# ADIC SPACES, LECTURE 3

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## 1. Adic spaces

For any Huber pair $(A, A^+)$ the space $X = \mathrm{Spa}(A, A^+)$ has a pre-sheaf of complete topological rings $\mathscr{O}_X$. The Huber pair is called sheafy if $\mathscr{O}_X$ is a sheaf. Moreover for any $x \in X$ the stalk $\mathscr{O}_{X,x}$ is a local ring equipped with a valuation $v_x : \mathscr{O}_{X,x} \to \Gamma_x \cup \{0\}$.

Let $\mathcal{V}$ be the category whose objects are triples $\mathscr{X} = (X, \mathscr{O}_X, (v_x)_{x\in X})$, where

- $X$ is a topological space,
- $\mathscr{O}_X$ is a sheaf of complete topological rings, which makes $X$ a locally ringed space
- $v_x : \mathscr{O}_{X,x} \to \Gamma_x \cup \{0\}$ is a continuous valuation.

A morphism in $\mathcal{V}$, $f : \mathscr{X} \to \mathscr{Y}$ is a morphism of locally ringed spaces $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ such that the maps $\mathscr{O}_Y(f^{-1}U) \to \mathscr{O}_X(U)$ are continuous for ever open $U \subset X$ and there exist order preserving homomorphisms of abelian groups $\Gamma_{f(x)} \to \Gamma_x$ such that the following diagram commutes

$$\begin{tikzcd} \mathscr{O}_{Y,f(x)} \arrow[r] \arrow[d] & \mathscr{O}_{X,x} \arrow[d] \\ \Gamma_{f(x)} \cup \{0\} \arrow[r] & \Gamma_x \cup \{0\} \end{tikzcd}$$

upto equivalence.

An adic space is an object $\mathscr{X}$ in $\mathcal{V}$ such that $X = \cup U_i$ where $(U_i, \mathscr{O}_X|_{U_i}, (v_x)_{x\in U_i})$ is isomorphic to $\mathrm{Spa}(A_i, A_i^+)$ for a sheafy Huber pair $(A_i, A_i^+)$ in $\mathcal{V}$. Morphisms of adic spaces are morphisms in $\mathcal{V}$.

For a sheafy Huber pair the triple $(X = \mathrm{Spa}(A, A+), \mathscr{O}_X, (v_x)_{x\in X})$ is called an affinoid adic space. We denote by $\mathcal{A}d$ the category of adic spaces.

## 2. The functor of points for adic spaces

Just like schemes any adic space $X$ gives a functor $\mathcal{A}d \to \mathrm{Sets}$, called its functor of points. If $Y$ is another adic space, then

$$X(Y) = \mathrm{Hom}_{\mathcal{A}d}(Y, X).$$

If $Y = \mathrm{Spa}(A, A^+)$, then $X(Y) = \mathrm{Hom}_{Hub}((A, A^+), (\mathscr{O}_X(X), \mathscr{O}_X^+(X)))$ in the category of Huber pairs. Hence $(A, A^+) \to \mathrm{Spa}(A, A^+)$ is a fully faithful functor from the category of sheafy Huber pairs to adic spaces.

## 3. Examples of Adic Spaces

3.1. **The final object.** $(A, A)$ is a Huber pair for any discrete ring $A$. Consider the ring $\mathbb{Z}$.

The final object in $\mathcal{A}d$ is $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$. This space has 3 types of points:

(1) $\eta$ corresponding trivial valuation on $\mathbb{Z}$.
(2) The points $s_p$ corresponding to the pull back of the trivial valuation on $\mathbb{F}_p$ by the quotient map $\mathbb{Z} \to \mathbb{F}_p$ for each prime $p \in \mathbb{Z}$.
(3) The points $\eta_p$ corresponding to the $p$-adic valuation, $|n| = p^{-\alpha}$ if $n = p^\alpha m$ where $p \nmid m$ and $|0| = 0$.

It is easy to see that $\eta$ is open while the points $s_p$ are closed. On the other hand, $\overline{\eta} = \mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$ and $\overline{\eta_p} = \{\eta_p, s_p\}$.

There is a unique map from $(\mathbb{Z}, \mathbb{Z})$ to any Huber pair $(A, A^+)$, hence a unique map $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$.

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This space can be represented by the following diagram

$$
\begin{tikzcd}
\eta \arrow[d] \arrow[dr] \arrow[drr] & & & \\
\eta_2 \arrow[d] & \eta_3 \arrow[d] & \eta_5 \arrow[d] & \cdots \\
s_2 & s_3 & s_5 & \cdots
\end{tikzcd}
$$

On the other hand if $K$ is a non-archimedian field with valuation ring $K^o$ then $\mathrm{Spa}(K, K^o)$ has a single point. An adic space over $K$ is an adic space $X$ with a morphism to $\mathrm{Spa}(K, K^o)$. A morphism between adic spaces over $K$ is a morphism $X \to Y$ such that the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
& \mathrm{Spa}(K, K^o) &
\end{tikzcd}
$$

The category of adic spaces over $K$ will be denoted by $\mathcal{A}d/K$, in which the final object clearly is $\mathrm{Spa}(K, K^o)$.

3.2. **Closed unit disc.** Over $\mathbb{Z}$ the closed unit disc is $D_{\mathbb{Z}} = \mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}[T])$, where $\mathbb{Z}[T]$ is discrete. Note that this is justified by its functor of points $D_{\mathbb{Z}}(\mathrm{Spa}(A, A^+)) = A^+$.

The closed adic unit disc over $\mathbb{Q}_p$ is the affinoid adic space

$$D_{\mathbb{Q}_p} := \mathrm{Spa}(\mathbb{Q}_p \langle T \rangle, \mathbb{Z}_p \langle T \rangle).$$

Here the topology on $\mathbb{Q}_p \langle T \rangle$ comes from the sup norm

$$\left| \sum_0^\infty a_n T^n \right| = \sup\{|a_n| : n \geq 0\}.$$

Note that for any $|\cdot| \in D_{\mathbb{Q}_p}$, $|T| \leq 1$. Moreover for any point $\alpha \in \overline{\mathbb{Q}_p}$, $|\alpha| \leq 1$ there is a valuation $|\cdot|_\alpha = D_{\mathbb{Q}_p}$, given by $|f|_\alpha = |f(\alpha)|$. Thus

$$D_{\mathbb{Q}_p} \supset \{\alpha \in \overline{\mathbb{Q}_p} : |\alpha| \leq 1\}/\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p).$$

However $D_{\mathbb{Q}_p}$ has many more points. For example if

$$B(\alpha, r) = \{\beta \in \overline{\mathbb{Q}_p} : |\beta - \alpha| \leq r\}$$

is the closed ball of radius $r$ around $\alpha$, where $\alpha \in \overline{\mathbb{Q}_p}$ with $|\alpha| \leq 1$ and $0 < r \leq 1$, then there is a point in $D_{\mathbb{Q}_p}$ corresponding to $B(\alpha, r)$ given by the valuation

$$|f| = \sup\{|f(\beta)| : \beta \in B(\alpha, r)\}.$$

This is called the Gauss point of $B(\alpha, r)$.

There is a natural map $D_{\mathbb{Q}_p} \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. If $\mathrm{Spa}(A, A^+)$ is an adic space over $\mathbb{Q}_p$ then

$$\mathrm{Hom}_{\mathcal{A}d/\mathbb{Q}_p}\left(\mathrm{Spa}(A, A^+), D_{\mathbb{Q}_p}^{\mathcal{A}d}\right) = A^+.$$

There is a nice classification of points of $D_K$ which can be found in Scholze's paper on perfectoid spaces in Publ. IHES, for an algebraically closed non-archimedian field $K$.

3.3. **Open Unit disc.** Over $\mathbb{Z}$ the open unit disc is $D^o_{\mathbb{Z}} = \mathrm{Spa}(\mathbb{Z}[[T]], \mathbb{Z}[[T]])$ where we give $\mathbb{Z}[[T]]$ the $T$-adic topology. Let $(A, A^+)$ be a complete Huber pair, it can be shown that

$$D^o_{\mathbb{Z}}(\mathrm{Spa}(A, A^+)) = A^{oo}$$

where $A^{oo}$ is the ideal of topologically nilpotent elements. Since $T$ is topologically nilpotent in $\mathbb{Z}[[T]]$ it has to go to a topologically nilpotent element in $A$. Conversely sending $T$ to any topologically nilpotent element of A gives a continuous ring homomorphism $\mathbb{Z}[[T]] \to A$.

The open unit disc over $\mathbb{Q}_p$ is a bit harder to define. Consider $\mathbb{Z}_p[[T]]$ with $(p, T)$-adic topology and let $X = \mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]])$. There is a natural map $X \to \mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. There are exactly two points in $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$,

- the point $\eta$ corresponding to the p-adic valuation obtained from the inclusion $\mathbb{Z}_p \to \mathbb{Q}_p$,

- the closed point $s$ corresponding to the pull-back of the trivial valuation on $\mathbb{F}_p$ by the map $\mathbb{Z}_p \to \mathbb{F}_p$.

The closure of $\eta$ is the entire space so $\eta$ is the generic point. Moreover $\{\eta\}$ is the rational open set $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)(p/p)$ and so is isomorphic to $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

The generic fiber $X_\eta$ with its natural map to $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, is the open unit disc over $\mathbb{Q}_p$ which we denote $D^o_{\mathbb{Q}_p}$.

If $|\cdot| \in D^o_{\mathbb{Q}_p}$ then since $T$ is topologically nilpotent, we have $|T^n| \to 0$, hence there is a $k > 0$ such that $|T^k| \leq |p| \neq 0$, thus $|\cdot| \in X(T^k/p)$. Hence

$$D^o_{\mathbb{Q}_p} = \bigcup_{k \geq 1} X(T^k/p).$$

This is an open cover which does not have a finite sub-cover, thus $D^o_{\mathbb{Q}_p}$ is not quasi-compact and hence not affinoid.

*Exercise.* Show that $\mathrm{Hom}_{\mathcal{A}d/\mathbb{Q}_p}(\mathrm{Spa}(A, A^+), D^o_{\mathbb{Q}_p}) = A^{oo}$.

3.4. **Affine Line.** The affine line over $\mathbb{Z}$ is $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z})$, where $\mathbb{Z}[T]$ has discrete topology. Clearly

$$\mathbb{A}^1_{\mathbb{Z}}(\mathrm{Spa}(A, A^+)) = A.$$

Over $\mathbb{Q}_p$ the affine line is given by a union over closed discs of increasing radii. Consider the following inclusion of Huber pairs

$$\cdots \subset (\mathbb{Q}_p \langle p^2 T \rangle, \mathbb{Z}_p \langle p^2 T \rangle) \subset (\mathbb{Q}_p \langle pT \rangle, \mathbb{Z}_p \langle pT \rangle) \subset (\mathbb{Q}_p \langle T \rangle, \mathbb{Z}_p \langle T \rangle).$$

This gives successive embeddings of adic spaces and we can take the union

$$\mathbb{A}^1_{\mathbb{Q}_p} = \bigcup_{n=0}^{\infty} \mathrm{Spa}(\mathbb{Q}_p \langle p^n T \rangle, \mathbb{Z}_p \langle p^n T \rangle)$$

which is manifestly an adic space that is not quasi-compact and hence not affinoid.

In fact $\mathrm{Spa}(\mathbb{Q}_p \langle p^n T \rangle, \mathbb{Z}_p \langle p^n T \rangle)$ is the closed disc of radius $1/|p|^n$, since $|p^n T| \leq 1 \Rightarrow |T| \leq 1/|p|^n$.

*Exercise.* Show that $\mathrm{Hom}_{\mathcal{A}d/\mathbb{Q}_p}(\mathrm{Spa}(A, A^+), \mathbb{A}^1_{\mathbb{Q}_p}) = A$.

3.5. **Projective Line.** The projective line over $\mathbb{Z}$ can be constructed by gluing two copies of $\mathbb{A}^1_{\mathbb{Z}}$, which I leave as an exercise.

In case of $\mathbb{Q}_p$, consider the the closed unit disc $D_{\mathbb{Q}_p}$ and take the unit circle

$$S^1_{\mathbb{Q}_p} = \{|\cdot| \in D_{\mathbb{Q}_p} : 1 = |T|\} = D_{\mathbb{Q}_p}(1/T).$$

This is a rational open subset isomorphic to $\mathrm{Spa}(\mathbb{Q}_p \langle T, T^{-1} \rangle, \mathbb{Z}_p \langle T, T^{-1} \rangle)$. The projective line is obtained by gluing two copies of the closed unit disc along the unit circle

$$\mathbb{P}^1_{\mathbb{Q}_p} = \mathrm{Spa}(\mathbb{Q}_p \langle T_1 \rangle, \mathbb{Z}_p \langle T_1 \rangle) \sqcup \mathrm{Spa}(\mathbb{Q}_p \langle T_2 \rangle, \mathbb{Z}_p \langle T_2 \rangle)/\sim,$$

where the identification $\sim$ is obtained by the isomorphism

$$\phi : (\mathbb{Q}_p \langle T_1, T_1 {-} 1 \rangle, \mathbb{Z}_p \langle T_1, T_1^{-1} \rangle) \to (\mathbb{Q}_p \langle T_2, T_2^{-1} \rangle, \mathbb{Z}_p \langle T_2, T_2^{-1} \rangle)$$

given by $\phi(T_1) = T_2^{-1}$.

Hence clearly $\mathbb{P}^1_{\mathbb{Q}_p}$ is quasi-compact, but it is again not affinoid. To see this let us investigate the ring of global sections of the structure sheaf. If $\sigma \in \mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}_p}}(\mathbb{P}^1_{\mathbb{Q}_p})$ then $\sigma = (\sigma_1, \sigma_2)$ where $\sigma_i \in \mathbb{Q}_p \langle T_i \rangle$ such that $\phi(s_1) = s_2$. This forces $s_i$ to be constants and $\mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}_p}}(\mathbb{P}^1_{\mathbb{Q}_p}) = \mathbb{Q}_p$ with $\mathscr{O}^+_{\mathbb{P}^1_{\mathbb{Q}_p}}(\mathbb{P}^1_{\mathbb{Q}_p}) = \mathbb{Z}_p$. Of course $\mathbb{P}^1_{\mathbb{Q}_p}$ is not $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

## 4. Schemes, Formal Schemes and Rigid spaces as Adic spaces

4.1. **Schemes.** If $A$ is a discrete ring then $(A, A)$ is a sheafy Huber pair and there are maps of locally ringed spaces

$$\operatorname{Spec} A \to \operatorname{Spa}(A, A) \to \operatorname{Spec} A.$$

The first map is given by sending a prime ideal $P \subset A$ to the trivial valuation on $A/P$, where as the second map is obtained by taking the support of a valuation. Clearly the composition is identity.

There is a fully faithful functor from the category of schemes to $\mathcal{A}d$ which sends $\operatorname{Spec} A$ to $\operatorname{Spa}(A, A)$ for a discrete ring $A$.

4.2. **Formal Schemes.** If $A$ is an adic ring (it is a complete topological ring with $I$-adic topology for some ideal $I$) then $(A, A)$ is again a sheafy Huber pair. The formal scheme associated to $A$ is a locally ringed space of topologically complete rings, denoted by $\operatorname{Spf}(A)$.

Again there is a fully faithful functor from formal schemes to adic spaces sending $\operatorname{Spf}(A)$ to $\operatorname{Spa}(A, A)$.

4.3. **Rigid spaces.** Let $K$ be a algebraically closed non-archimedian field, (think of $\mathbb{C}_p$), then an affinoid $K$-algebra is a complete normed $K$-algebra $A$ which is a quotient of $K \langle T_, \dots, T_n \rangle$ for some $n$. The rigid space associated to $A$ is a locally ringed Grothendieck-topologised space whose underlying set is

$$\operatorname{Spm}(A) = \{m \subset A \mid m \text{ maximal ideal } \}.$$

This space is given a Grothendieck topology and has a structure sheaf with respect to that topology.

The associated adic space to $\operatorname{Spm}(A)$ is $\operatorname{Spa}(A, A^o)$. This extends to a fully faithful functor from the category of rigid spaces over $K$ to $\mathcal{A}d/K$.

As an example consider the closed unit poly-disc $\operatorname{Spm}(K \langle T_1, \dots, T_n \rangle)$; the associated adic space is the closed unit poly disc $\operatorname{Spa}(K \langle T_1, \dots, T_n \rangle, K^o \langle T_1, \dots, T_n \rangle)$.