INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH (IISER), PUNE
(An Autonomous Institution, Ministry of Human Resource Development, Govt. of India)
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SPRING 2018

Mid-Semester Examination

Course name: Calculus on Manifolds

Course Code: MTH322 **Total marks: 60**

Date: 19 February 2018 **Time: 10 AM - 12 Noon**

Instructions: Please answer all questions. All problems are worth 10 points.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given
$$f(x) = \begin{cases} \dfrac{x_1|x_2|}{\sqrt{x_1^2 + x_2^2}}, & (x_1, x_2) \neq (0,0) \\ 0, & (x_1, x_2) = (0,0). \end{cases}$$
Show that f has all directional derivatives at $(0,0)$ but is not differentiable at the origin.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x^3 + y^3 - 3xy$, show that there are open intervals $U \ni 3/2$ and $V \ni 3/2$ and a differentiable function $g : U \to V$ with $g(3/2) = 3/2$ such that in $U \times V$ all solutions of $f(x,y) = 0$ are of the form $(x, g(x))$. Find $g'(3/2)$.

3. Find the volume of the region $D = \left\{(x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq z \leq h\right\} \subset \mathbb{R}^3$.

4. Let $v_1, \ldots, v_n$ be a basis of $\mathbb{R}^n$. Find the volume of the n-parallepiped
$$\mathcal{P} = \{c_1 v_1 + \cdots + c_n v_n \mid 0 \leq c_i \leq 1\} \subset \mathbb{R}^n.$$

5. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuously differentiable function and $m > n$, such that $Df(x)$ has rank n for all $x \in \mathbb{R}^m$. Show for any open set $U \subset \mathbb{R}^m$, $f(U)$ is also open in $\mathbb{R}^n$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $\mathcal{C}^2$. Show that for any $a \in \mathbb{R}^2$, $\partial_{1,2} f(a) = \partial_{2,1} f(a)$, that is the order in which we take the double derivative doesn't matter. (Hint. If they are unequal at a then they are unequal in a rectangle containing a. Now use Fubini's theorem).