No problems to be submitted this week. Concepts covered: Splitting Fields, Normal extensions Algebraic closure.

1. Let L/K be a degree 2 extension and $\operatorname{char} K \neq 2$. Show that L is generated by a square root, that is $L = K(\alpha)$ where $\alpha^2 \in K$ but $\alpha \notin K$.

2. Let $K/\mathbb{Q}$ be the splitting field of $x^p - 2$ where $p > 2$ is a prime, find the group of automorphisms of K.

3. What are all the field automorphisms of $\mathbb{Q}$ and $\mathbb{Z}/p\mathbb{Z}$?

4. (*Galois Theory, D. J. H Garling*) Find the splitting field $K/\mathbb{Q}$ for each of the following polynomials, determine $[K : \mathbb{Q}]$ and find $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$.

   (a) $x^4 - 5x^2 + 6$,

   (b) $x^4 + 5x^2 + 6$,

   (c) $x^4 + 5$,

   (d) $x^4 + 1$,

   (e) $x^4 + 4$,

   (f) $(x^4 + 1)(x^4 + 4)$.

5. (Garling) Let L/K be the splitting field of a degree n polynomial over K. Show that $[L : K]$ divides $n!$.

6. (Garling) If L/K is algebraic show that an algebraic closure of L is also an algebraic closure of K.

7. (Garling) Show that any algebraically closed field has to be infinite.

8. (*Algebra, Serge Lang*) Let F be a field and $F(x)$ the field of rational functions over F. Show that if $y = f(x)/g(x)$ with $f, g \in F[x]$ co-prime and $n = \max(\deg g, \deg f)$ then $[F(x) : F(y)] = n$.

9. (Lang) (**Bonus**) Let k be a finite field with $\operatorname{char} k = p$ and $k(x, y)$ the field of rational functions over k of 2 variables. Show that $[k(x, y) : k(x^p, y^p)] = p^2$, but there are infinitely many fields between $k(x, y)$ and $k(x^p, y^p)$.

10. (**Bonus**) If F is a finite field show that for any $a \in F$ there are $b, c \in F$ such that $a = b^2 + c^2$.