Submit problems 3 and 5 on Tuesday, 27 March.

1. Let $A \subset \mathbb{R}^2$ be the rectangle $[-1/2, 1/2] \times [0, 2\pi]$ and $f : A \to \mathbb{R}^3$ be given by
$$f(s,t) = \big(t\cos(s/2) + 1\big)u(s) + t\sin(s/2)v$$
where $u(s) = \big(\cos(s), \sin(s), 0\big)$ and $v = (0,0,1)$. Show that $M = f(A)$ is a manifold with boundary of dimension 2. What is ∂M?

2. Consider the cylinder $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ -1/2 \le z \le 1/2\}$. Show that C is a manifold with boundary of dimension 2. What is ∂M?

3. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^k$ be a submersion. Let $a \in f(U)$ and
$$M = \{x \in U \mid f(x) = a\}\ .$$
   (a) Let $p \in M$ and (V, ϕ) a chart for M at p, such that $\phi(x) = p$ for $x \in V$. Show that $f_* \circ \phi_* : T_x\mathbb{R}^{n-k} \to T_a\mathbb{R}^k$ is the zero map of vector spaces.

   (b) Show that $T_pM = \ker(f_* : T_p\mathbb{R}^n \to T_a\mathbb{R}^k)$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^1$ function and $\Gamma_f \subset \mathbb{R}^3$ be the graph of f as in problem 10 of assignment 9. Then Γ_f is a manifold.
   (a) Find the tangent space to Γ_f at $p = (0, 0, f(0,0))$.

   (b) Now let $f(x,y) = \dfrac{\cos(x^2+y^2)}{1+x^2+y^2}$ graph this function and repeat part (a) for this function.

5. Consider the following primitive model for a solar system. A star is located at the origin $O = (0,0,0)$, and is stationary. A planet P is revolving around the star at distance $10r$ (at constant speed) along the x-y plane. A satellite S of P is revolving around P at a distance r from P (with constant speed) and is always in the plane containing the z-axis and P. During one complete revolution of P, S revolves 10 times around P. Initially P is at $(10r, 0, 0)$ and S at $(11r, 0, 0)$.
   (a) Show that the trajectory of S is a manifold of dimension 1 and give an atlas for it.

   (b) Find the tangent spaces at the points where the three bodies are collinear.

Multilinear Algebra.

Let V be a vector space and $\mathfrak{T}^k(V)$ denote the space of k tensors over V. $\Lambda^k(V) \subset \mathfrak{T}^k(V)$ is the subset of all alternating k tensors.

1. Show that $\mathfrak{T}^k(V)$ forms a vector space and $\Lambda^k(V)$ is a vector subspace.

2. Recall the tensor product; if $S \in \mathfrak{T}^k(V)$ and $T \in \mathfrak{T}^\ell(V)$ then $S \otimes T \in \mathfrak{T}^{k+\ell}(V)$ is defined by
$$(S \otimes T)(v_1, \dots, v_{k+\ell}) = S(v_1, \dots, v_k)T(v_{k+1}, \dots, v_{k+\ell}).$$
Show that the tensor product satisfies the following properties:
(a) (associativity) $S \otimes (T \otimes U) = (S \otimes T) \otimes U$;

(b) (distributivity) $S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$ and $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$;

(c) (homogeneity) $(cS) \otimes T = S \otimes (cT) = c(S \otimes T)$ for any $c \in \mathbb{R}$.

(d) Give an example where $S \otimes T \neq T \otimes S$ (so commutativity fails).

Let the dimension of V be n and $v_1, \ldots, v_n$ be a basis of V. If $\phi_1, \ldots, \phi_n$ is the dual basis for $V^* = \mathfrak{T}^1(V)$. Then

$$\phi_{i_1} \otimes \cdots \otimes \phi_{i_k}, \quad 1 \leq i_1, \ldots, i_k \leq n$$

is a basis for $\mathfrak{T}^k(V)$. Hence $\mathfrak{T}^k(V)$ has dimension n^k.

3. Recall that a tensor $T \in \mathfrak{T}^2(V)$ is called an inner product if $T(v, w) = T(w, v)$ for all $v, w \in V$ and $T(v, v) > 0$ for all $v \in V$ and $v \neq 0$. Show that there is a basis $v_1, \ldots, v_n$ of V such that

$$T(v_i, v_j) = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Such a basis is called an orthonormal basis for T.

(a) Let $w_1, \ldots, w_n$ be any basis of V. Define $w_1' = w_1$, $w_2' = w_2 - \dfrac{T(w_1', w_2)}{T(w_1', w_1')} w_1'$,

$w_3' = w_3 - \dfrac{T(w_1', w_3)}{T(w_1', w_1')} w_1' - \dfrac{T(w_2', w_3)}{T(w_2', w_2')} w_2'$ and so on. Then show that $T(w_i', w_j') = 0$ if $i \neq j$.

(b) Let $v_i = \dfrac{1}{\sqrt{T(w_i', w_i')}} w_i'$ then show that $v_1, \ldots, v_n$ is an orthonormal basis for T.

4. Recall that if we have a linear transformation $f : V \to W$ between vector spaces, then there is a map $f^* : \mathfrak{T}^k(W) \to \mathfrak{T}^k(V)$.

(a) Show that f^* is a linear transformation.

(b) Show that $f^*\omega \in \Lambda^k(V)$ if $\omega \in \Lambda^k(W)$, So it gives a linear transformation $f^* : \Lambda^k(W) \to \Lambda^k(V)$. (Here there is a convenient abuse of notation.)

(c) If f is an isomorphism then show that f^* is also an isomorphism.

(d) If T is an inner product on V and $v_1, \ldots, v_n$ and orthonormal basis for T, then define the linear transformation $g : V \to \mathbb{R}^n$ given by $g(v_i) = e_i$ where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Show that $T = f^*\langle \, , \, \rangle$, where $\langle \, , \, \rangle$ is the standard inner product on $\mathbb{R}^n$.

5. Recall that $\text{Alt} : \mathfrak{T}^k(V) \to \mathfrak{T}^k(V)$ defined as follows, if $T \in \mathfrak{T}^k(V)$ the tensor $\text{Alt}(T)$ does the following

$$\text{Alt}\, T(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Show the following:

(a) $\text{Alt}(T) \in \Lambda^k(V)$. (Hint. Composing a 2-cycle with a permutation changes the sign of the permutation.)

(b) $\text{Alt}(\omega) = \omega$ if $\omega \in \Lambda^k(V)$. (Hint. Show that all the summands of Alt are the same.)

6. If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$ then their wedge product is defined as follows

$$\omega \wedge \eta = \frac{(k + \ell)!}{k!\ell!} \text{Alt}(\omega \otimes \eta)$$

Show that the wedge product satisfies the following properties:

(a) $\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega$ (graded commutativity);

(b) $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$ and $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$ (distributivity);

(c) $(a\omega) \wedge \eta = \omega \wedge (a\eta) = a(\omega \wedge \eta)$ for any $a \in \mathbb{R}$ (homogeneity);

The wedge product also satisfies associativity but it is a bit harder to prove. Refer to Theorem 4-4 on page 80 of Spivak.

Let the dimension of V be n and $v_1, \dots, v_n$ be a basis of V. If $\phi_1, \dots, \phi_n$ is the dual basis for $V^* = \Lambda^1(V)$. Then

$$\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}, \quad 1 \leq i_1 < \ldots < i_k \leq n$$

is a basis for $\mathfrak{T}^k(V)$. Hence $\mathfrak{T}^k(V)$ has dimension $\binom{n}{k}$. In particular $\Lambda^n(V)$ is 1-dimensional and $\Lambda^k(V)$ is trivial if $k > n$.

7. Prove the following for the wedge product:
   (a) If $\phi\Lambda^1(V)$ then $\phi \wedge \phi = 0$. (Hint. Use problem 6 (a).)

   (b) If $\phi_1, \dots, \phi_k \in \Lambda^1(V)$ then

$$\phi_1 \wedge \ldots \wedge \phi_k = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(n)}$$

Recall that any non-zero element ω of $\Lambda^n(V)$ gives an orientation of V. If $v_1, \dots, v_n$ is a (ordered) basis of V then it is positively oriented if $\omega(v_1, \dots, v_n) > 0$ otherwise it is negatively oriented.

8. If $\omega \in \Lambda^n(V)$ is non-zero then $\omega(v_1, \dots, v_n) \neq 0$ if and only if $v_1, \dots, v_n$ is a basis of V.

9. If $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$ and $\phi_1, \dots, \phi_n$ is the dual basis of $\Lambda^1(\mathbb{R}^n)$, show that det $=$ $\phi_1 \wedge \ldots \wedge \phi_n$, where det $\in \Lambda^n(\mathbb{R}^n)$ is the determinant.

10. det is the standard orientation of $\mathbb{R}^n$. Show that for $\mathbb{R}^2$ and $\mathbb{R}^3$ it gives the same definition of orientation that we have in physics.
    (a) If v_1, v_2 is a orthornormal basis of $\mathbb{R}^2$ (with respect to the standard inner product on $\mathbb{R}^n$), then it is positively oriented if v_1 has to be rotated by an angle of $\pi/2$ in counter-clockwise direction to get v_2.

    (b) If v_1, v_2, v_3 is an orthonormal basis for $\mathbb{R}^3$, then it is positively oriented if it satisfies the right hand cork screw rule. (Hint. Use the cross product.)

    (c) Notice that ordering of the basis elements is quite crucial for its orientation. e_1, e_2 is a positively oriented basis of $\mathbb{R}^2$ where as e_2, e_1 is negatively oriented. Show that in general $e_1, \dots, e_n$ is always positively oriented basis for $\mathbb{R}^n$ with respect to the standard orientation, and $e_{\sigma(1)}, \dots, e_{\sigma(n)}$ is positively oriented if $\sigma \in S_n$ is an even permutation.