Submission time 10:00, Friday 1 September. All problems are worth 20 points. You may work with one other person but please mention her/his name on your paper.

1. Let $p, q \in \mathbb{Z}$ be primes. Suppose E and F are finite fields with $|E| = q^m$ and $|F| = p^n$. If there is an embedding of E in F show that $p = q$ and m divides n.

2. Find the splitting field K of $(x^4 - 1)(x^4 + 2)$ over $\mathbb{Q}$. What is $[K : \mathbb{Q}]$? Find an element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$. (**6+6+8**)

3. Let L/K be a field extension.

   (a) If $\operatorname{char} K \neq 2$ and $[L : K] = 2$ show that $L = K(\alpha)$ where $\alpha^2 \in K$. (**8**)

   (b) Let $K = \mathbb{Q}$ and $L = \mathbb{Q}(\alpha)$ where $\alpha \in \mathbb{C}$ and $\alpha^3 - 5\alpha + 1 = 0$. Show that $[L : K] = 3$ and that there is no $\beta \in L$ such that $L = \mathbb{Q}(\beta)$ and $\beta^3 \in \mathbb{Q}$. (Hence the previous assertion is not true in general for degree 3.) (**12**)

4. Let K be a field and $\overline{K}$ be its algebraic closure:

   (a) What are all the prime ideals in $\overline{K}[x]$. (**5**)

   (b) If $\alpha \in \overline{K}$ and $m_{\alpha,K}$ the irreducible polynomial for α over K then how many embeddings $\sigma : K(\alpha) \to \overline{K}$ are there so that $\sigma|_K = \mathrm{Id}_K$. (**15**).

5. Let L/K be a finite, normal extension. Let $f(x) \in K[x]$ be an irreducible polynomial. Let $g(x), h(x) \in L[x]$ be two monic irreducible factors of f in $L[x]$. Show that there is an automorphism $\sigma \in \mathrm{Aut}(L/K)$ such that $\sigma^*(g) = h$. Give a counterexample to this fact when L/K is not normal. (**15 + 5**)