**Indian Institute of Science Education and Research, Pune**

MID SEMESTER EXAMINATION, FALL 2017

**Course name:** Galois Theory | **Course code:** MTH 410

*Date:* 21 September, 2017 | *Duration:* 2 hrs

*Instructor:* Chitrabhanu Chaudhuri | Total points: 100

**Instructions:** Answer all questions. Each question is worth 20 points. Prove all statements that have not been proven in class.

1. Let $K$ be the splitting field over $\mathbb{Q}$ of $x^5 - 7$. What is $[K : \mathbb{Q}]$? Find all sub fields of $K$ which are Galois extensions of $\mathbb{Q}$. (Justify why those are all.)

2. Let $F$ be an infinite field of characteristic $p$ for some prime number $p$. Let $L = F(x, y)$ be the field of rational functions in 2 variables over $F$. Let $K \subset L$ be the subfield $F(x^p, y^p)$. What is $[L : K]$? For $c \in F$ what is the degree of $K(x + cy)$ over $K$? Is $L/K$ a simple extension? (Justify all your answers.)

3. Let $E/F$ be a finite extension of fields. Let $f(x) \in F[x]$ be a separable polynomial. Suppose $K$ is the splitting field of $f$ over $F$ and $K'$ the spliting field of $f$ over $E$. If $|\mathrm{Gal}(K/F)|$ and $[E : F]$ are co-prime integers, what can you say about $\mathrm{Gal}(K'/E)$? (Justify.) Is $K'/F$ necessarily a Galois extension, prove or give a counter-example.

4. Let $F = \mathbb{C}(t)$ and $\tau \in \mathrm{Aut}(F)$ be given by $\tau(f(t)) = f(\frac{1}{1-t})$. Show that $\tau^3 = \mathrm{Id}_F$. Let $\langle \tau \rangle$ be the cyclic group generated by $\tau$. Show that the fixed field $F^{\langle \tau \rangle}$ is $\mathbb{C}(s)$ where $s = t + \dfrac{1}{1-t} + \dfrac{t-1}{t}$.

5. Let $\zeta_p = e^{2\pi i/p}$ be the primitive $p$-th root of 1 in $\mathbb{C}$. Show that $\mathbb{Q}\left(\zeta_5, \zeta_7 + \zeta_7^{-1}\right)$ is a Galois extension of $\mathbb{Q}$ and compute the Galois group.