - s') - (\mathrm{rank}(b) + r + s - r' - s')] \\ &= \mathrm{TJF}^{b'}[3b^+ - 2b^- - \mathrm{rank}(b) + (3r - 3r' - 2s + 2s' - r - s + r' + s')] \\ &= \mathrm{TJF}^{b'}[3b^+ - 2b^- - \mathrm{rank}(b) + (2r - 2r' - 3s + 3s')]. \end{aligned} \tag{6.8}$$

Step 5: Use the stable equivalence relation (6.5). From (6.5), $\mathrm{TJF}^{b'} \simeq \mathrm{TJF}^{b}[-2r + 3s + 2r' - 3s']$. Substitute into (6.8):

$$\begin{aligned}\mathcal{Z}(M)_{W'} &\simeq \mathrm{TJF}^{b}[-2r + 3s + 2r' - 3s'] \otimes [3b^+ - 2b^- - \mathrm{rank}(b) + (2r - 2r' - 3s + 3s')] \\ &= \mathrm{TJF}^{b}[3b^+ - 2b^- - \mathrm{rank}(b) + (-2r + 3s + 2r' - 3s' + 2r - 2r' - 3s + 3s')] \\ &= \mathrm{TJF}^{b}[3b^+ - 2b^- - \mathrm{rank}(b) + 0].\end{aligned}$$

The terms in r, s, r', s' cancel completely. Therefore,

$$\mathcal{Z}(M)_{W'} \simeq \mathrm{TJF}^{b}[3b^+ - 2b^- - \mathrm{rank}(b)] \simeq \mathcal{Z}(M)_W.$$

Thus $\mathcal{Z}(M)$ is independent of the choice of bounding four-manifold W. □

7.0.6 Examples

Example 7.9 (The three-sphere)**.** *For $M = S^3$, we can take $W = D^4$, the four-ball. Then $b(W)$ is the zero form on $\mathbb{Z}^0$, with $b^+ = b^- = 0$ and $\mathrm{rank} = 0$. Hence*

$$\mathcal{Z}(S^3) \simeq \overline{\mathrm{TJF}}_0[0] \simeq \mathrm{TJF}^0[0] \simeq \mathrm{TMF} \oplus \mathrm{TMF}[1].$$

Example 7.10 ($S^2 \times S^1$)**.** *For $M = S^2 \times S^1$, we can take W to be D^4 with a single 0-framed 2-handle attached to the unknot. Then $b(W) = (0)$, the zero form on $\mathbb{Z}$, with*

$b^+ = b^- = 0$, rank $= 1$. *Thus*

$$\mathcal{Z}(S^2 \times S^1) \simeq \overline{\mathrm{TJF}}_{(0)}[0] \simeq (\mathrm{TJF}^{(0)})^\vee[0] \simeq (\mathrm{TMF} \oplus \mathrm{TMF}[1])^\vee \simeq \mathrm{TMF} \oplus \mathrm{TMF}[-1].$$

Example 7.11 ($S^2 \times S^2$)**.** *For $M = \emptyset$ (the empty set as a boundary), we consider $W = S^2 \times S^2$. Its intersection form is the hyperbolic form $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with $b^+ = b^- = 1$, rank $= 2$. Then*

$$\mathcal{Z}(S^2 \times S^2) \simeq \overline{\mathrm{TJF}}_H[3 \cdot 1 - 2 \cdot 1] = \overline{\mathrm{TJF}}_H[1].$$

Since H is unimodular, $\overline{\mathrm{TJF}}_H \simeq \mathrm{TJF}^H[-2] \simeq \mathrm{TMF}[3 \cdot 1 - 2 \cdot 1 - 2] = \mathrm{TMF}[-1]$. Thus $\mathcal{Z}(S^2 \times S^2) \simeq \mathrm{TMF}[-1][1] = \mathrm{TMF}$. This means that the invariant is an element of $\pi_*\mathrm{TMF}$ (specifically $\eta \in \pi_1\mathrm{TMF}$), not a module, because the boundary is empty. The module TMF is the ground ring, and the element η corresponds to the specific map $\mathrm{TMF} \to \mathrm{TMF}$ induced by the cobordism.

7.0.7 Open questions: Functoriality and the Kirby category

The construction above defines $\mathcal{Z}(M)$ as an object in $\mathrm{Ho}(\mathrm{Mod}_{\mathrm{TMF}})$ for each closed oriented three-manifold M. A natural question is whether $\mathcal{Z}$ extends to a functor on the bordism category of three-manifolds. That is, given a four-manifold cobordism $W : M_0 \to M_1$, can we define a morphism $\mathcal{Z}(W) : \mathcal{Z}(M_0) \to \mathcal{Z}(M_1)$ in a functorial way?

This is precisely Questions 7.4 and 7.6 in [7]. The difficulty is that different handle decompositions of the same cobordism can lead to different sequences of stabilisation maps, and one must show that these give the same morphism in $\mathrm{Ho}(\mathrm{Mod}_{\mathrm{TMF}})$ up to the natural identifications coming from stable equivalence. A positive answer would require a deep understanding of the behaviour of TJF^b under the Kirby moves (handle slides and stabilisation/destabilisation). This remains an open problem, though the computations in this thesis provide evidence that such a functor exists.

Remark 7.12 (On the category $\mathrm{Bil}_\pm$)**.** *Some treatments of this material introduce a category* $\mathrm{Bil}_\pm$ *in which stabilisation maps are formally inverted, with the goal of making* $\mathrm{TJF}^{(-)}$ *a functor. However,* $\mathrm{TJF}^{(-)}$ *does not descend to a functor on such a category because stabilisation maps do not become isomorphisms under* $\mathrm{TJF}^{(-)}$*; they become shifts. The correct framework is to work with the* graded *homotopy category, where shifts are invertible, or to keep track of shifts explicitly as we have done here. The category* $\mathrm{Bil}_\pm$ *is therefore not needed for the definition of* $\mathcal{Z}(M)$*, though it may be a useful organisational tool for understanding stable equivalence.*

Remark 7.13. *The question of whether* $\mathcal{Z}$ *extends to a functor on the bordism category (i.e., whether four-manifold cobordisms induce morphisms of* TMF*-modules in a functorial way) is more subtle and remains an open problem (see GKMP Questions 7.4 and 7.6). The computations in this thesis provide a foundation for future work on this question.*

Chapter 8

Explicit Computations of TJF^b for Rank 2 Forms

We now compute TJF^b explicitly for each stable equivalence class identified in Theorem 5.4. These computations are the main results of this note.

8.0.1 Indefinite forms with signature $(1, 1)$

Example 8.1 (H_{even} and H_{odd})**.** *Both forms are unimodular with signature* $(1, 1)$*. By Proposition 6.1,*

$$\mathrm{TJF}^{H_{\text{even}}} \simeq \mathrm{TJF}^{H_{\text{odd}}} \simeq \mathrm{TMF}[1].$$

However, they are distinguished by the pointed element $\mathfrak{d}_b$*. In [7, Example 8.1], it is shown that* $\mathfrak{d}_{H_{\text{even}}} = \eta \in \pi_1\mathrm{TMF}$*, while* $\mathfrak{d}_{H_{\text{odd}}} = 0$*.*

Example 8.2 ($\langle 2\rangle \oplus \langle -2\rangle$)**.** *This form is stably equivalent to* H_{even} *(both have signature* $(1, 1)$ *and trivial linking form). By stable equivalence invariance,*

$$\mathrm{TJF}^{\langle 2\rangle\oplus\langle -2\rangle} \simeq \mathrm{TJF}^{H_{\text{even}}} \simeq \mathrm{TMF}[1].$$

8.0.2 Positive definite forms with determinant 2

Example 8.3 ($\langle 1\rangle \oplus \langle 2\rangle$)**.** *Using additivity and Proposition 6.8:*

$$\mathrm{TJF}^{(1)\oplus(2)} \simeq \mathrm{TJF}^{(1)} \otimes_{\mathrm{TMF}} \mathrm{TJF}^{(2)} \simeq \mathrm{TMF}[-2] \otimes_{\mathrm{TMF}} (\nu : \mathrm{TMF}[3] \to \mathrm{TMF})[-4].$$

Tensoring the cofiber sequence $\mathrm{TMF}[3] \xrightarrow{\nu} \mathrm{TMF} \to \mathrm{TJF}^{(2)}[4]$ *with* $\mathrm{TMF}[-2]$ *yields*

$$\mathrm{TMF}[1] \xrightarrow{\nu} \mathrm{TMF}[-2] \to \mathrm{TJF}^{(1)\oplus(2)}[4].$$

Hence

$$\mathrm{TJF}^{\langle 1\rangle\oplus\langle 2\rangle} \simeq (\nu : \mathrm{TMF}[1] \to \mathrm{TMF}[-2])[-4].$$

Let $M = (\nu : \mathrm{TMF}[1] \to \mathrm{TMF}[-2])$*. The long exact sequence in homotopy gives:*

$$\cdots \to \pi_{n+1}\mathrm{TMF} \xrightarrow{\nu} \pi_{n-2}\mathrm{TMF} \to \pi_n M \to \pi_{n+2}\mathrm{TMF} \xrightarrow{\nu} \pi_{n-1}\mathrm{TMF} \to \cdots$$

Thus $\pi_n M$ *is an extension:*

$$0 \to (\nu : \pi_{n+1}\mathrm{TMF} \to \pi_{n-2}\mathrm{TMF}) \to \pi_n M \to \ker(\nu : \pi_{n+2}\mathrm{TMF} \to \pi_{n-1}\mathrm{TMF}) \to 0.$$

After shifting by $[-4]$*, this gives the homotopy groups of* $\mathrm{TJF}^{\langle 1\rangle\oplus\langle 2\rangle}$.

8.0.3 Positive definite forms with determinant 3

The diagonal case $\langle 1\rangle \oplus \langle 3\rangle$

By additivity,

$$\mathrm{TJF}^{\langle 1\rangle\oplus\langle 3\rangle} \simeq \mathrm{TJF}^{\langle 1\rangle} \otimes_{\mathrm{TMF}} \mathrm{TJF}^{\langle 3\rangle} \simeq \mathrm{TMF}[-2] \otimes_{\mathrm{TMF}} (\mathrm{TMF} \wedge \mathbb{CP}^2[1] \to \mathrm{TMF})[-6].$$

Thus $\mathrm{TJF}^{\langle 1\rangle\oplus\langle 3\rangle} \simeq (\mathrm{TMF} \wedge \mathbb{CP}^2[1] \to \mathrm{TMF})[-8]$. This module has length 3 (i.e., three nontrivial summands in its associated graded), reflecting that $|\det| = 3$.

The non-diagonal case $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

This is the most interesting case. The linking form is $(\mathbb{Z}/3, 2/3)$, which is not isometric to the diagonal case. By Theorem 5.4, this form is not stably equivalent to any diagonal form of rank 2. However, we can stabilize it to a diagonal form of higher rank.

Consider $b \oplus \langle 1\rangle \oplus \langle -1\rangle$. This has rank 4, signature $(3, 1)$, and linking form $(\mathbb{Z}/3, 2/3)$. The diagonal form $b' = \langle 1\rangle \oplus \langle 1\rangle \oplus \langle 1\rangle \oplus \langle -3\rangle$ has the same signature and linking form. By Nikulin's theorem, they are stably equivalent. In fact, we can show that they're isomorphic by means of the matrix $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \end{pmatrix}$ Hence

$$\mathrm{TJF}^{b\oplus\langle 1\rangle\oplus\langle -1\rangle} \simeq \mathrm{TJF}^{\langle 1\rangle^{\oplus 3}\oplus\langle -3\rangle}.$$

Using additivity and the fact that $\mathrm{TJF}^{(1)} \simeq \mathrm{TMF}[-2]$ and $\mathrm{TJF}^{(-1)} \simeq \mathrm{TMF}[3]$, we obtain

$$\mathrm{TJF}^b \otimes \mathrm{TMF}[-2] \otimes \mathrm{TMF}[3] \simeq (\mathrm{TMF}[-2])^{\otimes 3} \otimes \mathrm{TJF}^{(-3)}.$$

Simplifying,

$$\mathrm{TJF}^b \otimes \mathrm{TMF}[1] \simeq \mathrm{TMF}[-6] \otimes \mathrm{TJF}^{(-3)}.$$

Now use the duality relation (Theorem 6.10) for rank 1: $\overline{\mathrm{TJF}}_3 \simeq \mathrm{TJF}^{(3)}[-1]$. But $\overline{\mathrm{TJF}}_3 \simeq (\mathrm{TJF}^{(-3)})^\vee$. Therefore,

$$\mathrm{TJF}^{(-3)} \simeq (\mathrm{TJF}^{(3)})^\vee[1].$$

Substituting,

$$\mathrm{TJF}^b \otimes \mathrm{TMF}[1] \simeq \mathrm{TMF}[-6] \otimes (\mathrm{TJF}^{(3)})^\vee[1].$$

Hence

$$\mathrm{TJF}^{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \simeq \mathrm{TMF}[-7] \otimes (\mathrm{TJF}^{(3)}[-1])^\vee.$$

This explicit formula shows how the nontrivial linking form $2/3$ manifests itself: the appearance of the dual $(\mathrm{TJF}^{(3)})^\vee$ rather than $\mathrm{TJF}^{(3)}$ reflects the fact that $2/3 \equiv -1/3$ modulo $\mathbb{Z}$, and the sign is absorbed by the duality.

8.0.4 Positive definite forms with determinant 4

Even case: $\langle 2\rangle \oplus \langle 2\rangle$

Using additivity,

$$\mathrm{TJF}^{(2)\oplus(2)} \simeq \mathrm{TJF}^{(2)} \otimes_{\mathrm{TMF}} \mathrm{TJF}^{(2)}.$$

Let $X = (\nu : \mathrm{TMF}[3] \to \mathrm{TMF})$, so that $\mathrm{TJF}^{(2)} \simeq X[-4]$. Then

$$\mathrm{TJF}^{(2)\oplus(2)} \simeq (X \otimes_{\mathrm{TMF}} X)[-8].$$

The tensor product $X \otimes_{\mathrm{TMF}} X$ fits into a cofiber sequence obtained by tensoring the defining cofiber sequence of X with X:

$$X[3] \xrightarrow{\nu\otimes 1_X} X \to X \otimes_{\mathrm{TMF}} X \to X[4].$$

Taking homotopy groups, one finds that $X \otimes_{\mathrm{TMF}} X$ has a direct sum decomposition corresponding to the two $\mathbb{Z}/2$ factors in the discriminant group. Since the linking form on $(\mathbb{Z}/2)^2$ is split, the module decomposes as a direct sum of two copies of $\mathrm{TJF}^{(2)}$ (up to shift). More precisely,

$$\pi_*(X \otimes_{\mathrm{TMF}} X) \cong (\pi_*\mathrm{TMF}[-4] \oplus \pi_*\mathrm{TMF}[-6]) \oplus (\pi_*\mathrm{TMF}[-8] \oplus \pi_*\mathrm{TMF}[-10]).$$

Odd case: $\langle 1\rangle \oplus \langle 4\rangle$

By additivity,

$$\mathrm{TJF}^{(1)\oplus(4)} \simeq \mathrm{TMF}[-2]\otimes_{\mathrm{TMF}}(\mathrm{TMF}\wedge\mathbb{CP}^3[1] \to \mathrm{TMF})[-8] \simeq (\mathrm{TMF}\wedge\mathbb{CP}^3[1] \to \mathrm{TMF})[-10].$$

Let $Y = (\mathrm{TMF}\wedge\mathbb{CP}^3[1] \to \mathrm{TMF})$. The cellular filtration of $\mathbb{CP}^3$ (cells in dimensions 0, 2, 4, 6) induces a filtration on Y with associated graded pieces:

$$\mathrm{gr}_0 Y \simeq \mathrm{TMF}[-8], \quad \mathrm{gr}_1 Y \simeq \mathrm{TMF}[-6], \quad \mathrm{gr}_2 Y \simeq \mathrm{TMF}[-4], \quad \mathrm{gr}_3 Y \simeq \mathrm{TMF}[-2].$$

After shifting by $[-10]$, $\mathrm{TJF}^{(1)\oplus(4)}$ has associated graded pieces in degrees $-18, -16, -14, -12$. The nontrivial cyclic linking form on $\mathbb{Z}/4$ forces nontrivial extensions in the spectral sequence, making this module indecomposable of length 4.

Chapter 9

Applications to Abelian Chern-Simons Theories

In this section, we interpret our classification of TJF^b for rank 2 bilinear forms with $|\det b| \leq 4$ in terms of the conjectural relationship between TMF-modules and three-dimensional abelian Chern–Simons theories outlined in [7, Section 10].

Remark 9.1 (Cautionary Note)**.** *The discussion in this section is largely heuristic and relies on conjectural relationships between quantum field theory and homotopy theory that have not been rigorously established. The Stolz–Teichner conjecture, Conjecture A of [7], and the identification of* $\mathcal{B}(\mathrm{CS}_b)$ *with* $\overline{\mathrm{TJF}}_b$ *are all currently mathematical conjectures. We present them here as guiding principles and as motivation for the algebraic constructions in the preceding sections. The reader should not mistake these physical arguments for rigorous mathematics; rather, they provide a rich source of predictions and a conceptual framework for understanding the algebraic structures we have computed.*

9.0.1 Mathematical preliminaries on Chern–Simons theory

We begin with a self-contained mathematical description of abelian Chern–Simons theory, treating it as a functorial topological field theory. This description avoids physical language as much as possible, though some physical terminology is unavoidable given the origins of the subject.

The Chern–Simons action as a differential character

Let M be a closed oriented three-dimensional manifold. Let $P \to M$ be a principal $U(1)^r$-bundle. Since $U(1)^r$ is a torus, isomorphism classes of such bundles are classified by $H^2(M;\mathbb{Z}^r)$. For simplicity, we will restrict to trivial bundles; this is sufficient for defining the theory on closed manifolds when the level is integral, as we assume.

Let $\mathcal{A}$ denote the space of connections on the trivial $U(1)^r$-bundle over M. A connection can be identified with a 1-form $A = (A^1, \dots, A^r)$ with values in the Lie algebra $\mathfrak{u}(1)^r \cong \mathbb{R}^r$. The curvature is $F = dA$, a closed 2-form.

Definition 9.2. *Let $b : \mathbb{Z}^r \otimes \mathbb{Z}^r \to \mathbb{Z}$ be a symmetric bilinear form, represented by a symmetric matrix $B = (b_{ij})$. The* Chern–Simons action *is the functional $S_{\mathrm{CS}} : \mathcal{A} \to \mathbb{R}/\mathbb{Z}$ defined by*

$$S_{\mathrm{CS}}(A) = \frac{1}{4\pi} \int_M \sum_{i,j=1}^{r} b_{ij}\, A^i \wedge dA^j.$$

The integral is taken over M, and the result is considered modulo $\mathbb{Z}$ because the action appears in the path integral as $e^{2\pi i S_{\mathrm{CS}}(A)}$.

Lemma 9.3. *Under a gauge transformation $A \mapsto A + d\phi$ for $\phi : M \to \mathbb{R}^r$, the Chern–Simons action changes by*

$$S_{\mathrm{CS}}(A + d\phi) - S_{\mathrm{CS}}(A) = \frac{1}{4\pi} \int_{\partial M} \sum_{i,j} b_{ij}\, \phi^i dA^j + \frac{1}{4\pi} \int_M \sum_{i,j} b_{ij}\, d\phi^i \wedge d\phi^j.$$

If $\partial M = \emptyset$, the first term vanishes. The second term is an integer multiple of $1/2$ in general, but if b is integral (i.e., $b_{ij} \in \mathbb{Z}$), it is an integer. Hence $e^{2\pi i S_{\mathrm{CS}}(A)}$ is gauge-invariant.

Sketch. The computation is a straightforward application of Stokes' theorem and the fact that $\int_M d\phi^i \wedge d\phi^j = \int_{\partial M} \phi^i d\phi^j = 0$ when $\partial M = \emptyset$. The integral $\int_M d\phi^i \wedge d\phi^j$ is an integer multiple of 2π when ϕ is a map to S^1, so the factor of $1/(4\pi)$ yields a half-integer. The integrality of b ensures the total is an integer. □

The partition function as a Gaussian integral

For a closed oriented three-manifold M, the *partition function* of the Chern–Simons theory is formally defined by the path integral

$$Z_{\mathrm{CS}_b}(M) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i S_{\mathrm{CS}}(A)}\, \mathcal{D}A,$$

where $\mathcal{G}$ is the gauge group and $\mathcal{D}A$ is a (nonexistent) Lebesgue measure on the infinite-dimensional space of connections modulo gauge transformations. This integral is not mathematically well-defined in general, but for abelian Chern–Simons theory, it can be defined using the method of Gaussian integration and Reidemeister torsion.

Definition 9.4. *For a closed oriented three-manifold M with $H_1(M) = 0$ (i.e., a homology sphere), the partition function of CS_b is given by*

$$Z_{\mathrm{CS}_b}(M) = \frac{1}{\sqrt{|\det b|^{b_1(M)}}} \sum_{\alpha \in H^1(M;A_b)} e^{2\pi i \langle \alpha \cup \alpha, [M]\rangle/2},$$

where the cup product is interpreted using the linking form. For $b_1(M) > 0$, the formula involves a sum over the torsion linking form.

Remark 9.5. *This formula can be made rigorous using the theory of Gaussian integration on finite-dimensional spaces after gauge-fixing. The result is a topological invariant of M that depends only on the linking form. For a mathematical treatment, see [27].*

9.0.2 The modular tensor category associated to CS_b

When one studies Chern–Simons theory on manifolds with boundary, one is led to assign algebraic data to surfaces. In particular, to a closed oriented surface Σ, one associates a finite-dimensional complex vector space $\mathcal{H}_{\mathrm{CS}_b}(\Sigma)$, called the *state space*. For the torus $\Sigma = T^2$, this vector space has a distinguished basis labeled by the discriminant group A_b.

The discriminant group as a label set for simple objects

Definition 9.6. *Let* $b : \mathbb{Z}^r \otimes \mathbb{Z}^r \to \mathbb{Z}$ *be a nondegenerate symmetric bilinear form with matrix* B*. The* discriminant group *is the finite abelian group*

$$A_b := \operatorname{coker}(b_{\mathrm{adj}}) = \mathbb{Z}^r / B\mathbb{Z}^r,$$

where $b_{\mathrm{adj}} : \mathbb{Z}^r \to (\mathbb{Z}^r)^* \cong \mathbb{Z}^r$ *is the adjoint map* $x \mapsto b(x, -)$*. The order of* A_b *is* $|\det B|$ *when* b *is nondegenerate.*

Flat connections on the torus and the discriminant group

Let us explain how the discriminant group A_b appears naturally in the quantisation of abelian Chern–Simons theory on the torus $T^2 = S^1 \times S^1$.

A flat $U(1)^r$-connection on T^2 is determined by its holonomies around the two generating cycles. Since $\pi_1(T^2) \cong \mathbb{Z}^2$, the space of flat connections modulo gauge transformations is

$$\operatorname{Hom}(\pi_1(T^2), U(1)^r) \cong \operatorname{Hom}(\mathbb{Z}^2, U(1)^r) \cong (U(1)^r)^2.$$

Explicitly, a flat connection is given by two vectors $\theta, \phi \in (\mathbb{R}/\mathbb{Z})^r$ representing the holonomies.

In the Hamiltonian formulation, the Chern–Simons theory on $T^2 \times \mathbb{R}$ has a finite-dimensional Hilbert space. The level matrix b imposes a quantization condition: the holonomies are not independent but must satisfy a certain compatibility condition. More precisely, the physical states correspond to those flat connections for which the exponentiated action is well-defined. This condition picks out a Lagrangian subspace of $(U(1)^r)^2$ with respect to the symplectic form determined by b. The resulting finite set of states is naturally identified with the discriminant group

$$A_b = \mathbb{Z}^r / B\mathbb{Z}^r.$$

For a detailed derivation, see [27]. The key point is that the holonomies are quantised to values in the finite abelian group A_b, which is why the simple objects of the modular tensor category are labelled by A_b.

The linking form and modular data

The linking form $\lambda_b : A_b \times A_b \to \mathbb{Q}/\mathbb{Z}$ determines the modular S and T matrices.

Modular tensor categories

We recall the definition of a modular tensor category, following [2].

Definition 9.7. *A* modular tensor category *is a semisimple ribbon category (i.e., a braided category with a twist) with a finite set of simple objects, such that the braiding is nondegenerate. Nondegeneracy means that the S-matrix, defined by*

$$S_{ij} = \mathrm{tr}(c_{j,i} \circ c_{i,j}),$$

where $c_{i,j} : V_i \otimes V_j \to V_j \otimes V_i$ is the braiding, is invertible over the ground field (typically $\mathbb{C}$).

For abelian Chern–Simons theory CS_b, the simple objects are labeled by the discriminant group A_b, the braiding is determined by the linking form λ_b, and the modular S and T matrices that we define now.

Definition 9.8. *For $x, y \in A_b$, define*

$$T_{x,y} := e^{\pi i \lambda_b(x,x)} \delta_{x,y}, \qquad S_{x,y} := \frac{1}{\sqrt{|A_b|}} e^{-2\pi i \lambda_b(x,y)}.$$

These matrices are unitary and satisfy the relations

$$S^2 = (ST)^3 = C, \qquad C^2 = 1,$$

where $C_{x,y} = \delta_{x,-y}$ is the charge conjugation matrix. In particular, they form a representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$ up to the central element C.

Sketch of the modular relations. The relations follow from the properties of the linking form: symmetry, nondegeneracy, and the fact that $\lambda_b(x,x) \equiv \lambda_b(-x,-x) \pmod{\mathbb{Z}}$. The verification is a standard computation in the theory of finite abelian groups and Gauss sums. □

Definition 9.9. *The collection of data* $\{A_b, \lambda_b, S, T\}$ *defines a* modular tensor category $\mathcal{C}_b$. *The simple objects are labeled by* A_b*, the fusion rules are given by the group law on* A_b *(since the theory is abelian), and the braiding and twist are encoded in* S *and* T*.*

Example 9.10 ($U(1)_k$)**.** *For* $r = 1$ *and* $b = (k)$*, we have* $A_b = \mathbb{Z}/k\mathbb{Z}$*. The linking form is* $\lambda(x,y) = xy/k$*. Then*

$$T_{x,y} = e^{\pi i x^2/k}\delta_{x,y}, \qquad S_{x,y} = \frac{1}{\sqrt{k}} e^{-2\pi i xy/k}.$$

These are the modular matrices for the $\widehat{U(1)}_k$ *Wess–Zumino–Witten model. This is a well-studied modular tensor category, often denoted* $\mathcal{C}(U(1)_k)$*.*

9.0.3 The Stolz–Teichner conjecture and its mathematical status

We now turn to the conjecture that connects TMF to two-dimensional quantum field theories. This is a deep and active area of research at the intersection of homotopy theory and mathematical physics.

Conjecture 9.11 (Stolz–Teichner [25])**.** *For each integer* $k \in \mathbb{Z}$*, let* $\mathcal{T}_k$ *denote the space of two-dimensional* $(0,1)$*-supersymmetric quantum field theories with gravitational anomaly* k *(i.e., with central charge* $c = 3k/2$*). Then there is a homotopy equivalence*

$$\Omega^\infty \mathrm{TMF}_k \simeq \mathcal{T}_k,$$

where TMF_k *is the* k*-th space in the* Ω*-spectrum* TMF *(so that* $\pi_0\Omega^\infty\mathrm{TMF}_k = \pi_k\mathrm{TMF}$*). Equivalently, the spectrum* TMF *is the classifying spectrum for deformation classes of such theories.*

Remark 9.12. *The original conjecture is often stated as* TMF $\simeq \mathcal{T}$*, where* $\mathcal{T}$ *is the spectrum whose* k*-th space is the space of theories with gravitational anomaly* k*. The more precise formulation above avoids conflating a spectrum with a single space.*

Remark 9.13 (What does this mean mathematically?)**.** *The right-hand side* $\mathcal{T}$ *is not a mathematical object in ordinary homotopy theory; it is a heuristic space whose points are isomorphism classes of* $(0,1)$*-supersymmetric field theories, with paths corresponding to continuous deformations. The conjecture asserts that this heuristic space has the homotopy type of the infinite loop space of the* TMF*-spectrum. This is an example of a "classification conjecture" that aims to give a homotopy-theoretic description of the moduli space of quantum field theories.*

Remark 9.14 (Evidence)**.** *There is substantial computational evidence supporting this conjecture. The Witten genus of a spin manifold takes values in modular forms, which are the homotopy groups of* TMF *after inverting* 2 *and* 3*. Moreover, both* TMF *and the conjectural classifying space of* $(0,1)$*-theories are* E_∞*-ring spectra and receive an orientation from* MSpin*. These formal properties are consistent with the conjecture. However, a complete mathematical proof remains elusive; the current status is that of a well-supported conjecture.*

9.0.4 Conjecture A: Boundary conditions as TMF-modules

Generalizing the Stolz–Teichner conjecture, [7] propose that for any three-dimensional quantum field theory $\mathfrak{T}$, the space of boundary conditions forms a module over TMF.

Definition 9.15 (Heuristic definition of boundary conditions)**.** *Let* $\mathfrak{T}$ *be a three-dimensional quantum field theory defined on oriented three-manifolds. A* boundary condition *for* $\mathfrak{T}$ *is a specification of the behaviour of the fields on a codimension-one* *boundary. Equivalently, it is a two-dimensional theory that can be coupled to* $\mathfrak{T}$ *along the boundary so that the combined system is well-defined.*

Conjecture 9.16 (Conjecture A of [7])**.** *Let* $\mathfrak{T}$ *be a three-dimensional* $(0,1)$*-supersymmetric quantum field theory (or, more generally, a three-dimensional topological field theory). Then the space* $\mathcal{B}(\mathfrak{T})$ *of two-dimensional* $(0,1)$*-supersymmetric boundary conditions for* $\mathfrak{T}$ *forms a module over* TMF*. The module structure is given by stacking: if* $B \in \mathcal{B}(\mathfrak{T})$ *is a boundary condition and* $E \in \mathrm{TMF}$ *is a two-dimensional theory (interpreted as a point in the space* $\mathcal{T}$*), then* $E \otimes B$ *(the theory obtained by placing* E *on top of* B*) is another boundary condition.*

Remark 9.17 (Mathematical status of Conjecture A)**.** *This conjecture is even more speculative than the Stolz–Teichner conjecture. It posits a functor from a hypothetical category of three-dimensional field theories to the category of* TMF*-modules. No rigorous construction of such a functor exists at present. The conjecture is motivated by:*

1. *The fact that* TMF *itself is the space of two-dimensional theories, so it should act on the space of boundary conditions by stacking.*

2.

   Remark 9.18 (Lower-dimensional analogues)**.** *The Stolz–Teichner conjecture fits into a larger pattern. For lower dimensions, analogous statements are known or conjectured:*

   - *For* 0*-dimensional* $(0,1)$*-supersymmetric theories, the classifying space is related to the spectrum* $HC[u^{\pm 1}]$ *representing* 2*-periodic de Rham cohomology; see [13] for details. The sphere spectrum* $\mathbb{S}$ *appears only in a heuristic sense.*

- *For* 1*-dimensional* $(0,1)$*-supersymmetric theories (i.e., supersymmetric quantum mechanics), the classifying space is conjectured to be* KO *(for real theories) or (for complex theories). Partial results are known, but a full proof is not yet available. See [24] for a discussion.*

Thus, while the pattern is clear and supported by evidence, rigorous proofs in all dimensions are still an active area of research.

Thus, while not proven, Conjecture A is a natural extrapolation of established patterns.

9.0.5 The boundary module for abelian Chern–Simons theory

For the specific case of an abelian Chern–Simons theory CS_b, [7] propose an explicit description of the boundary module in terms of the higher topological Jacobi forms we have been studying.

Conjecture 9.19 (Boundary module for CS_b Section 10.4 of [7])**.** *The* TMF*-module of* $(0,1)$*-supersymmetric boundary conditions for* CS_b *is equivalent to*

$$\mathcal{B}(\mathrm{CS}_b) \simeq \overline{\mathrm{TJF}}_b := \Gamma(\mathcal{L}_b^\vee)^\vee.$$

Remark 9.20 (Why this identification?)**.** *The justification for this conjecture comes from a combination of physical reasoning and mathematical analogy:*

1. *The partition function of* CS_b *on a three-manifold* M *can be expressed as a sum over* A_b*-valued flat connections. This sum is related to the linking form of* M*, which is exactly the data encoded in* $\mathcal{L}_b$*.*
2. *The Looijenga line bundle* $\mathcal{L}_b$ *arises naturally in the geometric quantisation of the moduli space of flat connections on a torus. Its global sections give the Hilbert space* $\mathcal{H}_{\mathrm{CS}_b}(T^2)$*, which is the vector space with basis* A_b*.*

3. *The "higher" version* $\overline{\mathrm{TJF}}_b$ *is a* TMF*-module refinement of this vector space, incorporating torsion information and the elliptic genus of the boundary theory.*

No rigorous derivation of this identification exists in the literature; it remains a conjecture.

Recall from Theorem 6.10 that $\overline{\mathrm{TJF}}_b$ is related to the cohomology module TJF^b by

$$\overline{\mathrm{TJF}}_b \simeq \mathrm{TJF}^b[-\operatorname{rank}(b)],$$

up to a sign. For rank 2 forms, this becomes $\overline{\mathrm{TJF}}_b \simeq \mathrm{TJF}^b[-2]$. Thus, our explicit computations of TJF^b give explicit descriptions of the conjectural boundary condition modules $\mathcal{B}(\mathrm{CS}_b)$ up to a shift.

9.0.6 Interpretation of the discriminant group and module structure

Our classification shows that TJF^b, and hence $\mathcal{B}(\mathrm{CS}_b)$, remembers strictly more information than the order $|A_b|$ alone. The structure of the module encodes the isomorphism class of the linking form (A_b, λ_b).

The split case: $\langle 2\rangle \oplus \langle 2\rangle$

Consider the form $b = \langle 2\rangle \oplus \langle 2\rangle$. Here:

$$A_b \cong \mathbb{Z}/2 \times \mathbb{Z}/2, \qquad \lambda((x_1, x_2), (y_1, y_2)) = \frac{1}{2}(x_1y_1 + x_2y_2).$$

This linking form is *split*: it decomposes as an orthogonal direct sum of two copies of the linking form on $\mathbb{Z}/2$. Concretely, there is an isomorphism of groups $A_b \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ such that $\lambda = \lambda_1 \oplus \lambda_2$ where each λ_i is the unique nontrivial linking form on $\mathbb{Z}/2$.

Lemma 9.21. *The form* $b = \langle 2\rangle \oplus \langle 2\rangle$ *is stably equivalent to the zero form. Consequently, it is stably equivalent to any unimodular form of signature* $(2, 0)$*, such as* $H_{\text{even}} \oplus H_{\text{even}}$*.*

Proof. The discriminant group of b is $A_b \cong (\mathbb{Z}/2)^2$ with linking form

$$\lambda((x_1, x_2), (y_1, y_2)) = \frac{1}{2}(x_1 y_1 + x_2 y_2) \pmod{\mathbb{Z}}.$$

We claim that λ is *metabolic*. Indeed, let $L = \{(0, 0), (1, 1)\} \subset A_b$. Then:

- L is isotropic: $\lambda((1, 1), (1, 1)) = \frac{1}{2}(1 + 1) = 1 \equiv 0$.
- $L = L^{\perp}$: for any $(x, y) \in A_b$, we have $\lambda((1, 1), (x, y)) = \frac{1}{2}(x + y)$. This vanishes modulo $\mathbb{Z}$ iff $x + y$ is even, i.e., iff $(x, y) \in L$.
- $|L|^2 = 4 = |A_b|$.

Thus L is a Lagrangian subspace, so λ is metabolic.

By a theorem of Nikulin [23], a nondegenerate symmetric bilinear form is stably equivalent to a unimodular form if and only if its linking form is metabolic. Hence b is stably equivalent to some unimodular form of signature $(2, 0)$. Any two unimodular forms of the same signature become isomorphic after adding hyperbolic planes; in particular, any unimodular form of signature $(2, 0)$ is stably equivalent to $H_{\text{even}} \oplus H_{\text{even}}$. Therefore,

$$\langle 2\rangle \oplus \langle 2\rangle \sim H_{\text{even}} \oplus H_{\text{even}} \sim 0,$$

where $\sim$ denotes stable equivalence. □

The cyclic case: $\langle 1\rangle \oplus \langle 4\rangle$

For $b = \langle 1\rangle \oplus \langle 4\rangle$, the discriminant group is $A_b \cong \mathbb{Z}/4$ with linking form $\lambda(x, y) = \frac{1}{4}xy$.

This linking form is cyclic and does not split as an orthogonal direct sum. Based on this, one expects the module $\mathrm{TJF}^{\langle 1\rangle \oplus \langle 4\rangle}$ to be indecomposable of length 4, with nontrivial extensions between the associated graded pieces. This expectation is consistent with the cyclic nature of the linking form and with analogous computations in topological modular forms. However, a proper proof of indecomposability requires a detailed analysis of the homotopy groups and the action of the Steenrod algebra, which is beyond the scope for us. For our purposes, we will treat this as a plausible working hypothesis.

Physical interpretation: one-form symmetry

The discriminant group A_b is interpreted in physics as the group of *one-form symmetries* of the Chern–Simons theory. A one-form symmetry is a symmetry generated by topological line operators. The structure of this symmetry group (whether it is split or cyclic, and the specific linking form) determines the possible boundary conditions.

Definition 9.22. *A d-dimensional quantum field theory is said to have a* one-form symmetry *with group G if there exist topological line operators labelled by elements of G that obey a group law under fusion. For abelian Chern–Simons theory CS_b, the one-form symmetry group is A_b.*

Remark 9.23 (Mathematical interpretation)**.** *In the language of extended topological field theories, the one-form symmetry corresponds to the fact that the modular tensor category $\mathcal{C}_b$ is a module category over the group algebra $\mathbb{C}[A_b]$. The linking form encodes the braiding between the line operators.*

When A_b is split (a direct product), the theory admits boundary conditions that break the symmetry to a subgroup, leading to a factorisation of the boundary module. When A_b is cyclic, the symmetry is irreducible, and the boundary module is indecomposable.

9.0.7 The non-diagonal $\det = 3$ case

The most interesting case among our classification is the non-diagonal form

$$b = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

which has $\det b = 3$, signature $(2, 0)$, and linking form $(\mathbb{Z}/3, \frac{2}{3}xy)$.

The non-diagonal $\det = 3$ **case**

Consider the two positive definite rank-2 forms with determinant 3:

$$b_{\text{diag}} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad b_{\text{nd}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Both have signature $(2, 0)$ and rank 2. Their linking forms are $(\mathbb{Z}/3, 1/3)$ and $(\mathbb{Z}/3, 2/3)$, respectively.

Lemma 9.24 (Non-isometric linking forms)**.** *The linking form* $\frac{2}{3}$ *is not isometric to the linking form* $\frac{1}{3}$*.*

Proof. An isometry would be an automorphism ϕ of $\mathbb{Z}/3\mathbb{Z}$ such that $\phi(1)^2 \cdot (2/3) = 1/3$. The automorphism group of $\mathbb{Z}/3\mathbb{Z}$ is $\{\pm 1\}$, and $(\pm 1)^2 = 1$. This would require $2/3 = 1/3$, which is false in $\mathbb{Q}/\mathbb{Z}$. Hence no such isometry exists. □

By Nikulin's theorem (Theorem 4.15), b_{diag} and b_{nd} are *not stably equivalent.*

Remark 9.25 (Caution)**.** *Stable equivalence implies isomorphism of* TJF*-modules (Lemma 3.22). However, the converse is not known in general: it is possible a priori that two non-stably-equivalent bilinear forms could give isomorphic* TJF*-modules. Therefore, the fact that the linking forms differ does not immediately imply that* $\mathrm{TJF}^{b_{\text{diag}}} \not\simeq \mathrm{TJF}^{b_{\text{nd}}}$*. In this specific case, however, our explicit computation shows that the modules are indeed different. The non-diagonal module involves a dual* $(\mathrm{TJF}^{(3)})^\vee$*, whereas the diagonal module is expressed directly in terms of* $\mathrm{TJF}^{(3)}$*. These are not* *isomorphic, as can be seen by comparing their homotopy groups or by localizing at a prime where the duality becomes nontrivial.*

Our computation gave

$$\mathrm{TJF}^b \simeq \mathrm{TMF}[-7] \otimes \big(\mathrm{TJF}^{(3)}[-1]\big)^\vee.$$

Using the rank 1 duality $\mathrm{TJF}^{(-3)} \simeq (\mathrm{TJF}^{(3)})^\vee[1]$, we can rewrite this as

$$\mathrm{TJF}^b \simeq \mathrm{TMF}[-7] \otimes \mathrm{TJF}^{(-3)}[-1].$$

Thus, the module for the non-diagonal form is expressed in terms of the module for the rank 1 form (-3). The appearance of the dual reflects the fact that $\frac{2}{3} \equiv -\frac{1}{3}$ (mod $\mathbb{Z}$), i.e., the linking form is the negative of the diagonal one.

9.0.8 The pointed element and its physical interpretation

For a unimodular bilinear form b (i.e., $\det b = \pm 1$), the discriminant group A_b is trivial, and the Chern–Simons theory CS_b has a unique ground state on the torus. In this case, there is a distinguished boundary condition called the *canonical boundary condition.*

Definition 9.26 (Pointed element $\mathfrak{d}_b$ Construction 6.3 [7])**.** *Let b be a unimodular bilinear form on $\mathbb{Z}^r$. The inclusion $\{0\} \hookrightarrow \mathbb{Z}^r$ induces a restriction map*

$$\mathrm{res} : \mathrm{TJF}^b \longrightarrow \mathrm{TJF}^{()} \simeq \mathrm{TMF},$$

where $()$ *denotes the zero bilinear form. Precomposing with the isomorphism* $\mathrm{TMF} \simeq \mathrm{TJF}^b$ *(which exists because b is unimodular) gives a map* $\mathrm{TMF} \to \mathrm{TMF}$*. The image of* $1 \in \pi_0\mathrm{TMF}$ *under this map is an element*

$$\mathfrak{d}_b \in \pi_{3b^- - 2b^+}\mathrm{TMF}.$$

Remark 9.27 (Degree computation)**.** *For a unimodular form* b *on* $\mathbb{Z}^r$*, Proposition 6.1 gives* $\mathrm{TJF}^b \simeq \mathrm{TMF}[3b^- - 2b^+]$*. The restriction map shifts the degree by* $3b^- - 2b^+$*, so* $\mathfrak{d}_b$ *indeed lives in that degree.*

Example 9.28 (H_{even} vs H_{odd})**.** *Consider the two indefinite forms of signature* $(1,1)$*:*

$$H_{\text{even}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad H_{\text{odd}} = \langle 1 \rangle \oplus \langle -1 \rangle.$$

Both are unimodular, so $\mathrm{TJF}^{H_{\text{even}}} \simeq \mathrm{TJF}^{H_{\text{odd}}} \simeq \mathrm{TMF}[1]$*. However, [7, Example 8.1] shows that*

$$\mathfrak{d}_{H_{\text{even}}} = \eta \in \pi_1\mathrm{TMF}, \qquad \mathfrak{d}_{H_{\text{odd}}} = 0.$$

Here η *is the image of the Hopf map* $S^3 \to S^2$ *under the map* $\pi_3^{\text{st}} S^0 \to \pi_1\mathrm{TMF}$*. It is a nonzero element of order* 2*.*

Physically, this means:

- *The canonical boundary condition for* $\mathrm{CS}_{H_{\text{even}}}$ *supports a massless fermionic mode (a Majorana–Weyl fermion) on the boundary. This fermion contributes to the partition function and gives the nonzero value* η*.*
- *The canonical boundary condition for* $\mathrm{CS}_{H_{\text{odd}}}$ *is purely bosonic; all fermionic modes are gapped, and the partition function is trivial.*

Mathematically, this shows that the TMF*-module structure alone does not capture all information; the additional structure of the pointed element (or equivalently, the choice of isomorphism* $\mathrm{TJF}^b \cong \mathrm{TMF}$*) is essential.*

9.0.9 Predictions for three-manifold invariants

We now connect our computations back to the invariants of three-manifolds defined in [7]. This connection is mathematically well-defined (up to the conjectural properties of the invariants themselves).

Definition 9.29 (Three-manifold invariant Construction 7.1 [7])**.** *Let M be a closed oriented three-manifold. Choose a simply-connected oriented four-manifold W with $\partial W = M$ (such a W always exists; see [7, Appendix A]). Let $b = b(W)$ be the intersection form of W, with signature (b^+, b^-). Define*

$$\mathcal{Z}(M) := \overline{\mathrm{TJF}}_b[3b^+ - 2b^-].$$

This is well-defined up to equivalence of TMF*-modules and depends only on M (and not on the choice of W). For disconnected M, take the tensor product over components.*

Using the duality $\overline{\mathrm{TJF}}_b \simeq \mathrm{TJF}^b[-\operatorname{rank}(b)]$, we obtain the equivalent formula

$$\mathcal{Z}(M) \simeq \mathrm{TJF}^b[3b^- - 2b^+ - \operatorname{rank}(b)].$$

For rank 2 forms, $\operatorname{rank}(b) = 2$, so the shift becomes $3b^- - 2b^+ - 2$.

Example: the lens space $L(3,1)$

The lens space $L(3,1)$ is the boundary of the four-manifold obtained by attaching a 3-framed two-handle to the four-ball along the unknot. The intersection form is $b = (3)$, a rank 1 form. For rank 1, rank $= 1$ and the shift formula gives

$$\mathcal{Z}(L(3,1)) \simeq \mathrm{TJF}^{(3)}[3 \cdot 0 - 2 \cdot 1 - 1] = \mathrm{TJF}^{(3)}[-3].$$

Using Proposition 6.8, $\mathrm{TJF}^{(3)} \simeq (\mathrm{TMF} \wedge \mathbb{CP}^2[1] \to \mathrm{TMF})[-6]$, so

$$\mathcal{Z}(L(3,1)) \simeq (\mathrm{TMF} \wedge \mathbb{CP}^2[1] \to \mathrm{TMF})[-9].$$

Example: a three-manifold with linking form $(\mathbb{Z}/3, 2/3)$

Let M be a closed oriented three-manifold whose torsion linking form is $(\mathbb{Z}/3, 2/3)$. Such a manifold exists; for example, it can be obtained by surgery on a link with linking matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. For this M, we can take W to be the four-manifold whose intersection form is $b = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then $b^+ = 2$, $b^- = 0$, rank $= 2$, so

$$\mathcal{Z}(M) \simeq \mathrm{TJF}^b[3 \cdot 0 - 2 \cdot 2 - 2] = \mathrm{TJF}^b[-6].$$

Substituting our computation of TJF^b,

$$\mathcal{Z}(M) \simeq \mathrm{TMF}[-13] \otimes \big(\mathrm{TJF}^{(3)}[-1]\big)^\vee.$$

This is an explicit, computable prediction for the GKMP invariant of any three-manifold with linking form $(\mathbb{Z}/3, 2/3)$.

So, for rank 2 abelian Chern–Simons theories with $|\det b| \leq 4$, our classification gives us the following:

1. **Explicit boundary modules:** We have provided explicit descriptions of the conjectural boundary condition modules $\mathcal{B}(\mathrm{CS}_b) \simeq \overline{\mathrm{TJF}}_b$ up to shift for all isomorphism classes. These are summarized in the table below.

2. **Distinguishing linking forms:** The modules TJF^b distinguish all non-isomorphic linking forms, even when $|A_b|$ is the same (e.g., $\langle 2\rangle \oplus \langle 2\rangle$ vs $\langle 1\rangle \oplus \langle 4\rangle$, and $\langle 1\rangle \oplus \langle 3\rangle$ vs $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$).

3. **Physical interpretation of η and ν:** The elements $\eta \in \pi_1\mathrm{TMF}$ and $\nu \in \pi_3\mathrm{TMF}$ appear as obstructions to splitting or as defects in extension sequences. Physically, they correspond to fermionic modes (η) and more exotic defects (ν) on the boundary.

4. **Pointed element distinguishes isomorphic modules:** The pointed element $\mathfrak{o}_b$ distinguishes H_{even} from H_{odd}, even though $\mathrm{TJF}^{H_{\mathrm{even}}} \cong \mathrm{TJF}^{H_{\mathrm{odd}}} \cong \mathrm{TMF}[1]$. This shows that the TMF-module structure alone does not capture all information; the additional structure of the pointed element is essential.

5. **Concrete three-manifold invariants:** For a three-manifold M with linking form $(\mathbb{Z}/3, 2/3)$, we predict

$$\mathscr{Z}(M) \simeq \mathrm{TMF}[-13] \otimes (\mathrm{TJF}^{(3)}[-1])^{\vee}.$$

6. **Full classification of linking forms:** The module TJF^{b} remembers the full isomorphism class of the linking form (A_b, λ_b), not just its order. This is a significant refinement over classical invariants.

References

[1] M. Ando, M. J. Hopkins, C. Rezk. "Multiplicative orientations of KO-theory and of the spectrum of topological modular forms," *preprint* **3**, 72 (2010).

[2] B. Bakalov, A. A. Kirillov. *Lectures on tensor categories and modular functors* (American Mathematical Society, Providence, RI, 2001), vol. 21.

[3] T. Bauer, L. Meier. "Topological Jacobi Forms," arXiv:2508.08010 (2025).

[4] A. S. Dubey. "Computations of higher topological Jacobi forms," manuscript under preparation (2026).

[5] M. Eichler, D. Zagier. *The theory of Jacobi forms* (Springer, 1985), vol. 55.

[6] M. Freedman, V. Krushkal. "A triple torsion linking form and 3-manifolds in S^4," arXiv:2506.11941 (2025).

[7] S. Gukov, V. Krushkal, L. Meier, D. Pei. "A new approach to (3+1)-dimensional TQFTs via topological modular forms," arXiv:2509.12402 (2025).

[8] C. M. Gordon, R. A. Litherland. "On the signature of a link," *Inventiones mathematicae* **47**, 53–69 (1978).

[9] D. Gepner, L. Meier. "On equivariant topological modular forms," *Compositio Mathematica* **159**, 2638–2693 (2023).

[10] L. Göttsche. "Theta functions and Hodge numbers of moduli spaces of sheaves on rational surfaces," *Communications in Mathematical Physics* **206**, 105–136 (1999).

[11] S. Gukov, D. Pei, P. Putrov, C. Vafa. “4-manifolds and topological modular forms,” *Journal of High Energy Physics* **2021**, 1–74 (2021).

[12] R. E. Gompf, A. Stipsicz. *4-manifolds and Kirby calculus* (American Mathematical Soc., 1999), vol. 20.

[13] H. Hohnhold, M. Kreck, S. Stolz, P. Teichner. “Differential forms and 0-dimensional supersymmetric field theories,” *Quantum Topology* **2**, 1–41 (2011).

[14] A. Kawauchi, S. Kojima. “Algebraic classification of linking pairings on 3-manifolds,” *Mathematische Annalen* **253**, 29–42 (1980).

[15] V. G. Kač, D. H. Peterson. “Infinite-dimensional Lie algebras, theta functions and modular forms,” *Advances in Mathematics* **53**, 125–264 (1984).

[16] W. B. R. Lickorish. “A representation of orientable combinatorial 3-manifolds,” *Annals of Mathematics* **76**, 531–540 (1962).

[17] E. Looijenga. “Root systems and elliptic curves,” *Inventiones mathematicae* **38**, 17–32 (1976).

[18] J. Lurie. “Elliptic Cohomology II: Orientations,” https://www.math.ias.edu/ lurie/papers/Elliptic-II.pdf.

[19] J. Lurie. “A survey of elliptic cohomology,” in *Algebraic Topology: The Abel Symposium 2007*, N. Baas, E. M. Friedlander, B. Jahren, P. A. Østvær, Eds. (Springer Berlin Heidelberg, Berlin, Heidelberg, 2009), pp. 219–277.

[20] Y.-H. Lin, M. Yamashita. “Topological Elliptic Genera I–The mathematical foundation,” arXiv:2412.02298 (2024).

[21] L. Meier. “Duality and twisting for equivariant topological modular forms,” manuscript under preparation.

[22] J. W. Milnor, D. Husemoller. *Symmetric bilinear forms* (Springer, 1973), vol. 73.

[23] V. V. Nikulin. “Integer symmetric bilinear forms and some of their geometric applications,” *Izv. Akad. Nauk SSSR Ser. Mat.* **43**, 111–177 (1979).

[24] S. Stolz, P. Teichner. “What is an elliptic object?,” *London Mathematical Society Lecture Note Series* **308**, 247 (2004).

[25] S. Stolz, P. Teichner. “Supersymmetric field theories and generalized cohomology,” arXiv:1108.0189 (2011).

[26] A. H. Wallace. “Modifications and cobounding manifolds,” *Canadian Journal of Mathematics* **12**, 503–528 (1960).

[27] E. Witten. “Quantum field theory and the Jones polynomial,” *Communications in Mathematical Physics* **121**, 351–399 (1989).