1. Vector Fields

1. Show that S^1 has a non-vanishing vector field. Obtain a non-vanishing 1-form on S^1. (Even dimensional spheres have no non-vanishing vector fields.)

2. Let $C \subset \mathbb{R}^3$ be a curve, that is a smooth one dimensional manifold with chart $(\gamma, (a, b))$ where $(a, b) \subset \mathbb{R}$ is an open interval. What is $\gamma_* E_1$ where E_1 is the standard vector field of $(a, b) \subset \mathbb{R}$. Now compute it for the cycloid
$$h : (0, 2\pi) \to \mathbb{R}^2, \qquad h(t) = \big(r(t - \sin t), r(1 - \cos t)\big).$$

3. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ be $\mathcal{C}^\infty$. Γ_f, the graph of f is a manifold. Consider the chart $(\mathbb{R}^2, \phi)$ of Γ_f, where $\phi(x_1, x_2) = (x_1, x_2, f(x_1, x_2))$. Calculate the vector fields $X_1 = \phi_* E_1$ and $X_2 = \phi_* E_2$. Show that there is a non-vanishing 2-form on Γ_f.

4. Suppose $U \subset \mathbb{R}^2$ is open and X is a $\mathcal{C}^1$ vector field on U. Suppose $\gamma : (a, b) \to U$ is $\mathcal{C}^1$, where (a, b) is an open interval in $\mathbb{R}$, further assume that $\gamma_*(t, e_1) = X(\gamma(t))$ (i.e. the tangent vectors to the curve are given by the vector field at each point).
   (a) Write down a differential equation satisfied by γ.
   (b) What can you say about the existence and uniqueness of the solution to the differential equation?

   In fact such a curve always exists. Through each point of U there is a unique such curve. These curves are called the integral curves of X (See `http://en.wikipedia.org/wiki/Vector_field#Flow_curves`).

5. Let $f : U \to \mathbb{R}^m$ be a differentiable map where $U \subset \mathbb{R}^n$ is open, and let $f(p) = q$.
   (a) If $E_1, \ldots, E_n$ are the standard vector fields on U and $E_1, \ldots, E_n$ the standard vector fields on $\mathbb{R}^m$, show that
$$(f_* E_i)(q) = \sum_{j=1}^{m} \partial_i f_j(p) E_i(q).$$
   (b) If $v = \sum_{i=1}^n v_i E_i(p) \in T_p\mathbb{R}^n$ find $f_*(v)$.

6. Let $M \subset \mathbb{R}^n$ be a $\mathcal{C}^\infty$ manifold of dimension k. Show that the tangent bundle
$$TM = \{T_pM\}_{p \in M} \subset \mathbb{R}^n \times \mathbb{R}^n$$
is a $\mathcal{C}^\infty$ manifold of dimension $2k$. (Hint. If (U, ϕ) is a chart for M, then $(U \times \mathbb{R}^k, \Phi)$ is a chart for TM, where $\Phi(x, y) = (\phi(x), D\phi(x) \cdot y)$).

   There is a map $TM \to M$ which sends a tangent vector at a point of M to the point of M. The fibers of this map are all vector spaces (tangent spaces).

2. Differential Forms

7. A differential form ω is called **closed** if $d\omega = 0$. Let $\omega = Pdx + Qdy$ be a 1 form on $\mathbb{R}^2$, calculate $d\omega$ and find conditions on P, Q for which ω is closed.

8. A differential r form η is called **exact** if $\eta = d\omega$ for some $r - 1$ form ω. Since $d^2 = 0$ it follows that any exact form is closed. Show that the 1 form $\eta = 3x^2ydx + (x^3 + 2y)dy$ on $\mathbb{R}^2$ is closed and exact.

9. * Consider smooth functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$, then these define the pull back maps $f^* : \Omega^r(\mathbb{R}^k) \to \Omega^r(\mathbb{R}^m)$ and $g^* : \Omega^r(\mathbb{R}^m) \to \Omega^r(\mathbb{R}^n)$. Show that $(g \circ f)^* = f^* \circ g^*$.

10. Let $U \subset \mathbb{R}^n$ be open and $f, g : U \to \mathbb{R}$ be a smooth functions then show that
   (a) $df = \partial_1 f dx_1 + \ldots \partial_n f dx_n$ (Hint. What is $df(p)((p, e_i))$ for $p \in U$?)

   (b) $d(fg) = gdf + fdg$.

11. Let $U = \mathbb{R}^2 - \{(0,0)\}$, and ω be the differential 1 form on U defined by

$$\omega(x,y) = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

   (a) Show that $d\omega = 0$.

   (b) Show that there is no function $f : U \to \mathbb{R}$ such that $\omega = df$. Thus ω is closed but not exact.

12. Let $U \subset \mathbb{R}^n$ be open and $\phi : U \to \mathbb{R}^m$ be a smooth function, then show that
   (a) $\phi^*(dx_j) = \sum_{i=1}^n \partial_i \phi_j dx_i$;

   (b) $\phi^*(\sum_{j=1}^m g_j dx_j) = \sum_{i=1}^n \sum_{j=1}^m (g_j \circ \phi) \partial_i \phi_j dx_i$

   (c) If $m = n$ show that $\phi^*(h\ dx_1 \wedge \cdots \wedge dx_n) = (h \circ \phi) \det(D\phi)\ dx_1 \wedge \cdots \wedge dx_n$.

3. Orientation

13. * Let $M \subset \mathbb{R}^n$ be a connected smooth manifold of dimension 1. Then show that M is orientable.

14. Let $M \subset \mathbb{R}^n$ be an oriented manifold of dimension $n-1$.
   (a) Let $p \in M$. Show that there is a unique vector $N(p) \in T_p\mathbb{R}^n$ of norm 1 such that for any positively oriented basis $v_1, \ldots, v_{n-1}$ of T_pM the basis $N(p), v_1, \ldots, v_{n-1}$ of $T_p\mathbb{R}^n$ is positively oriented with respect to the standard orientation of $\mathbb{R}^n$.

   (b) * Let $N(p) = (p, n(p)) \in T_p\mathbb{R}^n$. Show that $p \mapsto n(p)$ is a smooth function on M. In fact since $||n(p)|| = 1$ we actually get a map $n : M \to S^{n-1}$. This is called the Gauss Map. $n(p)$ is called the unit normal at p. (See Lemma 38.3 of Munkres.)

   (c) Show that the Möbius strip is not orientable, by demonstrating that there can not be a smooth function $n(p)$ as above.

15. Show that a connected orientable manifold $M \subset \mathbb{R}^n$ can have two possible orientations. If M has dimension $n-1$ how are the Gauss maps for the two different orientations related. Show that fixing the unit normal at a point of the manifold determines the orientation.

16. Compute the Gauss map for the following oriented 2-manifolds in $\mathbb{R}^3$.
   (a) S^2 with the unit normal at $(1,0,0)$ being $(-1,0,0)$.

   (b) The cylinder $C = \{(x,y,z) \mid x^2 + y^2 = 1\}$ with the unit normal at $(1,0,0)$ being $(1,0,0)$.

   (c) The ellipsoid $E = \{(x,y,z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ unit normal at $(a,0,0)$ being $(1,0,0)$.

   (d) * Can you calculate the Gauss map for the torus oriented outward (We described this as a surface of revolution.)

   The Gauss map is deeply related to the geometry of the surfaces. For instance it says a lot about the curvature and total curvature of a surface.

4. Integration of forms

17. Show that our definition of integral of a top form on a manifold is independent of the choice of an atlas and a partition of unity.

18. Show that any smooth top form is integrable on a compact manifold. (Hint. There is a finite atlas.) Similarly show that if a top form has compact support (i.e. it vanishes outside a compact subset of the manifold), then it is integrable.

19. Let $M \subset \mathbb{R}^n$ be a compact oriented k manifold. Show that integration is a linear functional on $\Omega^k(M)$, i.e.
$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$$
where $\omega, \eta \in \Omega^k(M)$ and $a, b \in \mathbb{R}$.

20. Show that the volume form on S^1 is given by $-ydx + xdy$. Consider the function $f : S^1 \to S^1$ given by $f(x, y) = (x^2 - y^2, 2xy)$. Calculate $\int_{S^1} f^* \operatorname{vol}_{S^1}$.

21. Let $M \subset \mathbb{R}^n$ be a compact oriented manifold and A be an orthogonal matrix $\det A = 1$, such that $N = A(M) = \{Ax \mid x \in M\} \subset \mathbb{R}^n$ is again a compact oriented manifold. Show that volume(N) = volume(M).

22. Let $M \subset \mathbb{R}^n$ be a compact oriented manifold and $A(t)$ be an orthogonal matrix for each $t \in [a, b]$ such that $\det A = 1$. Moreover the entries of $A(t)$ are smooth functions of $t \in [a, b] \subset \mathbb{R}$. The set $M' = \{(A(t)x, t) \mid t \in [a, b],\ x \in M\} \subset \mathbb{R}^{n+1}$ is then a manifold with boundary along with a natural orientation. Express the volume of M' in terms of the volume of M.

23. Let $M \subset \mathbb{R}^n$ be a compact k manifold. The centroid of M is the point $(\bar{x}_1, \ldots, \bar{x}_n)$ where $\bar{x}_i$ is the average of the i-th coordinate of all points in M.
$$\bar{x}_i = \frac{1}{\text{volume}(M)} \int_M x_i$$
Find the centroids of the following manifolds:
    (a) $S^2 \subset \mathbb{R}^3$.
    (b) The region in $\mathbb{R}^3$ bounded by the sphere of radius 1 and center $(1, 0, 0)$ and the sphere of radius 4 and center $(0, 0, 0)$.

24. The annulus $A = \{x \in \mathbb{R}^2 \mid 1 \le ||x|| \le 2\}$ is a 2 manifold with boundary. Suppose the orientation on A is given by the the two form $dx_1 \wedge dx_2$, that is the usual orientation of $\mathbb{R}^2$. Then what are the induced orientations on the two boundary components?

25. Let $U = (0, 1) \times (0, 1) \times (0, 1)$ and $\phi : U \to \mathbb{R}^4$ be given by $f(x, y, z) = (x, z, y, (2y - z)^2)$. $M = \phi(U) \subset \mathbb{R}^4$ is a manifold covered by the single chart (U, ϕ). Calculate $\int_M \omega$ where
$$\omega = x_1 dx_1 \wedge dx_4 \wedge dx_3 + 2x_2x_3 dx_1 \wedge dx_2 \wedge dx_3\ .$$