Throughout this lecture A will be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$.

We have seen that

$$\dim A = \operatorname{ht}\mathfrak{m} \leq \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Definition 1: The noetherian local ring A is called regular if $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$.

Exercise (i). Show that $\mathbb{C}[x]/(x^2)$ is not regular.

Example 1. Consider $R = \mathbb{C}[x, y]/(xy)$. The maximal ideals of R are in one to one correspondence with maximal ideals of $\mathbb{C}[x, y]$ that contain the ideal (xy).

Let $\mathfrak{n}$ be the maximal ideal corresponding to (x, y) and let $A = R_{\mathfrak{n}}$ and $\mathfrak{m} = \mathfrak{n}R_{\mathfrak{n}}$. Then $\mathfrak{m}^2 = (x^2, y^2)$, thus $\mathfrak{m}/\mathfrak{m}^2$ has x, y as basis over $\mathbb{C}$ and $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = 2$. On the other hand $\dim A = \operatorname{ht}\mathfrak{n} \leq \dim R \leq 1$ by Krull's principal ideal theorem. In fact $\dim A = 1$ since $(x) \subset (x, y)$ is a chain in A of length 1. Hence A is not regular.

On the other hand consider the maximal ideal $\mathfrak{n}'$ of R corresponding to $(x, y - a)$ where a is a non-zero complex number. Let $B = R_{\mathfrak{n}'}$ and $\mathfrak{m}' = \mathfrak{n}'B$. Then we again have $\dim B = 1$. Now $(\mathfrak{m}')^2 = (x^2, y^2 - 2ay + a^2, ax) = (x, y^2 - 2ay + a^2)$, so $x \in (\mathfrak{m}')^2$ and just y in this case forms a basis of $\mathfrak{m}'/(\mathfrak{m}')^2$ over $\mathbb{C}$. Thus $\dim B = \dim_{\mathbb{C}} \mathfrak{m}'/(\mathfrak{m}')^2$, hence B is regular.

We shall give two more useful criteria for the ring A to be regular.

Theorem 2. Let $d = \dim A$, then A is regular if and only if the following equivalent conditions are satisfied:

(a) The maximal ideal $\mathfrak{m}$ can be generated by d elements.
(b) The associated graded ring $G_{\mathfrak{m}}(A)$ is isomorphic to the polynomial ring $k[t_1, \ldots, t_d]$ over k in d variables.

Proof. We shall show A regular $\Rightarrow (a) \Rightarrow (b) \Rightarrow A$ regular.

Assume A is regular which means $\dim_k \mathfrak{m}/\mathfrak{m}^2 = d$. Let $x_1, \ldots, x_d \in \mathfrak{m}$ be such that their images $\overline{x}_1, \ldots, \overline{x}_d$ forms a basis of $\mathfrak{m}/\mathfrak{m}^2$ as a k vector space. Then by Nakayama's lemma $x_1, \ldots, x_d$ must generate $\mathfrak{m}$ as an ideal.

Now for $(a) \Rightarrow (b)$ let $\mathfrak{m} = (x_1, \ldots, x_d)$, and let $\overline{x}_i$ be the image of x_i in $\mathfrak{m}/\mathfrak{m}^2$, then $\overline{x}_1 \ldots, \overline{x}_d \in G_{\mathfrak{m}}(A)$ generate $G_{\mathfrak{m}}(A)$ as a $k = A/\mathfrak{m}$ algebra (see proof of Proposition 2 in the lecture on Associated Graded rings). Let

$$\phi : k[t_1, \ldots, t_d] \to G_{\mathfrak{m}}(A)$$

be the k algebra homomorphism given by $\phi(t_i) = \overline{x}_i$. Then ϕ is surjective, so if $\mathfrak{a} = \ker\phi$, we have $G_{\mathfrak{m}}(A) \cong k[t_1, \ldots, t_d]/\mathfrak{a}$. In fact ϕ is a homomorphism of graded rings, so $\mathfrak{a}$ is a homogeneous ideal (see Example 1 in the lecture on Hilbert polynomial). Suppose $\mathfrak{a} \neq 0$ and $f \in \mathfrak{a}$ is a non-zero homogeneous polynomial then $G_{\mathfrak{m}}(A)$ is isomorphic to a quotient of $k[t_1, \ldots, t_d]/(f)$. Thus

$$\dim A = \mathrm{d_{hilb}}(G_{\mathfrak{m}}(A)) \leq \mathrm{d_{hilb}}(k[t_1, \ldots, t_d]/(f)) = d - 1$$

(see Example 2 in the lecture on Hilbert polynomials). This is a contradiction since we assumed $\dim A = d$. Thus $\mathfrak{a} = (0)$ and $G_{\mathfrak{m}}(A) \cong k[t_1, \ldots, t_d]$.

Finally assume $G_{\mathfrak{m}}(A) \cong k[t_1, \ldots, t_d]$, then $\mathfrak{m}/\mathfrak{m}^2$ is the first graded component of $k[t_1, \ldots, t_d]$, which are all the polynomials of degree 1. Clearly that has dimension d as s k vector space. □

A regular local ring is always an integral domain. In fact it is also normal. Recall that an integral domain is called normal if it is integrally closed in its field of fractions. To prove normality we need an easy lemma first.

Lemma 3. Let R be a noetherian integral domain and K its field of fractions. An element $x \in K$ is integral over R if and only if there is an element $a \in R$ such that

$$ax^n \in R \text{ for all } n > 0.$$

Proof. Let $S = R[x]$, then S is an R submodule of K. If x is integral over R then $R[x]$ is finitely generated by say $1, x, \ldots, x^n$. Then if $x = a/b$ with $a, b \in R$ we have $b^n S \subset R$. Hence we may take $a = b^n$.

For the converse, we note that the condition on the powers of x implies that $aS \subset R$. Since it is a submodule of R, that is an ideal, and R is noetherian, aS is finitely generated as an R module. This shows $R[x]$ is a finitely generated R module, hence x is integral over R. □

Lemma 4. Let R be a noetherian ring and $\mathfrak{a} \subset R$ an ideal.

(a) If $\cap_{n=1}^{\infty} \mathfrak{a}^n = (0)$ and $G_{\mathfrak{a}}(R)$ is an integral domain, then R is also an integral domain.
(b) If $\mathfrak{a}$ is contained in the Jacobson radical of R and $G_{\mathfrak{a}}(R)$ is a normal integral domain, then so is R.

Remark. Assume $\cap_{n=1}^{\infty} \mathfrak{a}^n = (0)$, for any $r \in R - \{0\}$ there is an integer t such that $r \in \mathfrak{a}^t - \mathfrak{a}^{t+1}$. Define r^* to be the image of r in $\mathfrak{a}^t/\mathfrak{a}^{t+1}$. Then clearly

$$r^* \neq 0 \in G_{\mathfrak{a}}(R).$$

Also define $0^* = 0 \in G_{\mathfrak{a}}(R)$. Thus $r \mapsto r^*$ is a surjective map from $R \to G_{\mathfrak{a}}(R)$. This is not a ring homomorphism. However, if $G_{\mathfrak{a}}(R)$ is an integral domain it is easy to check that the map is multiplicative, that is

$$r_1^* r_2^* = (r_1 r_2)^*.$$

If $G_{\mathfrak{a}}(R)$ is not an integral domain this map may fail to be multiplicative.

Proof of Lemma 4. For part (a) if $a, b \in R$ are non-zero then $(ab)^* = a^* b^* \neq 0$. Hence $xy \neq 0$.

Part (b) is significantly longer. First note that by Corollary 16 of the lecture on Graded rings since $\mathfrak{a}$ is contained in the Jacobson radical we have

$$\cap_{n=1}^{\infty} \mathfrak{a}^n = (0).$$

Hence R is an integral domain by part (a). Now let $x = a/b$ be integral over R with $a, b \in R$ and $b \neq 0$. We have to show that $a = bc$ for some $c \in R$, that is $a \in (b)$. Consider the ring $S = R/(b)$. As an R module, S is hausdorff in the $\mathfrak{a}$-adic topology, that is $\cap_{n=1}^{\infty} \mathfrak{a}^n S = 0$, by Corollary 16 of the lecture on Graded rings. Since $\mathfrak{a}^n S = (\mathfrak{a}^n + (b))/(b)$, this means

$$\bigcap_{n=1}^{\infty} ((b) + \mathfrak{a}^n) = (b).$$

Hence it is enough to show that $a \in (b) + \mathfrak{a}^n$ for all n. We shall prove this by induction on n. Since $\mathfrak{a}^0 = R$, this is trivial for $n = 0$. Assume it is true for n, that is

$$a = cb + d \quad \text{where } c \in R \text{ and } d \in \mathfrak{a}^n.$$

Hence $y = x - c = d/b$ is integral over R. Thus there is $u \in R$ such that

$$uy^n \in R \text{ for all } n > 0.$$

Let $v_n = uy^n$, then $ud^n = v_n b^n$. Hence $u^*(d^*)^n = v_n^*(b^*)^n$ in $G_{\mathfrak{a}}(R)$. This means in the field of fractions of $G_{\mathfrak{a}}(R)$ we have

$$u^* \left(\frac{d^*}{b^*}\right)^n \in G_{\mathfrak{a}}(R) \quad \text{for all } n > 0.$$

Since $G_{\mathfrak{a}}(R)$ is noetherian it follows that d^*/b^* is integral over $G_{\mathfrak{a}}(R)$. But $G_{\mathfrak{a}}(R)$ is normal, so $d^* = e^* b^*$ for some $e \in R$. Which means $d - eb \in \mathfrak{a}^{n+1}$. Thus

$$a = (c + e)b + (d - eb) \Rightarrow a \in (b) + \mathfrak{a}^{n+1}$$

completing the proof. □

The next result is basically a corollary of the previous lemma but we write it as a theorem because of its importance.

Theorem 5. A regular local ring A is an normal integral domain.

Proof. Since A is local $\mathfrak{m}$ is the Jacobson radical. The associated graded ring $G_{\mathfrak{m}}(A)$ is isomorphic to a polynomial algebra over k, hence a normal integral domain. □

Remark. In fact a regular local ring is always a unique factorisation domain (hence normal). However, the proof is much more difficult. See for instance Section 19.4 of *Commutative Algebra with a view towards Algebraic Geometry* — David Eisenbud.

Example 2. A regular local ring of dimension 0 has to be a field since it is an artinian integral domain.

Example 3. If R is a noetherian integral domain of dimension 1 then R is regular $\iff$ R is a discrete valuation ring $\iff$ R is normal. This follows immediately from Theorem 2, Theorem 5 and Theorem 8 of the lecture on Dedekind domains.

As we see when A is a normal integral domain and $\dim A \leq 1$, then A is regular if and only A is normal. However when $\dim A > 1$ this is not true. The following is an example when $\dim A = 2$ and A is normal but not regular.

Example 4. Let k be a field of characteristic not equal to 2 and let

$$R = \frac{k[x,y,z]}{(z^2 - xy)}.$$

The ring R is a noetherian integral domain since the ideal $z^2 - xy$ is an irreducible polynomial, (Eisenstein's criterion). However it is not a unique factorization domain.

We claim that R is normal. Consider the homomorphism $\phi : k[x,y,z] \to k[s,t]$ given by $\phi(x) = s^2$, $\phi(y) = t^2$, $\phi(z) = st$. It is easy to see that the $\ker\phi = (z^2 - xy)$, hence $R \cong k[s^2, t^2, st]$. Thus R is isomorphic to the subring of $k[s,t]$ generated by monomials of even degrees. Let F be the field of fractions of $R = k[s^2, t^2, st]$ and E be the field of fractions of $k[s,t]$. Then, E has the automorphism

$$\psi : E \to E \quad \text{given by } \psi(s) = -s \text{ and } \psi(t) = -t.$$

Since $\psi^2 = \mathrm{Id}_E$, so $G = \{\mathrm{Id}, \psi\}$ is a group of automorphisms of E. Clearly F is the fixed field of E and $R = F \cap k[s,t]$ are the fixed elements of $k[s,t]$. If $f \in F$ is integral over R, then f is of course integral over $k[s,t]$. But $k[s,t]$ is normal since it is a unique factorisation domain. Hence $f \in k[s,t]$. But then

$$f \in F \cap k[s,t] = R.$$

This proves R is normal. Let $\mathfrak{n}$ be the maximal ideal corresponding to the maximal ideal (x,y,z) of $k[x,y,z]$. Then $R_{\mathfrak{n}}$ is is a normal local integral domain which is of course noetherian.

We claim that A is not regular. The maximal ideal of A is $\mathfrak{m} = \mathfrak{n}A$ and is generated by $\overline{x}, \overline{y}$ and $\overline{z}$ the images of x, y and z. On one hand

$$\dim A = \dim R = \text{ transcendence degree of } F \text{ over } k = 2,$$

(see Theorem 17 of the lecture on Dimension of Rings). However, notice that $(z^2 - xy) \subset \mathfrak{n}^2$, hence

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \mathfrak{n}/\mathfrak{n}^2 = 3.$$

Exercise (ii). Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and $x \in \mathfrak{m}$ such that $x \notin \mathfrak{m}^2$ then show that $A/(x)$ is also regular and $\dim A/(x) = \dim A - 1$.

Proposition 6. A noetherian local ring A is regular if and only in the $\mathfrak{m}$-adic completion $\widehat{A}$ is regular.

Proof. Note that $\widehat{A}$ is also a noetherian local ring with maximal ideal $\widehat{\mathfrak{m}}$. Moreover, $G_{\mathfrak{m}}(A) \cong G_{\widehat{\mathfrak{m}}}(\widehat{A})$. Now use Theorem 2. □

Remark. It follows that for a regular local ring A, the completion $\widehat{A}$ is also an integral domain.

Exercise (iii). Show that the power series ring $A = k[[x_1, \ldots, x_d]]$ over a field k is a regular local ring of dimension d. (Hint. Show that $G_{\mathfrak{m}}(A) \cong k[x_1, \ldots, x_d]$ where $\mathfrak{m} = (x_1, \ldots, x_d)$).

The next result shows that for a regular closed point of Spec$R$, where R is a finitely generated algebra over a field, the complete local ring is just a power series ring.

Proposition 7. Let R be a finitely generated algebra over a field k and $\mathfrak{n} \subset R$ a maximal ideal such that $R/\mathfrak{n} = k$. If $A = R_{\mathfrak{n}}$, the localisation at $\mathfrak{n}$, is regular then

$$\widehat{A} \cong k[[t_1, \dots, t_d]]$$

where $d = \dim A$ and $\widehat{A}$ is the completion of A with respect to its maximal ideal $\mathfrak{n}A$.

Proof. Let $\mathfrak{m} = \mathfrak{n}A$, then by an application of the Noether normalisation theorem $A/\mathfrak{m} \cong R/\mathfrak{n}$ is a finite extension of k and by assumption it is in fact k.

Let $k[t_1, \dots, t_d]$ be the polynomial ring over k in d variables and $\mathfrak{m} = (x_1, \dots, x_d)$. Consider the k algebra homomorphism, (since we have an inclusion $k \to A$),

$$\phi : k[t_1, \dots, t_d] \to A \quad \text{given by } \phi(t_i) = x_i.$$

Let $\mathfrak{m}' = (t_1, \dots, t_d)$ then $\phi((\mathfrak{m}')^s) \subset \mathfrak{m}^s$ for all $s > 0$. Thus we get homomorphisms of k algebras

$$\phi_n : \frac{k[t_1, \dots, t_d]}{(\mathfrak{m}')^s} \to \frac{A}{\mathfrak{m}^s}.$$

Hence taking inverse limits we get a homomorphism of the completions

$$\widehat{\phi} : k[[t_1, \dots, t_d]] \to \widehat{A}.$$

We shall show that each ϕ is an isomorphism. Hence so is $\widehat{\phi}$.

Then since $A/\mathfrak{m} = k$ and A is a regular local ring, we have an isomorphism

$$\psi : k[t_1, \dots, t_d] \to G_{\mathfrak{m}}(A) \quad \text{such that } \psi(t_i) = \overline{x}_i \in \mathfrak{m}/\mathfrak{m}^2.$$

To show surjectivity of ϕ_n note that by surjectivity of ψ, for each $s \geq 0$, and $a \in \mathfrak{m}^s$ there is $f \in k[t_1, \dots, t_d]$ such that $a - \phi(f) \in \mathfrak{m}^{s+1}$. Hence, for any $a \in A$ there is $f_0 \in k[t_1, \dots, t_d]$ such that $a - \phi(f_0) \in \mathfrak{m}$. Similarly there is f_1 such that $a - \phi(f_0) - \phi(f_1) \in \mathfrak{m}^2$. Continuing in this manner we can get $f_0, f_1, \dots, f_{n-1} \in k[t_1, \dots, t_d]$ such that

$$a - \phi(f_0 + \ldots + f_{n-1}) \in \mathfrak{m}^n.$$

Hence, $\overline{a} = \phi_n(f_0 + \ldots + f_{n-1}) \in A/\mathfrak{m}^n$

Moreover, ϕ_n is injective $\iff$ $\phi^{-1}(\mathfrak{m}^n) = (\mathfrak{m}')^n$. Let $f \in k[t_1, \dots, t_d]$ be a homogeneous polynomial of degree s, then clearly $\phi(f) \in \mathfrak{m}^s$. However, if $\phi(f) \in \mathfrak{m}^{s+1}$ then $\psi(f) = 0 \in \mathfrak{m}^s/\mathfrak{m}^{s+1} \Rightarrow f = 0$. Now let $g \in k[t_1, \dots, t_d]$ be any polynomial such that $\phi(g) \in \mathfrak{m}^n$. Let $s \geq 0$ be the smallest degree for which the homogeneous component g_s of g of degree s is non-zero. If $s \geq n$ then $g \in (\mathfrak{m}')^n$ so assume $s < n$. Clearly $\phi(g - g_s) \in \mathfrak{m}^{s+1}$ and $\phi(g) \in \mathfrak{m}^n \subset \mathfrak{m}^{s+1}$. Hence

$$\phi(g_s) = \phi(g) - \phi(g - g_s) \in \mathfrak{m}^{s+1}$$

But g_s is homogeneous of degree s, hence $g_s = 0$. Contradicting our assumption $s < n$. □

Definition 8 (Regular ring): A noetherian ring R is called regular if for any prime ideal $\mathfrak{p} \subset R$, the localisation $R_{\mathfrak{p}}$ is a regular local ring.

Remark. If A is a regular local ring and $\mathfrak{p} \subset A$ a prime ideal then the localisation $A_{\mathfrak{p}}$ is also a regular local ring. The proof of this is quite difficult and uses methods we shall not be able to discuss in this course, you may refer to *Commutative Algebra* — Matsumura, Chapter 7, Theorem 45 and its corollary. It follows that in the definition of a regular ring we may replace prime ideal by maximal ideal.

Example 5. Clearly a field is a regular ring of dimension 0 and a Dedekind domain is a regular ring of dimension 1.

Example 6. If R is regular then $S = R \times R$ is also regular. A prime ideal of $R \times R$ is of the form $\mathfrak{q} = \mathfrak{p} \times R$ or $R \times \mathfrak{p}$ for a prime ideal $\mathfrak{p} \subset R$. Thus

$$S_{\mathfrak{q}} \cong R_{\mathfrak{p}}$$

which is a regular local ring. Hence $\mathbb{Z} \times \mathbb{Z}$ is regular. This shows that a regular ring need not be an integral domain.

Proposition 9. If R is a regular ring then so is $R[x]$ the polynomial ring over R in one variable.

Proof. If $\mathfrak{q}$ is a prime ideal of $B = R[x]$ then $\mathfrak{p} = \mathfrak{q} \cap R$ is a prime ideal of R. Then $S = R - \mathfrak{p}$ is a multiplicatively closed subset of $R[x]$ and

$$S^{-1}R[x] \cong R_{\mathfrak{p}}[x].$$

Let $\mathfrak{q}' = S^{-1}\mathfrak{q}$ then it is easy to show that

$$B_{\mathfrak{q}} \cong R_{\mathfrak{p}}[x] \text{ localised at } \mathfrak{q}'.$$

Now $R_{\mathfrak{p}}$ is a local ring and $\mathfrak{q}' \cap R_{\mathfrak{p}} = \mathfrak{p}R\mathfrak{p}$. Thus we may assume R it self is a regular local ring. Moreover we have $\mathfrak{q} \cap R = \mathfrak{p}$ the maximal ideal of R. We have to show $B_{\mathfrak{q}}$ is a regular local ring. Clearly $\mathfrak{q} \supset \mathfrak{p}B$ hence the image of $\mathfrak{q}$ in

$$\frac{B}{\mathfrak{p}B} \cong k[x] \quad \text{where } k = R/\mathfrak{p} \text{ is the residue field.}$$

is a principal ideal. It follows that $\mathfrak{q} = \mathfrak{p}B$ or $\mathfrak{q} = \mathfrak{p}B + (f)$ for some monic polynomial $f(x) \in B$. Suppose $\dim R = d$. If $\mathfrak{q} = \mathfrak{p}B$ then it is generated by d elements, and $\operatorname{ht}\mathfrak{q} = \operatorname{ht}\mathfrak{p} = d$. On the other hand if $\mathfrak{q} = \mathfrak{p}B + (f(x))$ then $\operatorname{ht}\mathfrak{q} = \operatorname{ht}\mathfrak{p} + 1 = d + 1$ and it is generated by $d + 1$ elements. Hence, in both cases $B_{\mathfrak{q}}$ is regular. □

Remark. The previous proposition shows that, in particular any polynomial ring $k[x_1, \ldots, x_n]$ over a field k is regular. Similarly if D is a Dedekind domain, for instance $D = \mathbb{Z}$, then $D[x_1, \ldots, x_n]$ is a regular ring.

The next exercise gives an extremely useful and convenient method of checking whether a local ring is regular.

Exercise (iv) (Jacobian criterion). Let K be a field and $k[x_1, \ldots, x_n]$ a polynomial ring. Let $f_1, \ldots, f_k \in k[x_1, \ldots, x_n]$ and $R = k[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$. Suppose $\mathfrak{m}$ is a maximal ideal of R such that $k = R/\mathfrak{m}$ and $A = R_{\mathfrak{m}}$ the localisation at $\mathfrak{m}$. Consider the matrix

$$M = \left(\left(\overline{\frac{\partial f_i}{\partial x_j}} \right) \right)$$

where $\overline{\frac{\partial f_i}{\partial x_j}}$ is the image of $\frac{\partial f_i}{\partial x_j}$ in $R/\mathfrak{m}$. Then show that R is regular if and only if

$$\operatorname{rank}(M) = n - \dim A.$$

(Hint. Show that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n - \operatorname{rank}(M)$.)