Submit problems 4 and 10 on Thursday, March 1. Reading, chapter 4 of Munkres.

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. For $a \in \mathbb{R}$ we defined the a-th level set of f to be
$$L_f(a) = \{x \in \mathbb{R}^n \mid f(x) = a\}$$
Suppose $Df(x) \neq 0$ for all $x \in L_f(a)$, then show that $L_f(a)$ has measure 0.
(Hint. Use the implicit function theorem and Sard's theorem. Show first that $L_f(a)$ intersected with any closed rectangle has measure 0. Then show that $L_f(a)$ is a countable union of measure 0 sets.)

2. Show that the following sets are Jordan domains and find their volumes.
   (a) The 4 dimensional closed unit ball of radius 1 $\overline{B_1(0)} \subset \mathbb{R}^4$.
   (The volume of an n-dimensional unit ball goes to 0 as n goes to ∞. See `http://en.wikipedia.org/wiki/Volume_of_an_n-ball`)
   (b) $E \subset \mathbb{R}^3$ given by
$$E = \left\{ (\lambda x, \lambda y, \lambda) \mid 0 \leq \lambda \leq 1,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\} .$$

3. What is the mass of a solid hemisphere or radius 1 meter centred at the origin whose mass density is $\mu(x, y, z) = \dfrac{1}{1 + z^2}$ kg/m^3.

4. Evaluate the following integrals (convince yourself that the sets are Jordan domains):
   (a) $\int_D x_1^2 e^{x_2}$ where $D \subset \mathbb{R}^2$ is the triangle with vertices $(0, 0)$, $(0, 2)$ and $(2, 2)$;
   (b) $\int_E x_1^2$ where $E = \{(x_1, x_2) \mid 1 \leq x_1^2 + x_2^2 \leq 2\} \subset \mathbb{R}^2$;
   (c) $\int_F z$ where $F = \{(x, y, z) \mid x \geq 0, y \geq 0, 0 \leq z \leq 1, x^2 + y^2 \leq 1\}$.

5. Let $V \subset \mathbb{R}^n$ be an open set. Show that there are compact sets $C_1, C_2, \dots$ such that $C_k \subset \operatorname{Int} C_{k+1}$ and $\cup_{k=1}^{\infty} C_k = V$.

6. Let $J \subset \mathbb{R}^2$ be a Jordan domain such that if $(x, y) \in J$ then $x > 1$. Let $E = \{(x, 0, y) \in \mathbb{R}^3 \mid (x, y) \in J\} \subset \mathbb{R}^3$ and let F be the set obtained by rotating E about the z-axis. Find an expression for the volume of F in terms of the volume of J.

7. Calculate the following integrals (using a change of variables if necessary).
   (a) $\int_E e^{x^2+y^2}$ where $E = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1,\ x \geq 0\}$.
   (b) $\int_E \frac{1}{\sqrt{x^2+y^2}}$ in $\mathbb{R}^2$ where E is the annulus $1 \leq ||x|| \leq 2$.

8. Let $V \subset \mathbb{R}^n$ be an open set and $\{\phi_i\}_{i=1}^{\infty}$ be a locally finite partition of unity on V (subordinate to some open cover). If $C \subset V$ is compact show that all but finitely many ϕ_i are zero on C.

9. Let $U \subset \mathbb{R}^n$. A function function $g : U \to \mathbb{R}^m$ is called Lipschitz if there is a positive real number k such that $||g(x) - g(y)|| \leq k||x - y||$.
   (a) Show that a Lipschitz function takes measure 0 sets to measure 0 sets.
   (b) Suppose U is open and g is $\mathcal{C}^1$, show that g is Lipschitz on any closed rectangle $V \subset U$.
   (c) Any diffeomorphism takes a Jordan domain to a Jordan domain.

10. Suppose $f : U \to \mathbb{R}^n$ is a diffeomorphism such that $v(W) = v(g(W))$ for any Jordan domain $W \subset U$, what can you say about f. Such a diffeomorphism is called volume preserving. (Hint. Think of the change of variables formula.)

11. Let $g : U \to \mathbb{R}^n$ be a diffeomorphism, such that for any $t \in \mathbb{R}^n$ and $x \in U$ we have $||Dg(x)t|| = ||t||$. Such a diffeomorphism is called an isometry. Show that g is volume preserving.

12. (**Polar Coordinates**). Show that $g : (0, \infty) \times (0, 2\pi) \to \mathbb{R}^2$ given by $g(r, \theta) = (r\cos\theta, r\sin\theta)$ is a diffeomorphism. The pair $g^{-1}(x, y)$ is called the polar coordinates of the point in $\mathbb{R}^2$ whose Cartesian coordinates are (x, y).
    Let $B_a = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2\}$ and $C_a = [-a, a] \times [-a, a]$. Show that
$$\int_{B_a} e^{-x^2-y^2} = \pi(1 - e^{-a^2})$$
and
$$\int_{C_a} e^{-x^2-y^2} = \left[\int_{-a}^{a} e^{-x^2}\right]^2.$$
Show that
$$\int_{\mathbb{R}^2} e^{-x^2-y^2} = \lim_{a\to\infty} \int_{B_a} e^{-x^2-y^2} = \lim_{a\to\infty} \int_{B_a} e^{-x^2-y^2}$$
Infer that
$$\int_{\mathbb{R}} e^{-x^2} = \sqrt{\pi}.$$

13. (**Spherical Coordinates**). Show that $h : (0, \infty) \times (0, 2\pi) \times (0, pi) \to \mathbb{R}^3$ given by $h(r, \theta, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos$ is a diffeomorphism. The triple $h^{-1}(x, y, z)$ are called the spherical coordinates of the point in $\mathbb{R}^3$ whose Cartesian coordinates are (x, y, z). Use these coordinates to calculate the following.
    (a) Volume of the region
$$\{(x, yz) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2} \leq z,\ 1 \leq x^2 + y^2 + z^2 \leq 2\}.$$
    (b) If $R = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x, 0 \leq y, 0 \leq z, x^2 + y^2 + z^2 \leq 1\}$ calculate
$$\int_R e^{(x^2+y^2+z^2)^{3/2}}.$$

14. (**Cylindrical Coordinates**). Show that $f : (0, \infty) \times (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$ given by $f(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$ is a diffeomorphism. The triple $f^{-1}(x, y, z)$ are called the cylindrical coordinates of the point in $\mathbb{R}^3$ whose Cartesian coordinates are (x, y, z). Use these coordinates to calculate the following.
    (a) Volume of the region given by $E = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq x^2 + y^2 + 1, x^2 + y^2 \leq 1\}$.
    (b) Mass of a bullet with density $2z$ g/cm^3 and shape $B = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq 1 - x^2 - y^2\}$ where units are again in centimetres.

15. (**Barycentric Coordinates**). Let a_0, a_1, a_2, a_3 be four points in $\mathbb{R}^3$ not lying on a plane. Then these points determine a tetrahedron and the points of the tetrahedron have Cartesian coordinates $(t_0a_0+t_1a_1+t_2a_2+t_3a_3)$ where $t_i \geq 0$ and $t_0+t_1+t_2+t_3 = 1$. The triple (t_1, t_2, t_3) are the barycentric coordinates of the point. Show that $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(t_1, t_2, t_3) = a_0 + t_1(a_1 - a_0) + t_2(a_2 - a_0) + t_3(a_3 - a_0)$ is a diffeomorphism. Integration of functions on the tetrahedron can be done quite easily using these coordinates.
    (a) Find the volume of the tetrahedron determined by the points a_0, a_1, a_2, a_3.
    (b) There are similar barycentric coordinates on the convex polyhedron determined by $n + 1$ points in $\mathbb{R}^n$ in general positions (i.e. not contained in a hyperplane). Find the volume of such a polyhedron.

16. Let $f : U \to \mathbb{R}^n$ be a diffeomorphism and $0 \in U$. Let $A_t = [0, t]^n \subset \mathbb{R}^n$ be the closed rectangle which is the n-fold product of the closed interval $[0, t]$. If t is small enough then $A_t \subset U$. Let $g(t) = v(f(A_t))$ (note that $f(A_t)$ is a Jordan domain by problem 1). Show that
$$\lim_{t\to 0} \frac{g(t)}{t^n} = |\det Df(0)|.$$

17. Let $v_1, v_2, \ldots, v_n$ be linearly independent vectors in $\mathbb{R}^n$. The n-parallelepiped $\mathcal{P}(v_1, \ldots, v_n)$ is the set
$$\{c_1v_1 + \cdots + c_nv_n \mid 0 \leq c_i \leq 1\} \subset \mathbb{R}^n.$$
Find the volume of $\mathcal{P}(v_1, \ldots, v_n)$. (Hint. Choose a suitable linear change of coordinates.)