Submit problems 2 and 7 on Monday, February 12. Reading Munkres chapter 3.

1. Show that the function $g : \mathbb{R}^2 \to \mathbb{R}^3$ given by $g(x, y) = (\cos(x+2y), \sin(x+2y), 2x+4y)$ is not an immersion. Write g as a composition of a submersion and an immersion.

2. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $g(x) = ||x||^2 x$. Show that g is $\mathcal{C}^1$ and a bijection on $\mathbb{R}^n$ however g^{-1} is not differentiable at 0.

3. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $g(x, y) = (2ye^x, xe^y)$ and $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x, y) = (3x - y^2, 2x + y, xy + y^3)$.
   (a) Show that g maps an open neighbourhood $U$of $(0, 1)$ bijectively onto an open neighbourhood V of $(2, 0)$.
   (b) Show that $f \circ g^{-1}$ is differentiable on V and calculate $D(f \circ g^{-1})(2, 0)$.
   (c) Show that $f \circ g^{-1}$ is an immersion in an open neighbourhood of $(2, 0)$.

4. Show that the following sets have measure 0
   (a) The cantor set $C \subset [0, 1]$. Recall that C is uncountable.
   (b) The subset $A = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\} \subset \mathbb{R}^n$.
   (c) The set of rational numbers in $\mathbb{R}$.

5. Show that the following sets do not have measure 0.
   (a) The closed annulus $A = \{x \in \mathbb{R}^2 \mid 1 \leq ||x|| \leq 2\} \subset \mathbb{R}^n$.
   (b) The set of irrational numbers in $[0, 1]$ inside $\mathbb{R}$.

6. Show that the following functions are integrable.
   (a) $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1/q & x = \frac{p}{q} \text{ where } p, q \text{ are coprime integers} \\ 0 & \text{otherwise .} \end{cases}$$

   Show that f is integrable on $[0, 1]$ and evaluate the integral. (Hint. What is the set of discontinuities of f?)
   (b) If $f, g : [0, 1] \to \mathbb{R}$ are increasing functions then show that the function $h(x, y) = f(x)g(y)$ on $[0, 1] \times [0, 1]$ is integrable.

7. Let $f : A \to \mathbb{R}$ be an integrable function where $A \subset \mathbb{R}^n$ is a closed rectangle. If $\int_A f = 0$ and $f(x) \geq 0$ for all $x \in A$ show that the set $\{x \in A \mid f(x) > 0\}$ has measure 0. (We say f vanishes almost everywhere.) In the other direction show that if $g : A \to \mathbb{R}$ vanishes almost everywhere and g is integrable on A then $\int_A g = 0$.

8. Let A be a closed rectangle in $\mathbb{R}^n$ and $f : A \to \mathbb{R}$ a bounded function. Let $B \subset A$ be a closed rectangle such that $f(x) = 0$ if $x \in A \backslash B$. Show that f is integrable on A if and only if f is integrable on B and that the integrals over the two rectangles are the same.

   Let $S \subset \mathbb{R}^n$ be bounded and $f : S \to \mathbb{R}$ be an integrable function. Show that the integral of f on S is well defined, i.e. it does not depend on the rectangle $A \supset S$.

9. Show that any continuous function is integrable on a Jordan domain.

10. Let $f : A \to \mathbb{R}$ be a bounded function and A a closed rectangle in $\mathbb{R}^n$. If f vanishes outside a closed set of measure 0, show that f is integrable on A and $\int_A f = 0$.

11. Show that the fat cantor set is not a Jordan domain. For reference see `http://en.wikipedia.org/wiki/Smith-Volterra-Cantor_set#Other_fat_Cantor_sets` . This is an example of a compact set which is not a Jordan domain.

12. Note that $A = (\mathbb{Q} \times \mathbb{Q}) \cup ((0, 1) \times (0, 1))$ is countable. Number the points of A as $a_1, a_2, \dots$. Let T_k be an open rectangle such that $a_k \in T_k \subset (0, 1) \times (0, 1)$ and $v(T_k) \leq 1/4^k$. Let $T = \cup_{i=1}^{\infty} T_k$, then show that T is a bounded open set which is not a Jordan domain.