Submit problems 3, 4, 5 on Jan 16. Reading Chapter 2 of Calculus on Manifolds by M. Spivak.

1. Let $F : \mathbb{R}^n \to \mathbb{R}^m$, with $F(x) = (F_1(x), \ldots, F_m(x))$. Show that
   (a) F is continuous at $a \in \mathbb{R}^n$ if and only if all the coordinate functions F_i are continuous at a.
   (b) If F is a linear transformation show that F is continuous.

2. Recall that if a function is differentiable at a point then it is continuous at that point. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by
$$F(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2^2}{x_1^2 + x_2^4} & (x_1, x_2) \neq (0,0) \\ 0 & (x_1, x_2) = (0,0) \end{cases}$$
   Show that all directional derivatives of F exist at the origin but F is not differentiable at the origin.

3. Calculate the Jacobian matrix for the following functions
   (a) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(x) = Ax$ for some $m \times n$ matrix A. (Here as usual we treat $x \in \mathbb{R}^n$ as a column vector.)
   (b) $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (x^y, z)$
   (c) $G : \mathbb{R}^2 \to \mathbb{R}$ defined by $G(x_1, x_2) = \sin(x_1 \sin x_2)$
   (d) $H : \mathbb{R}^2 \to \mathbb{R}^2$ given by $H(x, y) = (x^2 + 2y, x + 3y^2)$ find the derivative at $(1, 1)$ as a linear transformation.

4. Show that the function $G : \mathbb{R}^2 \to \mathbb{R}$ defined by $G(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$.

5. Let $H : \mathbb{R}^n \to \mathbb{R}$ such that $|H(x)| \leq ||x||^2$. Show that H is differentiable at 0 and find the derivative.

6. $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$ and a is a local maxima of F, i.e. there exists $\epsilon > 0$ such that for any $x \in \mathbb{R}^n$ if $||x - a|| < \epsilon$ then $F(x) \leq F(a)$. Show that $\mathrm{D}\, F(a) = (0, \ldots, 0)$.

7. A function $H : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree m if $H(tx) = t^m H(x)$ for any $x \in \mathbb{R}^n$ and any $t \in \mathbb{R}$. Show that if H is also differentiable then
$$\sum_{i=1}^{n} x_i \partial_i H(x) = mH(x)$$
   (Hint. If $g(t) = F(tx)$ what is $g'(1)$?)

8. Let $G : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable and $F : \mathbb{R}^3 \to \mathbb{R}$ be $F(x_1, x_2, x_3) = G(x_1, G(x_1, G(x_2, x_3)))$. Find $DF(0, 0, 0)$ given that $G(0, 0) = 0$, $\partial_1 G(0, 0) = 1$ and $\partial_2 G(0, 0) = 2$.

9. Let $G : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable. Show that $\partial_1 \partial_2 G(x) = 0$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ if and only if $G(x) = g_1(x_1) + g_2(x_2)$ for differentiable functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$.

10. Let $G : \mathbb{R}^2 \to \mathbb{R}^3$ be the function $G(x_1, x_2) = (x_1 + x_2, x_1 - x_2, sin(x_1 x_2))$. The image of G is a surface in $\mathbb{R}^3$. Plot this surface with your favorite graphing program. Find an equation for the tangent plane to the surface at $(2\sqrt{\pi}, 0, 0)$. (Hint. The partial derivatives, and hence all directional derivatives, are tangent to the surface.)