Please answer all questions. Total time 2 hrs, total points 60.

1. Let R be a ring and Γ a finite group of automorphisms of R. Denote the fixed ring of Γ by $R^\Gamma = \{a \in R \mid \gamma(a) = a \text{ for all } \gamma \in \Gamma\}$. Show that R is integral over R^Γ. (10)

2. An integral domain R is called normal if it is integrally closed in its field of fractions $Q(R)$. Show that R is normal if and only if $R_{\mathfrak{p}}$ is normal for all prime ideals $\mathfrak{p} \subset R$. (10)

3. Prove or disprove the following statements:

   (a) The integral closure of $\mathbb{Z}$ in $\mathbb{Q}[i]$ is $\mathbb{Z}[i]$. (5)

   (b) The ring $R = \dfrac{\mathbb{Q}[x,y]}{(y^2 - x^3)}$ is a normal integral domain. (5)

4. Let $\phi : A \to B$ be a ring homomorphism. For any ideal $\mathfrak{a}$ of A we denote the extended ideal $\phi(\mathfrak{a})B$ by $\mathfrak{a}^e$ and for any ideal $\mathfrak{b}$ of B we denote the contracted ideal $\phi^{-1}(\mathfrak{b})$ by $\mathfrak{b}^c$.

   (a) If $\mathfrak{p} \subset R$ is a prime ideal such that $\mathfrak{p}^{ec} = \mathfrak{p}$ and $S = A - \mathfrak{p}$ show that $\phi(S)$ is a multiplicatively closed subset of B, and $\phi(S) \cap \mathfrak{p}^{ec} = \emptyset$. (5)

   (b) Show that a prime ideal $\mathfrak{p}$ of A is the contraction of a prime ideal of B if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$. (5)

5. Let R be a ring, a multiplicatively closed subset $S \subset R$ is called saturated if for any $a, b \in R$, $ab \in S \Rightarrow a$ and $b \in S$. Let $S \subset R$ be multiplicatively closed, show that S is saturated if and only if $R - S$ is a union of prime ideals of R. (10)

6. Let $S \subset R$ be multiplicatively closed and let $\widetilde{S}$ be the set of of all $a \in R$ such that $ab \in S$ for some $b \in R$. Show that $\widetilde{S}$ is the smallest saturated multiplicatively closed set containing S and $S^{-1}R \cong (\widetilde{S})^{-1}R$. (10)