Submit problems 1, 2, 3 by Monday, November 13.

1. Show that the following two statements of the Lindemann-Weirstrass theorem are equivalent.

   A. If $a_1, \ldots, a_n \in \mathbb{C}$ are distinct algebraic numbers then $e^{a_1}, \ldots, e^{a_n}$ are linearly independent over $\mathbb{Q}$.

   B. If $a_1, \ldots, a_n \in \mathbb{C}$ are algebraic numbers, linearly independent over $\mathbb{Q}$ then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent over $\mathbb{Q}$.

2. L/K is an extension of fields. Show that $a_1, \ldots, a_n \in L$ are algebraically independent over K if and only if a_1 is transcendental over K and a_i is transcendental over $K(a_1, \ldots, a_{i-1})$ for $2 \leq i \leq n$.

3. Let $K(x_1, \ldots, x_n)$ be the field of rational functions in n variables over a field K. Let $A \in M_n(K)$ be an $n \times n$ matrix over K with entries $a_{i,j}$. Let
$$y_i = \sum_{j=1}^{n} a_{i,j} x_j.$$
Show that $y_1, \ldots, y_n$ is a transcendence basis for $K(x_1, \ldots, x_n)$ if and only if A is invertible.

4. Prove that with ruler and compass we can not draw a square whose area is the same as a circle with unit radius.

5. Using the Lindemann-Weirstrass theorem show that if u is a non-zero algebraic number then $\sin u$, $\cos u$ and $\tan u$ are transcendental. Moreover if $u \neq 1$ then show that any complex value of $\log u$ is also transcendental.

6. Look up Schanuel's conjecture: `https://en.wikipedia.org/wiki/Schanuel%27s_conjecture`. Prove that the conjecture implies that e, π are algebraically independent over $\mathbb{Q}$.

7. Let K/F have finite transcendence degree and L_1 and L_2 be sub-fields of K containing F, show that
$$\operatorname{trdeg}_F L_1 L_2 \leq \operatorname{trdeg}_F L_1 + \operatorname{trdeg}_F L_2.$$
Give an example where the inequality is strict.

8. Let L/K be a field extension with $\alpha, t \in L$ such that α is algebraic over K but t is transcendental over K, show that
$$m_{\alpha,K}(x) = m_{\alpha,K(t)}(x).$$

9. If L/K and K/F are field extensions such that L is finitely generated over F, prove that K is finitely generated over F.

10. Let K be an algebraically closed field and F a sub-field such that $\operatorname{trdeg}_F K < \infty$. Suppose $\phi : K \to K$ is an embedding such that $\phi|_F$ is identity. Show that ϕ is surjective, that is it is an automorphism of K.

11. Let $K = \mathbb{R}(x)$ and $p(t) = t^2 - x^2 - 1,\ q(t) = t^2 + x^2 + 1 \in K[t]$. Let L_1 and L_2 be the splitting fields of p and q over K. Show there is $t \in L_1$ such that $L_1 = \mathbb{R}(t)$, but this is not true for L_2. What are the transcendence degrees of L_1 and L_2 over $\mathbb{R}$?

12. Prove that $x_1^{r_1}, \ldots, x_n^{r_n}$ forms a transcendence basis of the field of rational functions $K(x_1, \ldots, x_n)$ over K.

13. Let $\mathbb{C}(x)$ be the field of rational functions over $\mathbb{C}$. If K is its algebraic closure, show that $K \cong \mathbb{C}$.