1. Manifold

A subset $M \subset \mathbb{R}^n$ is $\mathcal{C}^r$ a manifold of dimension k if for any point $p \in M$, there is an open set $U \subset \mathbb{R}^k$ and a $\mathcal{C}^r$ function $\phi : U \to \mathbb{R}^n$, such that

(1) ϕ is injective and $\phi(U) \subset M$ is an open subset of M containing p;
(2) $\phi^{-1} : \phi(U) \to U$ is continuous;
(3) $\mathrm{D}\,\phi(x)$ has rank k for every $x \in U$.

Such a pair (U, ϕ) is called a chart of M. An atlas for M is a collection of charts $\{(U_i, \phi_i)\}_{i\in I}$ such that

$$\bigcup_{i\in I} \phi_i(U_i) = M$$

(that is the images of ϕ_i cover M). (In some texts, like Spivak, the pair $(\phi(U), \phi^{-1})$ is called a chart or a coordinate system.)

We shall mostly study $\mathcal{C}^\infty$ manifolds since they avoid any difficulty arising from the degree of differentiability. We shall also use the word smooth to mean $\mathcal{C}^\infty$.

If we have an explicit atlas for a manifold then it becomes quite easy to deal with the manifold. Otherwise one can use the following theorem to find examples of manifolds.

Theorem 1 (Regular value theorem)**.** *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^k$ be a $\mathcal{C}^r$ function. Let $a \in \mathbb{R}^k$ and*

$$M_a = \{x \in U \mid f(x) = a\} = f^{-1}(a).$$

The value a is called a regular value of f if f is a submersion on M_a; that is for any $x \in M_a$, $\mathrm{D}\,f(x)$ has rank k. Then M_a is a $\mathcal{C}^r$ manifold of dimension $n - k$.

The set M_a is called a level set of f; thus any level set of a regular value of f is a manifold. The proof of this theorem easily follows from the implicit function theorem. See PS5 for some examples of manifolds.

If (U, ϕ) and (V, ψ) are two charts of M and $\phi(U) \cap \psi(V) \neq \emptyset$ then the function

$$\psi^{-1} \circ \phi : \phi^{-1}(\psi(V)) \to V$$

is a diffeomorphism. It is called a transition function between the two charts.

2. Manifold with Boundary

Let $\mathbb{H}^k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_k \geq 0\}$ be the upper half space in $\mathbb{R}^k$. Note that the boundary of $\mathbb{H}^k$ as a subset of $\mathbb{R}^k$ is the set $\mathrm{Bd}\,\mathbb{H}^k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_k = 0\} = \mathbb{R}^{k-1} \times \{0\}$.

A subset $M \subset \mathbb{R}^n$ is a $\mathcal{C}^r$ manifold with boundary of dimension k if for any point $p \in M$, there is an open set $U \subset \mathbb{H}^k$ or $U \subset \mathbb{R}^k$ and a $\mathcal{C}^r$ function[1] $\phi : U \to \mathbb{R}^n$, such that

(1) ϕ is injective and $\phi(U) \subset M$ is an open subset of M containing p;
(2) $\phi^{-1} : \phi(U) \to U$ is continuous;
(3) $\mathrm{D}\,\phi(x)$ has rank k for every $x \in U$.

[1] Recall that if $S \subset \mathbb{R}^m$ is an arbitrary subset then we say $f : S \to \mathbb{R}^p$ is $\mathcal{C}^r$ if there is an open set $V \supset S$ and a $\mathcal{C}^r$ function $g : V \to \mathbb{R}^p$ such that $f(x) = g(x)$ for all $x \in S$.

Again the pair (U, ϕ) is called a chart for M, and a collection of charts which cover M is called an atlas. Note that a chart (U, ϕ) for a manifold with boundary is defined in the exact same way as a chart for a manifold (without boundary), however now the open set can come from $\mathbb{H}^k$.

Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold with boundary. Then $p \in M$ is an interior point if there is a chart (U, ϕ) of M with $U \subset \mathbb{R}^k$ open and $p \in \phi(U)$.

Any point of M which is not an interior point is called a boundary point and the set of all boundary points of M denoted by ∂M. In fact $\partial M \subset \mathbb{R}^n$ is a manifold (without boundary) of dimension $k-1$ (one less than dimension of M).

A point $p \in M$ is in ∂M if there is a chart (U, ϕ) where $U \subset \mathbb{H}^k$ is open and $p = \phi(x)$ for some $x \in U \cap \operatorname{Bd} \mathbb{H}^k$ (i.e. if $x = (x_1, \ldots, x_k)$ then $x_k = 0$).

Notice that the concept of boundary of a manifold M in $\mathbb{R}^n$ denoted ∂M is different from the concept of the boundary of a subset S of $\mathbb{R}^n$ denoted $\operatorname{Bd} S$. Similarly the interior of M namely $M - \partial M$ is a different concept to the interior of a set S denoted $\operatorname{Int} S$.

One can generalise Theorem 1 as follows:

Theorem 2. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^k$ be a $\mathcal{C}^r$ submersion, then*

$$M = \{x \in U \mid f_1(x) = \ldots = f_{k-1}(x) = 0, f_k(x) \geq 0\}$$

is a $\mathcal{C}^r$ manifold with boundary of dimension $n - k + 1$ and

$$\partial M = \{x \in U \mid f(x) = 0\}.$$

3. Maps between Manifolds

Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be $\mathcal{C}^\infty$ manifolds, and $f : M \to N$ a function. f is differentiable if for any two charts (U, ϕ) of M and (V, ψ) of N, the composite function $\psi^{-1} \circ f \circ \phi$ is differentiable. f is smooth if all the composite functions are smooth.

$f : M \to N$ is a diffeomorphism if f is smooth and there is a smooth function $g : N \to M$ for which $f \circ g = \mathrm{Id}_N$ and $g \circ f = \mathrm{Id}_M$.

Proposition 3. *$f : M \to N$ is a diffeomorphism if and only if f is a smooth bijection and for any for any two charts (U, ϕ) of M and (V, ψ) of N, the derivative of the composite function $\psi^{-1} \circ f \circ \phi$ is invertible at every point.*

This proposition follows easily from the inverse function theorem which will say that the inverse of f is also smooth.

For manifolds with boundary the definitions are completely analogous. It is easy to see that in that case a diffeomorphism takes boundary of domain manifold to the boundary of the range manifold and induces a diffeomorphism of the two boundaries.

See assignment 10 for examples of diffeomorphic manifolds.

4. Tangent spaces and Vector fields

Let $p \in \mathbb{R}^n$, then the tangent space of $\mathbb{R}^n$ at p is the set $\{p\} \times \mathbb{R}^n$.

$$T_p\mathbb{R}^n = \{(p, v) \mid v \in \mathbb{R}^n\}$$

Some times the vector (p,v) is also written as v_p and usually the vector v is written as a column vector whereas the point p is a point of $\mathbb{R}^n$. The tangent space forms a vector space.

If $f:\mathbb{R}^n\to\mathbb{R}^n$ is a differentiable map then the derivative of f gives a map between tangent spaces. If $p\in\mathbb{R}^n$, then define $f_*:T_p\mathbb{R}^n\to T_{f(p)}\mathbb{R}^m$ as

$$f_*((p,v))=(f(p),\mathrm{D}\,f(p)v).$$

Note that $\partial_v f(p)=\mathrm{D}\,f(p)v$ is the directional derivative of f at p along v. Hence the tangent vectors are directions along which we differentiate functions.

From the chain rule it follows that if $g:\mathbb{R}^n\to\mathbb{R}^m$ and $f:\mathbb{R}^m\to\mathbb{R}^k$ then $(f\circ g)_*=f_*\circ g_*$.

Let $M\subset\mathbb{R}^n$ be a manifold of dimension k, and $p\in M$. Suppose (U,ϕ) is a chart of M such that $p=\phi(x)$ then the tangent space of M at p is defined as

$$T_pM=\phi_*(T_x\mathbb{R}^k).$$

Since $D\phi(x)$ has rank k, so T_pM is a vector subspace of $T_p\mathbb{R}^n$ of dimension k.[2]

Theorem 4. [3] *If $f:\mathbb{R}^n\to\mathbb{R}^k$ and $a\in\mathbb{R}^k$ is a regular value of f, then $M=f^{-1}(a)$ is a manifold. Let $p\in M_a$, then*

$$T_pM=\ker(f_*:T_p\mathbb{R}^n\to T_a\mathbb{R}^k).$$

A differentiable function $f:M\to N$ between manifolds induces a linear map between tangent spaces. If $p\in M$ then we get an induced linear map $f_*:T_pM\to T_{f(p)}N$ which is the linearisation of f at p.

The set

$$T\mathbb{R}^n=\bigcup_{p\in\mathbb{R}^n}T_p\mathbb{R}^n=\mathbb{R}^n\times\mathbb{R}^n, \tag{1}$$

is called the tangent bundle of $\mathbb{R}^n$ and is naturallyidentified with $\mathbb{R}^{2n}$.

The set

$$TM=\bigcup_{p\in M}T_pM\subset T\mathbb{R}^n$$

is called the tangent bundle of M. Although we haven't proved this, but it is a manifold of dimension $2k$ in $\mathbb{R}^n\times\mathbb{R}^n$.

If M is a manifold, a continuous (respectively smooth) vector field X on M is a continuous (respectively smooth) function $X:M\to TM$, such that $X(p)\in T_pM$ for each $p\in M$. Thus a vector field is a function form the manifold to its tangent bundle which assigns a tangent vector at each point on M.

If $U\subset\mathbb{R}^n$ is an open subset (hence an n dimensional manifold) then we have the standard vector fields on U. Let $e_1,\dots,e_n$ be the standard basis of $\mathbb{R}^n$. Then define the vector fields $E_1,\dots,E_n$ on U by

$$E_1(p)=(p,e_1),\dots,E_n(p)=(p,e_n).$$

These vector fields form a *frame* for the tangent bundle of U in the sense that at each point the vector fields form a basis for the tangent space at that point.

Any other vector field X can be written as

$$X(p)=f_1(p)E_1(p)+\cdots+f_n(p)E_n(p).$$

Thus a vector field on U is equivalent to a function $f=(f_1,\dots,f_n):U\to\mathbb{R}^n$. The vector field is continuous or smooth if and only if the corresponding function is continuous or smooth.

[2] In ordinary vector calculus the tangent space to a point on a manifold is an affine space. For example the tangent line to a point on a curve or the tangent plane to a point on a surface. If $p\in M$ note that the tangent space at p to M according to usual vector calculus is the set of points $p+v$ where $(p,v)\in T_pM$. However this concept has the draw back that the tangent is not a vector space (may not contain 0).

[3] This is just the generalisation of the fact in vector calculus that if a surface is given by an equation, that is the zero set of a function then the tangent plane at a point to the surface is the plane perpendicular to the gradient of that function.

On an arbitrary manifold there may not exist such a frame for vector fields. For example in class we saw that there is a frame in the case of S^1 but not in the case of S^2. However there is a frame when we restrict to the image of a chart.

If M is a manifold of dimension k, and (U, ϕ) is a chart of M, then the vector fields $X_1, \ldots, X_k$ defined by

$$X_i(p) = (p, \partial_i \phi) = \phi_* E_i(\phi^{-1}(p))$$

form a basis of the tangent space of M at each point of $\phi(U)$.

Any vector field X on M then can be written as $X = f_1 X_1 + \ldots + f_k X_k$ on $\phi(U)$ and X is continuous or smooth if and only if all the functions f_i are continuous or smooth for each chart of M.

4.1. **Tangent space at boundary of a manifold.** If $M \subset \mathbb{R}^n$ is a k manifold with boundary then our previous definition of tangent space sort of works also at the boundary points.

If $p \in \partial M$, and (U, ϕ) is a chart for M with $U \subset \mathbb{H}^k$, $\phi(x) = p$, then $x \in \operatorname{Bd} \mathbb{H}^k \cap U$. Now U is not an open set of $\mathbb{R}^k$, but there is some open set $V \subset \mathbb{R}^k$ containing U and an extension $\tilde{\phi}$ of ϕ to V which is differentiable (or smooth). Then we define $T_p M = \tilde{\phi}_* T_x \mathbb{R}^k$.

There is a distinguished unit vector $n(p) \in T_p M$ called the outward normal. Consider the tangent vector $v = (x, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}) \in T_x \mathbb{R}^k$. Also note that $T_p \partial M \subset T_p M$ is a sub-vector space of dimension $k-1$. $n(p)$ is the unique vector in $T_p M$ such that

(1) $||n(p)|| = 1$,
(2) $n(p)$ is orthogonal to $T_p \partial M$,
(3) $\langle n(p), \tilde{\phi}_* v \rangle > 0$.

5. Differential forms

For a vector space V let $\mathfrak{T}^k(V)$ denote the vector space of k-tensors on V and let $\Lambda^k(V) \subset \mathfrak{T}^k(V)$ be the subspace of alternating k-tensors.

Let $M \subset \mathbb{R}^n$ be a manifold. A differential k-form on M is a function ω on M such that

$$\omega(p) \in \Lambda^k(T_p M) \quad \text{for all } p \in M.$$

Functions on M are conveniently treated as 0-forms. The set of all differential k forms on M will be denoted by $\Omega^k(M)$.

Note that if we have k vector fields $X_1, \ldots, X_k$ then applying ω to these we can get a function $g : M \to \mathbb{R}$ as follows:

$$g(p) = \omega(p)(X_1(p), \ldots, X_k(p)).$$

The form ω is smooth if for every set of k smooth vector fields the function thus obtained is smooth (similar definition for continuity).

For $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ we have the wedge product $\omega \wedge \eta \in \Omega^{k+\ell}(M)$ defined point wise

$$(\omega \wedge \eta)(p) = \omega(p) \wedge \eta(p).$$

If $f \in \Omega^0(M)$ is a function on M then we define $f \wedge \omega = \omega \wedge f = f\omega$.

The wedge product is linear in both the arguments and for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ we have $\omega \wedge \eta = (-1)^{k\ell}\eta \wedge \omega$.

Let $M \subset \mathbb{R}^n$ be a manifold and $U \subset \mathbb{R}^n$ be an open set containing M. Then any smooth differential form on U restricts to a differential form on M. On the other hand any smooth differential form on M can be extended to an open set $V \subset \mathbb{R}^n$ containing M.

If $\phi : M \to N$ is a differentiable map of manifolds, then we define $\phi^* : \Omega^r(M) \to \Omega^r(N)$, as follows: If ω is a differential r form on N, $p \in M$ is a point and $v_1, \ldots, v_r \in T_pM$ are r tangent vectors at p, then

$$(\phi^*\omega)(p)(v_1, \ldots, v_r) = \omega(\phi(p))(\phi_* v_1, \ldots, \phi_* v_r).$$

If $r = 0$, that is $\omega = f$ is a function on N, then this definition forces that $\phi^* f = f \circ \phi$. Then ϕ^* has the following properties:

(1) $\phi^*(a\omega + b\eta) = a\phi^*\omega + b\phi^*\eta$ where $a, b \in \mathbb{R}$ (linearity);
(2) $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ (contra-variance);
(3) $\phi^*(\omega \wedge \eta) = (\phi^*\omega) \wedge (\phi^*\eta)$

The differential operator is a linear operator $d : \Omega^r(M) \to \Omega^{r+1}(M)$ satisfying the following properties.

(1) If $\phi : M \to N$ is differentiable function of manifolds then $\phi^*(d\omega) = d(\phi^*\omega)$;
(2) $d(a\omega + b\eta) = ad\omega + bd\eta$ where $a, b \in \mathbb{R}$ (linearity);
(3) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^r \omega \wedge d\eta$ if ω is an r form and η an ℓ form;
(4) $d(d\omega) = 0$, written often as $d^2 = 0$.

First we define d for $U \subset \mathbb{R}^n$ open. Now by the above properties it suffices to define d for 0 forms or functions. For that we make the following definition. If $f : U \to \mathbb{R}$ is a smooth function then df is the 1 form on U given by

$$df(p)((p, v)) = \mathrm{D}\, f(p)v = \partial_v f(p) \quad \text{where } p \in U \text{ and } (p, v) \in T_p\mathbb{R}^n.$$

If $\pi_i : U \to \mathbb{R}$ is the projection onto the i-th coordinate, i.e. $\pi_i(x_1, \ldots, x_n) = x_i$ then we have the differentials $d\pi_i = dx_i$. More over note that

$$dx_i(p)(E_j(p)) = \partial_j x_i(p) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Thus $dx_1(p), \ldots, dx_n(p)$ is the dual basis of $\Lambda^1(T_p\mathbb{R}^n) = (T_p\mathbb{R}^n)^*$ to the basis $E_1(p), \ldots, E_n(p)$ of $T_p\mathbb{R}^n$. Hence $dx_1, \ldots, dx_n$ form a frame for $\Omega^1(U)$.

So any 1 form ω on U can be uniquely written as $\omega = g_1 dx_1 + \ldots + g_n dx_n$ where $g_i : U \to \mathbb{R}$ are smooth functions.

If $f : U \to \mathbb{R}$ is a differentiable function then it is easy to show that $df = \partial_1 f dx_1 + \ldots + \partial_n f dx_n$.

The r forms $dx_{i_1} \wedge \ldots \wedge dx_{i_r}$ for $1 \leq i_i < \ldots < i_r \leq n$ thus form a frame for $\Omega^r(U)$ and any r form η can be uniquely written as

$$\eta = \sum_{1 \leq i_1 < \ldots < i_r \leq n} g_{i_1 \cdots i_r}\, dx_{i_1} \wedge \ldots \wedge dx_{i_r}$$

where again $g_{i_1 \cdots i_r} : U \to \mathbb{R}$ are smooth functions.

Now the the properties of d force the following formula for $d\eta$:

$$\begin{aligned} d\eta &= \sum_{1 \leq i_1 < \ldots < i_r \leq n} dg_{i_1 \cdots i_r} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} \\ &= \sum_{1 \leq i_1 < \ldots < i_r \leq n} \sum_{\alpha=1}^{n} \partial_\alpha g_{i_1 \cdots i_r}\, dx_\alpha \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} \end{aligned}$$

Let $M \subset \mathbb{R}^n$ be a k dimensional manifold and ω be a differential r form on M. If (U,ϕ) is a chart of M, then $\phi^*\omega \in \Omega^r(U)$. Clearly we can write

$$\phi^*\omega = \sum_{1\le i_1<\ldots<i_r\le k} g_{i_1\cdots i_r}\; dx_{i_1}\wedge\ldots\wedge dx_{i_r}.$$

So we know how to explicitly write $d\phi^*\omega$.

Theorem 5. *There is a unique operator $d : \Omega^r(M) \to \Omega^{r+1}(M)$ such that for any chart (U,ϕ) of M we have $d(\phi^*\omega) = \phi^*(d\omega)$, for any r form ω.*

Explicitly define $d\omega$ point-wise as follows: for $p \in M$, choose a chart (U,ϕ) of M with $\phi(x) = p$ for some $x \in U$; $\phi_* : T_x\mathbb{R}^k \to T_pM$ is an isomorphism of vector spaces so for any $v_1,\ldots,v_r \in T_pM$ choose the unique tangent vectors $w_1,\ldots,w_r$ at x such that $\phi_*w_i = v_i$; finally set

$$d\omega(p)(v_1,\ldots,v_r) = (d\phi^*\omega)(x)(w_1,\ldots,w_r).$$

To see that this definition of $d\omega$ is chart independent, let (V,ψ) be another chart with $\psi(y) = p$. Also let $w'_1,\ldots,w'_r$ be the unique tangent vectors at y such that $\psi_*w'_i = v_i$. Note that $g = \phi^{-1}\circ\psi$ is a diffeomorphism of open sets in $\mathbb{R}^k$, and that $\psi = \phi\circ g$ (so $\psi^* = g^*\circ\phi^*$). Moreover we have $g(y) = x$ and $g_*(w'_i) = w_i$. Thus

$$\begin{aligned}(d\psi^*\omega)(y)(w'_1,\ldots,w'_r) &= d(g^*\circ\phi^*\omega)(y)(w'_1,\ldots,w'_r)\\ &= g^*(d\phi^*\omega)(y)(w'_1,\ldots,w'_r)\\ &= (d\phi^*\omega)(g(y))(g_*w'_1,\ldots,g_*w'_r)\\ &= (d\phi^*\omega)(x)(w_1,\ldots,w_r).\end{aligned}$$

6. Orientability

6.1. **Orientation of a vector space.** If V is a vector space of dimension n, then $\Lambda^n(V)$ is of dimension 1. Define the equivalence relation $\sim$ on $\Lambda^n(V) - \{0\}$ by $\alpha \sim \beta$ for $\alpha,\beta \in \Lambda^n(V) - \{0\}$ if $\alpha = c\beta$ where $c > 0$. The equivalence classes of $\sim$ are called orientations of V. Notice that there are two possible orientations.

If $\omega \in \Lambda^n(V) - \{0\}$, then ω determines an orientation on V. We say that a basis $v_1,\ldots,v_n$ of V is positively oriented if $\omega(v_1,\ldots,v_n) > 0$. Otherwise we say that the basis is negatively oriented. If $v_1,\ldots,v_n$ is positively oriented, and $w_1,\ldots,w_n$ is another basis such that $w_j = \sum_{i=1}^n a_{i,j}v_i$ then note that

$$\omega(w_1,\ldots,w_n) = \det A\ \omega(v_1,\ldots,v_n)$$

where $A = ((a_{i,j}))$ is the change of basis matrix. Thus $w_1,\ldots,w_n$ is also positively oriented if $\det A > 0$ otherwise it is negatively oriented.

On the other hand given any basis $v_1,\ldots,v_n$ of V, take the dual basis $\phi_1,\ldots,\phi_n$ of $\Lambda^1(V) = V^*$, then $\phi_1\wedge\ldots\wedge\phi_n$ is a non-zero alternating n tensor and gives an orientation on V for which $v_1,\ldots,v_n$ is of course positively oriented.

A vector space along with a choice of orientation is called an oriented vector space. A linear transformation is called orientation preserving if it takes any positively oriented basis to a positively oriented basis.

There is the standard orientation on $\mathbb{R}^n$ given by the determinant $\det \in \Omega^n(\mathbb{R}^n)$. This is just a generalisation of our notion of orientation for $\mathbb{R},\mathbb{R}^2$ and $\mathbb{R}^3$.

6.2. **Orientable manifolds.** A k dimensional manifold $M \subset \mathbb{R}^n$ is orientable if there is a non-vanishing k form on M, i.e. some $\omega \in \Omega^k(M)$ such that $\omega(p) \neq 0$ for any $p \in M$. A k form on a k dimensional manifold is called a top form.

Any two non-vanishing forms $\omega, \eta \in \Omega^k(M)$ are related by $\omega = f\eta$ for a smooth function $f : M \to \mathbb{R}$ such that $f(p) \neq 0$ for any p. We define an equivalence relation $\sim$ on the set of non-vanishing top forms on M, $\omega \sim \eta$ if $\eta = f\omega$ such that $f(p) > 0$ for any $p \in M$. The equivalence classes under this equivalence relation are called orientations on M.

Thus any non-vanishing top form gives an orientation on M, which is essentially a smoothly varying orientation for all the tangent spaces of M. Notice that if M is connected and orientable then there are just two possible orientations of M. If we fix the orientation of the tangent space at a point then the orientation of the whole manifold is determined in this case.

Let $M \subset \mathbb{R}^n$ be a k manifold with an orientation given by a top form ω. A chart (U, ϕ) is called positively oriented if $\phi^*\omega = f dx_1 \wedge \ldots \wedge dx_n$ for a positive function $f : M \to \mathbb{R}$. This is equivalent to saying that for any $x \in U$ such that $p = \phi(x)$ the basis $\phi_*((x, e_1)), \ldots, \phi_*((x, e_n))$ is a positively oriented basis of T_pM with respect to the orientation given by $\omega(p)$ (i.e positively oriented bases of $T_x\mathbb{R}^k$ are taken to positively oriented bases of T_pM by ϕ_*).

If (U, ϕ) and (V, ψ) are both positively oriented then $\det D(\phi^{-1} \circ \psi) > 0$. We have the following practical way of checking orientability of a manifold.

Theorem 6. *A manifold $M \subset \mathbb{R}^n$ is orientable if and only if there is an atlas $\{(U_i, \phi_i)\}_{i \in I}$ of M such that $\det D(\phi_i^{-1} \circ \phi_j) > 0$ for all $i, j \in I$.*

Any open set of $U \subset \mathbb{R}^n$ has the standard orientation given by $dx_1 \wedge \ldots \wedge dx_n$.

If $M \subset \mathbb{R}^n$ is an oriented manifold of dimension $n-1$, then for any $p \in M$, there is a unique vector $N(p) \in T_p\mathbb{R}^n$ satisfying

(1) $||N(p)|| = 1$,
(2) $N(p)$ is orthogonal to T_pM,
(3) $N(p), v_1, \ldots, v_n$ is a positively oriented basis of $T_p\mathbb{R}^n$ if $v_1, \ldots, v_n$ is a positively oriented basis of T_pM.

This vector $N(p)$ is called the unit normal of M, and is smoothly varying on M. On the other hand if we have a smoothly varying unit normal vector $N(p)$ for each point $p \in M$, then that determines an orientation on M.

6.3. **Induced orientation of the boundary.** Let $M \subset \mathbb{R}^n$ be a oriented k manifold with boundary, with the orientation given by $\omega \in \Omega^k(M)$. Then ∂M is an orientable manifold and the orientation of M determines an orientation of ∂M.

Consider the unit outward normal $n(p)$ for any $p \in \partial M$. We define the top form $\eta \in \Omega^{k-1}(\partial M)$ as follows:

$$\eta(v_1, \ldots, v_{k-1}) = \omega(n(p), v_1, \ldots, v_{k-1}) \quad \text{for any } v_1, \ldots, v_{k-1} \in T_pM$$

Then η is non-vanishing, since ω is, and gives an orientation of ∂M. According to this orientation $v_1, \ldots, v_{k-1}$ is a positively oriented basis of $T_p\partial M$ if $n(p), v_1, \ldots, v_{k-1}$ is a positively oriented basis of T_pM.

$\mathbb{H}^n \subset \mathbb{R}^n$ is an n manifold with boundary and $\partial\mathbb{H}^n = \mathbb{R}^{n-1} \times \{0\}$. Consider the standard orientation on $\mathbb{H}^n$ given by $dx_1 \wedge \ldots \wedge dx_n$. The induced orientation on $\partial\mathbb{H}^n$ is the standard orientation on $\mathbb{R}^{n-1}$ if n is even, otherwise the induced orientation is the opposite one. This is so designed to get the nice form of the Stokes' theorem.

7. Integration of forms

We shall define integral of top forms on manifolds.

For open sets of $\mathbb{R}^n$ this is quite easy. Let $\omega \in \Omega^n(U)$, then $\omega = f dx_1 \wedge \ldots \wedge dx_n$ for some smooth function $f : U \to \mathbb{R}$. Define

$$\int_U \omega = \int_U f.$$

Note that if $v_1, \ldots, v_n$ is a positively oriented orthonormal basis of $\mathbb{R}^n$, then

$$f(p) = \omega(p)((p, v_1), \ldots, (p, v_n)).$$

Thus the above integral can be written as

$$\int_U \omega = \int_U \omega(x)((x, v_1), \ldots, (x, v_n))dx_1 \cdots dx_n.$$

Now let $M \subset \mathbb{R}^n$ be an oriented k manifold (with or without boundary).

If (U, ϕ) is a positively oriented chart of M, and ω is a top form on M which vanishes outside $\phi(U)$, that is $\omega(p) = 0$ for $p \notin \phi(U)$, we define the integral of ω as follows:

$$\int_M \omega = \int_U \phi^* \omega.$$

It is an easy exercise to check that this definition is independent of the choice of the chart.

Now to define integral of arbitrary top forms on M we need to use partitions of unity. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Then $U_\alpha = V_\alpha \cap M$ for some open set $V_\alpha \subset \mathbb{R}^n$. Let

$$V = \bigcup_{\alpha \in A} V_\alpha.$$

V is an open set in $\mathbb{R}^n$ containing M and $\{V_\alpha\}_{\alpha \in A}$ is an open cover of V. Let $\phi_1, \phi_2, \ldots$ be a partition of unity for V subordinate to the open cover $\{V_\alpha\}_{\alpha \in A}$. The functions $\psi_i = \phi_i|_M$, obtained from ϕ_i by restricting the domain to M are a partition of unity on M subordinate to the open cover $\{U_\alpha\}_{\alpha \in A}$ of M. They satisfy the following properties:

- $\psi_i : M \to \mathbb{R}$ is $\mathcal{C}^\infty$ and $\psi_i(x) \geq 0$ for all $x \in M$;
- there exists an $\alpha \in A$ such that $\psi_i(x) = 0$ for $x \notin U_\alpha$;
- for any $x \in M$ there is an open set $U \ni x$ such that all but finitely many ϕ_i vanish on U;
- $\sum_{i=1}^\infty \psi_i(x) = 1$ for any $x \in M$.

Now if $\omega \in \Omega^k(M)$ we define the integral of ω on M in the following manner. Choose an atlas $\{U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ of positively oriented charts on M. Let $\psi_1, \psi_2, \ldots$ be a partition of unity subordinate to the open cover $\{\phi_\alpha(U_\alpha)\}_{\alpha \in A}$ of M. Define

$$\int_M \omega = \sum_{i=1}^\infty \int_M \psi_i \omega,$$

if the above sum converges absolutely. In that case we say that ω is integrable.

Note that $\psi_i \omega$ vanishes outside a single chart of the above atlas, so the integral $\int_M \psi_i \omega$ makes sense by our previous definition.

It is not a hard, although quite tedious, exercise to show that this definition is independent of the choice of the atlas and the partition of unity. Also this definition makes sense for forms that are not continuous on M.

If the manifold M is compact then any continuous top form is integrable. In fact integration of forms is a linear map $\Omega^k(M) \to \mathbb{R}$ in that case.

For any oriented k manifold $M \subset \mathbb{R}^n$, if ω and η are integrable top forms on M then $a\omega + b\eta$ is also integrable for any $a, b \in \mathbb{R}$ and

$$\int_M (a\omega + b\eta) = a\int_M \omega + b\int_M \eta.$$

The above definition is not very practical for actually evaluating integrals. The following theorem however gives us a procedure to compute the integrals by cutting up the manifold into disjoint pieces, integrating the form on each piece and finally summing all the integrals to get the integral on the whole manifold.

A subset $S \subset M$ is said to be of measure 0, if there is an atlas $\{U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ of M for which $\phi_\alpha^{-1}(S \cap \phi_\alpha(U_\alpha))$ is a measure 0 subset of U_α for each $\alpha \in A$.

For example any finite subset of M has measure 0 in M, or if $N \subset M$ is also a manifold (with or without boundary) such that $\dim N < \dim M$, then N has measure 0 in M.

Note that if $M \subset \mathbb{R}^n$ is a k manifold with $k < n$, then any subset of M is measure 0 in $\mathbb{R}^n$ however it may not have measure 0 in M, so this is a different concept that being measure 0 in Euclidean space.

Theorem 7. *Let $M \subset \mathbb{R}^n$ be a compact k manifold and $\{(U_1, \phi_1), \ldots, (U_N, \phi_N)\}$ be a collection of charts on M, such that $S = M - \Big(\phi_1(U_1) \cup \ldots \cup \phi_N(U_N)\Big)$ has measure 0 in M and $\phi_i(U_i) \cap \phi_j(U_j) = \emptyset$ for $i \neq j$ then*

$$\int_M \omega = \sum_{i=1}^N \int_{U_i} \phi_i^*\omega \ .$$

7.1. **Volume form.** Let $M \subset \mathbb{R}^n$ be an oriented k manifold, then there is a distinguished top form on M called the volume form.

Theorem 8. *There is a smooth k form Vol_M on M:*

(1) For any $p \in M$ and any positively oriented orthonormal basis $v_1, \ldots, v_k$ of T_pM

$$\mathrm{Vol}_M(p)(v_1, \ldots, v_k) = 1$$

(2) If (U, ϕ) is a chart on M, define the matrix $G(x)$ for any $x \in U$ by setting $G(x) = ((g_{i,j}(x)))$ where $g_{i,j}(x) = \langle \partial_i\phi(x), \partial_j\phi(x)\rangle$, then

$$\phi^* \mathrm{Vol}_M = \sqrt{\det G}\ dx_1 \wedge \ldots \wedge dx_k \ .$$

Proof. Choose a non-vanishing top form ω on M. Let $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ be two positively oriented orthonormal bases of T_pM. Writing w_j as a linear combination of $v_1, \ldots, v_k$ we get

$$w_j = \sum_{i=1}^k a_{i,j} v_i \ .$$

The matrix $A = ((a_{i,j}))$ then is an orthogonal matrix with positive determinant, hence $\det A = 1$. Thus

$$\omega(w_1, \ldots, w_k) = (\det A)\omega(v_1, \ldots, v_k) \ .$$

Hence $\rho(p) = \omega(v_1, \ldots, v_k)$ is the same if we choose any positively oriented orthonormal basis of T_pM. We thus get a function $\rho : M \to \mathbb{R}$ such that $\rho(p) \neq 0$ for any $p \in M$. Define

$$\mathrm{Vol}_M(p) = \frac{1}{\rho(p)}\omega(p) \ .$$

This is a top form on M and it clearly satisfies (1). It remains to show that this is a smooth form.

Let (U, ϕ) be a positively oriented chart on M. Clearly

$$\phi^* \mathrm{Vol}_M = f\ dx_1 \wedge \ldots \wedge dx_k \ .$$

for some function $f : U \to \mathbb{R}$. We want to find this function f and show that it is smooth. That will show that Vol_M is smooth on each chart, hence smooth on M. For $x \in U$ consider the basis $E_1(x), \ldots, E_k(x)$ of $T_x\mathbb{R}^k$. $\phi_* E_1(x), \ldots, \phi_* E_k(x)$ is a positively oriented basis of T_pM where $p = \phi(x)$ but it may not be orthonormal. Let $v_1, \ldots, v_k$ be an orthonormal basis of T_pM as before and write $\phi_* E_j(x)$ as a linear combination of the basis $v_1, \ldots, v_k$.

$$\phi_* E_j(x) = \sum_{i=1}^{k} b_{i,j} v_i \ .$$

Let B be the $k \times k$ matrix $B = ((b_{i,j}))$, then

$$\begin{aligned} f(x) &= (\phi^* \mathrm{Vol}_M)(x)(E_1(x), \ldots, E_k(x)) \\ &= \mathrm{Vol}_M(\phi(x))(\phi_* E_1(x), \ldots, \phi_* E_k(x)) \\ &= (\det B) \, \mathrm{Vol}_M(v_1, \ldots, v_k) = \det B \ . \end{aligned}$$

Now $\phi_* E_i(x) = \phi_*(x, e_i) = (p, \mathrm{D}\,\phi \ e_i) = (p, \partial_i \phi)$, thus $g_{i,j}(x) = \langle \partial_i\phi(x), \partial_j\phi(x)\rangle = \langle \phi_* E_i(x), \phi_* E_j(x)\rangle$. Expressing $g_{i,j}(x)$ in terms of $v_1, \ldots, v_k$ we have

$$\begin{aligned} \langle \phi_* E_i(x), \phi_* E_j(x)\rangle &= \langle \sum_{l=1}^{k} b_{l,i} v_l, \sum_{m=1}^{k} b_{m,j} v_m \rangle \\ &= \sum_{l=1}^{k}\sum_{m=1}^{k} b_{l,i} b_{m,j} \langle v_i, v_j\rangle \ . \end{aligned} \tag{2}$$

Note that $\langle v_i, v_j\rangle = 1$ if $i = j$ but 0 otherwise. Hence equation (2) yields $G(x) = B^t I B = B^t B$. Hence $\det B = \sqrt{\det G(x)}$. Since the entries of $G(x)$ are all smooth and $\det G(x) > 0$ we see that $f(x) = \sqrt{\det G(x)}$ is a smooth function on U. ☐

Now we define the integral of any function $f : M \to \mathbb{R}$ as

$$\int_M f = \int_M f \, \mathrm{Vol}_M$$

if this integral exists.

We define the volume of a compact oriented manifold M as

$$\mathrm{volume}(M) = \int_M 1 = \int_M \mathrm{Vol}_M \ .$$

Intuitively the volume form is an infinitesimal volume element in the sense of vector calculus. It is the volume element of the correct orientation on all the tangent spaces of M smoothly varying on M.

7.2. **Special cases.** As special cases let us look at line and surface integrals.

A parametrized curve $C \subset \mathbb{R}^n$ is a 1 manifold covered by a single chart (U, ϕ) hence naturally oriented if we declare the chart to be positively oriented, if further C is connected then we can take $U = (a, b) \subset \mathbb{R}$ an open interval. Now if t is the variable in U, then by Theorem 8 we have

$$\phi^* \mathrm{Vol}_C = ||\phi'(t)|| dt \ .$$

So if $f : C \to \mathbb{R}$ is a function then the integral of f on C is given by

$$\int_C f = \int f \, \mathrm{Vol}_C = \int_U \phi^*(f \, \mathrm{Vol}_C) = \int_a^b f \circ \phi(t) ||\phi'(t)|| dt \ .$$

This is the usual line integral formula.

A parametrized surface $S \subset \mathbb{R}^3$ is a 2 manifold covered by a single chart (U, ϕ). Now we have

$$G = \begin{pmatrix} \partial_x\phi \cdot \partial_x\phi & \partial_x\phi \cdot \partial_y\phi \\ \partial_y\phi \cdot \partial_x\phi & \partial_y\phi \cdot \partial_y\phi \end{pmatrix} \ .$$

Thus

$$\det G = ||\partial_x\phi||^2||\partial_y\phi||^2 - (\partial_x\phi\cdot\partial_y\phi)^2 = ||\partial_x\phi\times\partial_y\phi||^2\ .$$

Hence if $f : S \to \mathbb{R}$ is a function, the integral of f on S is given by

$$\int_S f = \int_S f\,\mathrm{Vol}_S = \int_U (f\circ\phi)\phi^*\,\mathrm{Vol}_S = \int_U (f\circ\phi)||\partial_x\phi\times\partial_y\phi||.$$

8. Where to go from here

The Differential Geometry course of semester VIII is a natural continuation of this course. However before learning about general manifolds, one could study curves and surfaces first and "Differential Geometry of Curves and Surfaces" by Do Carmo is an excellent reference for that and should be an easy read.

Here are some references for Differential Geometry.

- Topology from a Differential Point of View, by Milnor: This is an excellent introductory text, but it is less rigorous and leaves out a lot of topics. On the other hand it is very short and makes for a god bed time reading.
- Differential Topology, by Guillemin and Pollack: This is the easiest book and does things is good details, also it is fairly short.
- A Comprehensive Introduction to Differential Geometry, by Spivak Volume 1: This is the first of a 5 volume book which is an absolute classic and a must read for Differential Geometers. The first volume lays out the foundations and geometry really starts from volume 2. See MAA review for a detailed review of these books.
- Morse theory, by Milnor: Morse theory is also a central topic in Differential Geometry and Milnor's expository book on this topic is the best reference available.

Manifolds and Differential geometry is just as important in Physics and here are some good references to venture further into these topics:

- Mathematical Methods of Classical Mechanics, V.I. Arnold: This is again a classic book and it works through most of the differential geometry needed in Physics.
- General Relativity, Robert Wald: This is by far the most popular book in general relativity, although it may be a bit too mathematical and a bit out dated.