Submit problems 3 and 6 on Wednesday, 14 March by 17:00.

1. Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a submersion and $a \in \mathbb{R}^p$, show that the level set
$$M_a = \{x \in \mathbb{R}^n \mid f(x) = a\}$$
is a manifold of dimension $n - p$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a submersion and $a \in \mathbb{R}$, show that
$$H_a = \{x \in \mathbb{R}^n \mid f(x) \geq a\}$$
is an n dimensional manifold with boundary. What is ∂H_a?

3. Show that the ellipsoid
$$E = \left\{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$
is a manifold of dimension 2 in $\mathbb{R}^3$. Give an explicit atlas for E.

4. Show that the following sets are not manifolds.
   (a) The union of the x and y axes, $A = \{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}$. (Hint. Let U be a connected open neighbourhood of $(0, 0)$ in A, then $U \backslash \{(0, 0)\}$ has 4 connected components.)
   (b) The square $S = \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| = 1\}$
   (c) The surface $\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^3\}$.

5. The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$ is a manifold of dimension n. Show that the following is an atlas for S^n: $\{(U_1, \phi_1), (U_2, \phi_2)\}$ where $U_1 = U_2 = \mathbb{R}^n$,
$$\phi_1(x) = \frac{1}{||x||^2 + 1}(2x_1, \dots, 2x_n, ||x||^2 - 1) \quad \text{and} \quad \phi_2(x) = \frac{1}{1 + ||x||^2}(2x_1, \dots, 2x_n, 1 - ||x||^2).$$

6. The space of $n \times n$ matrices $\mathrm{M}_n(\mathbb{R})$ can be identified with $\mathbb{R}^N$ where $N = n^2$. Show that the subset of orthogonal matrices $\mathrm{O}_n(\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R}) \mid AA^t = I\}$ is a compact manifold and find its dimension. (Here A^t denotes the transpose of A, and I is the identity matrix.)

7. Show that the upper hemisphere $M = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1 \text{ and } x_n \geq 0\}$ is an n dimensional manifold with boundary. What is ∂M?

8. Let $M \subset \mathbb{R}^n$ be a manifold with boundary of dimension k show that ∂M is a manifold of dimension $k - 1$.

9. Suppose $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are manifolds of dimension k and ℓ respectively. Show that $M \times N \subset \mathbb{R}^{m+n}$ is a manifold of dimension $k + \ell$. Is the result true if M and N are manifolds with boundary?

10. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a $\mathcal{C}^1$ function. Show that the graph of f denoted by Γ_f
$$\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+m}$$
is a manifold of dimension n.