Submit problems 1, 6, 9 by Thursday, 14 Sep. Concepts covered: Galois extensions, and Galois groups.

1. Let L/K be an algebraic field extension, show that any embedding $\sigma : L \to L$ such that $\sigma|_K = \mathrm{Id}_K$ is an automorphism of L. Is it true if the extension is not algebraic?

2. Let $L = K(x)$ be the field of rational functions of a field K. Investigate $\mathrm{Aut}(L/K)$.

3. Let L/K be a Galois extension and $G = \mathrm{Gal}(L/K)$ for any $\alpha \in L$ show that there are only finitely many conjugates of α and they are exactly the roots of the irreducible polynomial of α over K.

4. Demonstrate explicitly a Galois extension of $\mathbb{Q}$ of degree 5 and write down the polynomial of which this is the splitting field.

5. Let $p \in \mathbb{Q}[x]$ be an irreducible degree 3 polynomial and L its splitting field. Suppose $\mathrm{Gal}(L/\mathbb{Q})$ is a cyclic group, show that p has all real roots.

6. Let $K = \mathbb{Q}(\sqrt{2+\sqrt{2}})$, show that $K/\mathbb{Q}$ is a Galois extension and determine $\mathrm{Gal}(K/\mathbb{Q})$.

7. Find the irreducible polynomials of the following algebraic numbers over $\mathbb{Q}$.

   (a) $1 + \sqrt[3]{2} + \sqrt[3]{4}$.

   (b) $\sqrt[3]{2} + i$.

8. Determine the Galois group of the splitting field of $x^4 + 8$ over $\mathbb{Q}$.

9. Give an example of a tower of field extension $\mathbb{Q} \subset F_1 \subset F_2 \subset F_3$ where $F_1/\mathbb{Q}$ and $F_3/\mathbb{Q}$ are Galois extensions but $F_2/\mathbb{Q}$ is not a Galois extension. Prove your assertion.