Concepts covered: Separable extensions, automorphism groups and fixed fields. Reading: Lang .

1. Let E/F be a finite extension of fields. Let $\overline{F}$ be the algebraic closure of F. The separable degree of E over F denoted by $[E:F]_s$ is the number of embeddings of E in $\overline{F}$.

   (a) If $E = F(\alpha)$ is a simple extension, show that $[E:F]_s$ divides $[E:F]$ if α is separable over F.

   (b) Prove in general that $[E:F]_s$ divides $[E:F]$ and equality holds if E is a separable extension of F.

2. Show that E/F is a finite separable extension if and only if $E = F(\alpha_1, \dots, \alpha_n)$ where each α_i is separable over F.

3. Suppose E/F is a finite extension then show that E is a normal and separable extension of F if and only if E is the splitting field of a separable polynomial in $F[x]$.

4. State and prove analogous statements as in Problem 3 for arbitrary extensions (not necessarily finite).

5. Show that $\alpha = 2\cos\frac{2\pi}{7}$ is a root of $x^3 + x^2 - 2x - 1$ and the other two roots are $\alpha^2 - 2$ and $\alpha^3 - 3\alpha$. Prove that $\mathbb{Q}(\alpha)$ is a degree 3 Galois extension of $\mathbb{Q}$ which is not generated by a cube root. (Check out `https://en.wikipedia.org/wiki/Casus_irreducibilis`.)

6. Show that any algebraically closed field is perfect.

7. Let $\zeta_7 = e^{2\pi i/7}$ be the primitive 7th root of unity. Find a generator for $\mathrm{Aut}(\mathbb{Q}(\zeta_7))$ and show that it is cyclic. If we number $\alpha_i = \zeta_7^i,\ i = 1, \dots, 6$ what is the sub-group of S_6 that $\mathrm{Aut}(\mathbb{Q}(\zeta_7))$ maps to?

8. What are $\mathrm{Aut}\left(\mathbb{Q}(\sqrt{3}+\sqrt{5})\right)$ and $\mathrm{Aut}\left(\mathbb{Q}(\sqrt{3}+\sqrt{5})/\mathbb{Q}(\sqrt{3})\right)$?

9. Let K be the splitting field of $x^3 - 2 \in \mathbb{Q}[x]$, list all the subgroups of $\mathrm{Aut}(K)$ and their fixed fields.

10. Let K be the splitting field of $x^7 - 1 \in \mathbb{Q}[x]$, list all the subgroups of $\mathrm{Aut}(K)$ and their fixed fields.