1. Direct Limits

Definition 1: A directed set I is a set with a partial order $\leq$ such that for every $i, j \in I$ there is $k \in I$ such that $i \leq k$ and $j \leq k$. Let R be a ring. A directed system of R-modules indexed by I is a collection of R modules $\{M_i \mid i \in I\}$ with a R module homomorphisms $\mu_{i,j} : M_i \to M_j$ for each pair $i, j \in I$ where $i \leq j$, such that

(i) for any $i \in I$, $\mu_{i,i} = \mathrm{Id}_{M_i}$ and
(ii) for any $i \leq j \leq k$ in I, $\mu_{i,j} \circ \mu_{j,k} = \mu_{i,k}$.

We shall denote a directed system by a tuple $(M_i, \mu_{i,j})$.

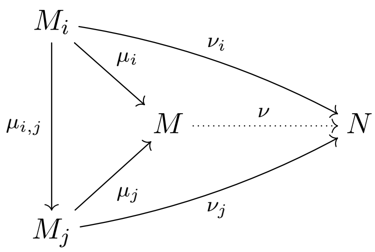
The direct limit of a directed system is defined using a universal property. It exists and is unique up to a unique isomorphism.

Theorem 2 (Direct limits)**.** *Let $\{M_i \mid i \in I\}$ be a directed system of R modules then there exists an R module M with the following properties:*

(i) *There are R module homomorphisms $\mu_i : M_i \to M$ for each $i \in I$, satisfying $\mu_i = \mu_j \circ \mu_{i,j}$ whenever $i < j$.*
(ii) *If there is an R module N such that there are R module homomorphisms $\nu_i : M_i \to N$ for each i and $\nu_i = \nu_j \circ \mu_{i,j}$ whenever $i < j$; then there exists a unique R module homomorphism $\nu : M \to N$, such that $\nu_i = \nu \circ \mu_i$.*

The module M is unique in the sense that if there is any other R module M' satisfying properties (i) and (ii) then there is a unique R module isomorphism $\mu' : M \to M'$. This module M is called the direct limit of the system $\{M_i \mid i \in I\}$ and denoted by $M = \varinjlim M_i$.

Proof. The conditions of the theorem can be summed up by the following commutative diagram.



Let

$$M' = \bigoplus_{i \in I} M_i$$

and $\mu'_i : M_i \to M'$ be the map such that $\mu'_i(x)$ has x at the i-th place and 0 everywhere else. Let K be the submodule generated by the set $\{\mu'_i(x_i) - \mu'_j(\mu_{i,j}(x_i)) \mid i \in I,\ x_i \in M_i\}$. Let $M = M'/K$ and $\Pi : M' \to K$ be the quotient map also let $\mu_i = \Pi \circ \mu'_i$.

Clearly M satisfies property (i) of the theorem. If there is an R module N as in property (ii) then there is a R module homomorphism $\nu' : M' \to N$ given by $\nu'((x_i)_{i \in I}) = \sum_{i \in I} \nu_i(x_i)$.

Since $\nu_i = \nu_j \circ \mu_{i,j}$ we see that K is contained in the kernel of ν' and we get an induced map $\nu : M \to N$. In fact this is the unique map which satisfies $\nu_i = \nu \circ \mu_i$. $\square$

Exercise (i). Prove the uniqueness part of Theorem 2.

Exercise (ii). If I has a largest element i_0, that is $i \leq i_0$ for all $i \in I$ then show that $\mu_{i_0} : M_{i_0} \to \varinjlim M_i$ is an isomorphism.

Remark. Any element of M' is of the form $\mu'_{i_1}(x_{i_1}) + \ldots + \mu'_{i_k}(x_{i_k})$ for finitely many elements $i_1, \ldots, i_k \in I$. Since $\Pi : M' \to M$ is surjective any $x \in M$ is of the form

$$x = \mu_{i_1}(x_{i_1}) + \ldots + \mu_{i_k}(x_{i_k}).$$

Choose $j \in I$ such that $i_1, \ldots, i_k \leq j$, and let $y_j = \mu_{i_1,j}(x_{i_1}) + \ldots + \mu_{i_k,j}(x_{i_k})$. Then $x = \mu_j(y_j)$. Thus any element of M is of the form $\mu_i(y_i)$ for some $i \in I$ and $y_i \in M_i$.

Moreover suppose $\mu_i(x_i) = 0$, then $\mu'_i(x_i) \in K$ hence for some $j \geq i$, $\mu_{i,j}(x_i) = 0$.

Exercise (iii). Let N be an R module and $\{M_n \mid n \in \mathbb{N}\}$ a collection of submodules of N such that $M_n \subset M_{n+1}$ for any integer n. Then $\{M_n \mid n \in \mathbb{N}\}$ forms a directed system of R modules with $\mu_{i,j} : M_i \to M_j$ being the inclusion map whenever $i < j$. Show that

$$\varinjlim M_n = \bigcup_{n \in \mathbb{N}} M_n.$$

Definition 3 (Homomorphism of directed systems)**:** Let $\mathscr{N} = (N_i, \nu_{i,j})$, and $\mathscr{M} = (M_i, \mu_{i,j})$ be two directed systems of R modules indexed by I. A homomorphism $\Phi : \mathscr{N} \to \mathscr{M}$ consists of R module homomorphisms $\phi_i : N_i \to M_i$ such that $\mu_{i,j} \circ \phi_i = \phi_j \circ \nu_{i,j}$ for each $i < j$.

The definition basically says that the following diagram commutes.

$$\begin{array}{ccc} N_i & \xrightarrow{\nu_{i,j}} & N_j \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \phi_j} \\ M_i & \xrightarrow{\mu_{i,j}} & M_j \end{array}$$

Let

$$N = \varinjlim N_i \quad \text{and} \quad M = \varinjlim M_i$$

be the direct limits of the two directed systems of the previous definition. Then we have maps $\nu_i : N_i \to N$ and $\mu_i : M_i \to M$. Moreover for each $i \in I$ we have $\mu_i \circ \phi_i : N_i \to M$ which by the universal property of the direct limit gives rise to a unique R module homomorphism $\phi : N \to M$. Moreover we have $\phi \circ \nu_i = \mu_i \circ \phi_i$.

Let $\mathscr{P} = (P_i, \pi_{i,j})$ be another directed systems of R modules. A sequence of homomorphisms of directed systems

$$\mathscr{M} \xrightarrow{\Phi} \mathscr{N} \xrightarrow{\Psi} \mathscr{P}$$

is called exact if for each $i \in I$, $M_i \xrightarrow{\phi_i} N_i \xrightarrow{\psi_i} P_i$ is exact.

Proposition 4. With the above notation if $\mathscr{M} \xrightarrow{\Phi} \mathscr{N} \xrightarrow{\Psi} \mathscr{P}$ is exact then the induced sequence on direct limits

$$\varinjlim M_i \xrightarrow{\phi} \varinjlim N_i \xrightarrow{\psi} \varinjlim P_i$$

is also exact.

Proof. Let $x \in \varinjlim N_i$ such that $\psi(x) = 0$. There is some $i \in I$ and $x_i \in N_i$ such that $x = \nu_i(x_i)$. We must have $\pi_i(\psi_i(x_i)) = \psi(\nu_i(x_i)) = 0$. Thus there exists $j > i$ such that $\pi_{i,j}(\psi_i(x_i)) = 0 \Rightarrow \psi_j(\nu_{i,j}(x_i)) = 0$. By assumption we have $y_j \in M_j$ such that $\phi_j(y_j) = \nu_{i,j}(x_i)$. Thus $\phi(\mu_j(y_j)) = \nu_j(\phi_j(y_j)) = \nu_j(\nu_{i,j}(x_i)) = \nu_i(x_i) = x$. This completes the proof. □

Example 1 (Germs of continuous functions)**.** Let X be a hausdorff topological space and for every $U \subset X$ open $\mathcal{C}(U)$ be the set of continuous functions $U \to \mathbb{R}$. Let $x \in X$, then the set $I = \{U \subset X \mid U \text{ open}, x \in U\}$ is a directed set with the partial order $U \leq V$ if $U \supset V$. Whenever $U \supset V$, we have the map $\rho_{U,V} : \mathcal{C}(U) \to \mathcal{C}(V)$ given by restriction of functions. Thus

$$\{\mathcal{C}(U) \mid U \subset X \text{ open}, x \in U\}$$

is a directed system of $\mathbb{R}$ modules (in fact rings). The direct limit

$$\mathcal{C}_x = \varinjlim \mathcal{C}(U)$$

is called the stalk of continuous functions to $\mathbb{R}$ at x. Its elements are called germs of continuous functions at x. This is in fact a ring. The details are left as an exercise.

Exercise (iv) (Direct limit of rings)**.** There is another way of constructing the direct limit. Let $(M_i, \mu_{i,j})$ be a directed system of R modules indexed by I. Let

$$M = \left(\bigsqcup_{i \in I} M_i\right) / \sim \quad \text{where} \quad x_i \sim \mu_{i,j}(x_i) \text{ for any } i \in I \text{ and } x_i \in M_i.$$

The set M has an R module structure.

a. If $x_i \in M_i$ and $x_j \in M_j$ then let $i, j \leq k$ and define $[x_i] + [x_j] = [\mu_{i,k}(x_i) + \mu_{j,k}(x_j)]$. Show that this is well defined and that $M \cong \varinjlim M_i$.

b. If M_i are R algebras and $\mu_{j,k}$ are R algebra homomorphisms then M has the structure of an R algebra. Show that the multiplication $[x_i][x_j] = [\mu_{i,k}(x_i)\mu_{j,k}(x_j)]$ is well defined. Hence this gives a way of constructing direct limit of rings.

2. Inverse Limits

An inverse system is a directed system with arrows reversed.

Definition 5 (Inverse system)**:** Let I be a directed set, a collection of R modules $\{M_i \mid i \in I\}$ indexed by I is an inverse system if for every $i < j$ in I, there is an R module homomorphism $\mu_{j,i} : M_j \to M_i$. Moreover, there homomorphisms satisfy:

(i) $\mu_{i,i} = \mathrm{Id}$ for all $i \in I$ and
(ii) $\mu_{k,i} = \mu_{j,i} \circ \mu_{k,j}$ for any $i \leq j \leq k$.

We denote the inverse system as a tuple $(M_i, \mu_{j,i})$. The inverse system is called surjective if $\mu_{j,i}$ is surjective for all $i \leq j$.

The inverse limit is defined exactly like a direct limit with the arrows reversed. The construction and properties are quite different though.

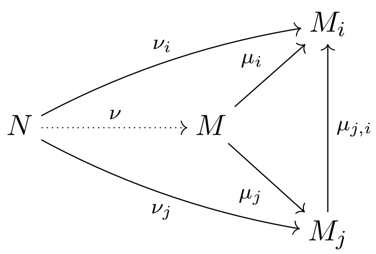
Theorem 6 (Inverse limit)**.** Let $\{M_i \mid i \in I\}$ be a inverse system of R modules. Then there exists an R module M with the following properties:

(i) There are R module homomorphisms $\mu_i : M \to M_i$ for each $i \in I$, such that for $i < j$, $\mu_i = \mu_{j,i} \circ \mu_j$.
(ii) If N is an R module with homomorphisms $\nu_i : M \to M_i$ for each $i \in I$, such that for $i < j$, $\nu_i = \mu_{j,i} \circ \nu_j$ then there is a unique R module homomorphism $\nu : N \to M$ satisfying $\nu_i = \mu_i \circ \nu$.

If M' is another R module with properties (i) and (ii) then there is a unique isomorphism $\mu' : M' \to M$. This module M is called the inverse limit of the inverse system $\{M_i \mid i \in I\}$and denoted by

$$M = \varprojlim M_i.$$

Proof. Properties (i) and (ii) of the theorem can be summed up in the following commutative diagram.



Let

$$M = \left\{ (x_i) \in \prod_{i \in I} M_i \;\middle|\; x_i = \mu_{j,i}(x_j) \text{ for all } i \leq j \right\}.$$

There are projection maps $\mu_i : M \to M_i$ and it is easy to check that this module satisfies all the properties mentioned in the theorem. □

Exercise (v). Complete the proof of Theorem 6.

Exercise (vi). If I has a largest element i_0, that is $i \leq i_0$ for all $i \in I$ then show that $\mu_{i_0} : \varprojlim M_i \to M_{i_0}$ is an isomorphism.

Exercise (vii). Let $(M_n, \mu_{m,n})$ be an inverse system indexed by the set of positive integers $\mathbb{N}$. Let $M' = \prod_{n\in\mathbb{N}} M_n$ and $d : M' \to M'$ be

$$d((a_1, \dots, a_n, \dots)) = (a_1 - \mu_{2,1}(a_2), \dots, a_n - \mu_{n+1,n}(a_{n+1}), \dots) = (a_n - \mu_{n+1,n}(a_{n+1}))_{n\in\mathbb{N}}.$$

Show that $\varprojlim M_n \cong \mathrm{Ker}(d)$.

Remark. It is easy to see that if we have an inverse system of rings the previous construction of inverse limit actually produces a ring.

Example 2 (Completion)**.** Let $\mathfrak{a} \subset R$ be an ideal of R. Consider the inverse system of rings

$$\{R_n = R/\mathfrak{a}^n \mid n \in \mathbb{N}\}.$$

We have the quotient maps $R/\mathfrak{a}^n \to R/\mathfrak{a}^m$ for $m \leq n$. The inverse limit

$$\varprojlim R/\mathfrak{a}^n$$

is called the $\mathfrak{a}$ - adic completion of R. We shall see more on this in the next lecture.

Example 3. Let $R = k[x]$ the polynomial ring in one variable over a field k and $\mathfrak{a} = (x)$ as in the previous example. Then the inverse limit $\varprojlim R/\mathfrak{a}^n$ is isomorphic to $k[[x]]$. There are quotient maps $\mu_n : k[[x]] \to k[x]/(x^n)$ for every $n \in \mathbb{N}$. If there is any ring N with maps $\nu_n : N \to k[x]/(x^n)$ satisfying property (ii) of Theorem 6, then clearly we have a unique ring homomorphism $\nu : N \to k[[x]]$ such that $\nu_i = \mu_i \circ \nu$. Hence by uniqueness $k[[x]]$ is isomorphic to the inverse limit.

Example 4 (Profinite groups)**.** A profinite groups is an inverse limit of a inverse system of finite abelian groups. They occur in Galois theory as Galois groups of infinite Galois extensions.

If $\mathscr{M} = (M_i, \mu_{j,i})$, $\mathscr{N} = (N_i, \nu_{j,i})$ are inverse systems of R modules indexed by I, then a homomorphism of inverse systems $\Phi : \mathscr{M} \to \mathscr{N}$ is a collection of homomorphisms $\phi_i : M_i \to N_i$ such that the following diagram is commutes for any $i \leq j$.

$$\begin{array}{ccc} M_j & \xrightarrow{\mu_{j,i}} & M_i \\ {\scriptstyle\phi_j}\downarrow & & \downarrow{\scriptstyle\phi_i} \\ N_j & \xrightarrow{\nu_{j,i}} & N_i \end{array}$$

In this case there is a induced homomorphism of R modules $\phi : \varprojlim N_i \to \varprojlim M_i$ obtained as follows: For each $i \in I$ we have a map $\phi\mu_i : \varprojlim M_i \to M_i$ and these maps satisfy property (ii) of Theorem 6, hence we get a unique map ϕ as required and the following diagram commutes.

$$\begin{array}{ccc} \varprojlim M_i & \xrightarrow{\mu_i} & M_i \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi_i} \\ \varprojlim N_i & \xrightarrow{\nu_i} & N_i \end{array}$$

Let $\mathscr{P} = (P_i, \pi_{j,i})$ be another inverse system of R modules indexed by I and $\Psi : \mathscr{N} \to \mathscr{P}$ a homomorphism of inverse systems consisting of maps $(\psi_i \mid i \in I)$. The sequence

$$\mathscr{M} \xrightarrow{\Phi} \mathscr{N} \xrightarrow{\Psi} \mathscr{P}$$

is called exact if $M_i \xrightarrow{\phi_i} N_i \xrightarrow{\psi_i} P_i$ is exact for each $i \in I$. In this case the induced sequence of the inverse limits may not be exact as in the case of direct limits. However we have to following partial result.

Proposition 7 (Exactness under inverse limit)**.** Let the indexing set be $\mathbb{N}$. If $0 \to \mathscr{M} \xrightarrow{\Phi} \mathscr{N} \xrightarrow{\Psi} \mathscr{P} \to 0$ is an exact sequence of inverse systems of R modules indexed by $\mathbb{N}$, then

$$0 \to \varprojlim M_n \xrightarrow{\phi} \varprojlim N_n \xrightarrow{\psi} \varprojlim P_n$$

is exact. Moreover, if $\mathscr{M}$ is a surjective system then

$$0 \to \varprojlim M_n \xrightarrow{\phi} \varprojlim N_n \xrightarrow{\psi} \varprojlim P_n \to 0$$

is also exact.

Proof. Let $M' = \prod_{n\in\mathbb{N}} M_n$ and $d_{M'} : M' \to M'$ be

$$d_{M'}((a_1, \ldots, a_n, \ldots)) = (a_1 - \mu_{2,1}(a_2), \ldots, a_n - \mu_{n+1,n}(a_{n+1}), \ldots) = (a_n - \mu_{n+1,n}(a_{n+1}))_{n\in\mathbb{N}}$$

as in Exercise (vii). Similarly define $N', d_{N'}$ and $P', d_{P'}$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\prod \phi_n} & N' & \xrightarrow{\prod \psi_n} & P' & \longrightarrow & 0 \\ & & \downarrow d_{P'} & & \downarrow d_{P'} & & \downarrow d_{P'} & & \\ 0 & \longrightarrow & M' & \xrightarrow[\prod \phi_n]{} & N' & \xrightarrow[\prod \psi_n]{} & P' & \longrightarrow & 0 \end{array}$$

where the rows are exact. Hence by snake's lemma we have an exact sequence

$$0 \to \mathrm{Ker}(d_{M'}) \to \mathrm{Ker}(d_{N'}) \to \mathrm{Ker}(d_{P'}) \to \mathrm{Coker}(d_{M'}) \to \mathrm{Coker}(d_{N'}) \to \mathrm{Coker}(d_{P'}) \to 0.$$

This proves the first part using the isomorphism $\varprojlim M_n \cong \mathrm{Ker}(d_{M'})$ of Exercise (vii). Moreover if $\mathscr{M}$ is a surjective system them $d_{M'}$ is surjective so $\mathrm{Coker}(d_{M'}) = 0$ which completes the proof. □

Remark. Most of the inverse systems that we shall encounter will be surjective.