Submit problems 5 and 9 on Wed, Jan 24. Reading Chapter 2 of Spivak.

1. (Norm of a linear transformation) Let $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Define its norm by
$$||\lambda|| = \sup\{||\lambda(x)|| \mid x \in \mathbb{R}^n, ||x|| = 1\}$$
   (a) If A is the matrix for λ with respect to the standard bases, show that $||\lambda|| \leq \sqrt{\sum_{i,j} A_{i,j}^2}$ and $|A_{i,j}| \leq ||\lambda||$ for any i, j.
   (b) Show that $||\lambda(x)|| \leq ||\lambda|| \times ||x||$.

2. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $G(x_1, x_2) = (\sin(x_1 x_2), \cos(x_1 + x_2))$. Calculate all the second order derivatives and show that G is $\mathcal{C}^2$. Calculate $\dfrac{\partial^3 G_1}{\partial^2 x_1 \partial x_2}$.

3. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ where $n < m$ be continuously differentiable. Let $a \in \mathbb{R}^n$ be such that $DF(a)$ has rank n. Show that there is an open neighborhood V of $F(a)$ and a differentiable function $G : V \to \mathbb{R}^n$ such that $G(F(x)) = x$ for any $x \in F^{-1}(V)$. What goes wrong if $n > m$?

   (Hint. Cook up a function $H : \mathbb{R}^m \to \mathbb{R}^m$ of the form $H(x_1, \ldots, x_m) = F(x_1, \ldots, x_n) + A(x_{n+1}, \ldots, x_m)$ such that you can use inverse function theorem for H, and would give the result in this case.)

4. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be $\mathcal{C}^1$ and $\det \mathrm{D} f(x) \neq 0$ for all $x \in U$. Show that $f(A)$ is open for any $A \subset U$ open.

5. Consider the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by $g(x, y) = (e^x \cos y, e^x \sin y)$.
   (a) Show that $\det \mathrm{D} g(x, y) \neq 0$ for all (x, y) and infer that g is an open mapping.
   (b) Show that g is not injective or surjective hence there is no global inverse function.
   (c) Note that $g(0, 0) = (1, 0)$ find open neighborhoods V and W of $(0, 0)$ and $(1, 0)$ such that $g : V \to W$ is a bijection and expicitly write down an inverse $g^{-1} : W \to V$.

6. Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}^m$ be $\mathcal{C}^1$ with $m < n$, we will show that f can not be injective.
   (a) If $n = 2$ and $m = 1$, let $\partial_1 f(a) \neq 0$ at some point $a \in A$ show that the function $F(x, y) = (f(x, y), y)$ is $\mathcal{C}^1$ and has inveritble derivative at a, infer that f can not be injective near a.
   (b) Now generalise the proof for arbitrary $m < n$.

7. * Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}^m$ be $\mathcal{C}^1$ with $m > n$. Show that $f(A)$ can not contain an open set.

8. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be $\mathcal{C}^1$ and invertible. Suppose $\det \mathrm{D} F(a) = 0$, show that F^{-1} is not differentiable at $F(a)$.

9. Consider the function $f : \mathbb{R} \to \mathbb{R}$
$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin \frac{1}{x} & x \neq 0, \\ 0 & x = 0 \end{cases}$$
   Show that f is differentiable at 0 and $f'(0) \neq 0$. Show also that there is no open neighborhood of 0 on which f is injective. Why can we not apply the inverse function theorem?

10. Let $f : \mathbb{R} \to \mathbb{R}$ be given by
$$f(t) = \begin{cases} e^{-1/t} & x > 0, \\ 0 & x \leq 0 \end{cases}$$
   (a) Show that f is $\mathcal{C}^\infty$ and all the derivatives vanish at 0.
   (b) Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be given by $\gamma(t) = (f(t), f(-t))$. Plot the image of γ. What is $D\gamma(0)$?
   (c) Show that there can not be a $\mathcal{C}^1$ parametrization $\phi : \mathbb{R} \to \mathbb{R}^2$ such that $\phi(\mathbb{R}) = \gamma(\mathbb{R})$, $\phi(a) = 0$ and $D\phi(a) \neq 0$.