1. Graded Rings and Modules

Definition 1: A graded ring R is a ring with a collection of additive subgroups $R_0, R_1, R_2, \dots$ such that $R = \bigoplus_{n=1}^{\infty} R_n$ and $R_n R_m \subset R_{n+m}$ for all $n, m \geq 0$. An element $a \in R$ is called homogeneous if $a \in R_n$ for some n.

A graded R module is an R module M with a collection of subgroups $M_0, M_1, M_2, \dots$ such that $M = \bigoplus_{n=1}^{\infty} M_n$ and $R_n M_m \subset M_{n+m}$ for all $n, m \geq 0$. An element $x \in M$ is called homogeneous if $x \in M_n$ for some n.

Remark. Note that R_0 is a subring of R and contains the multiplicative identity. Moreover each R_n is an R_0 module. On the other hand $R_+ = R_1 \oplus R_2 \oplus \dots$ is an ideal of R.

Any $a \in R$ can be uniquely written as finite sum $a = \sum_n a_n$ of homogeneous elements $a_n \in R_n$. Only finitely many a_n are non-zero and those non-zero a_n are called the homogeneous components of a.

Similarly each M_n is an R_0 module and any $x \in M$ can be uniquely written as a finite sum of homogeneous elements.

Any ring A can be regarded as a graded ring with the trivial grading $A_0 = A$ and $A_n = 0$ for $n \geq 1$.

Example 1. The polynomial ring $R = A[x_1, x_2, \dots, x_n]$ in n variables over a ring A is a graded ring where R_n is the set of homogeneous polynomials of degree n.

However the power-series ring $A[[x_1, \dots, x_n]]$ does not have any graded ring structure other than the trivial one (see `https://math.stackexchange.com/questions/2522568/is-the-formal-power-series-ring-a-graded-ring`).

Definition 2: Let R, S be graded rings then a $\phi : R \to S$ is called a homomorphism of graded rings is ϕ is a ring homomorphism such that $\phi(R_n) \subset S_n$.

Similarly if M, N are R modules a map $f : M \to N$ is called a homomorphism of graded R modules if f is a homomorphism of R modules such that $f(M_n) \subset N_n$.

Exercise (i). Show that if R is a graded integral domain then any unit must be homogeneous of degree 0. However this is not true when R is not an integral domain.

Exercise (ii). If A is a graded local integral domain then show that $A_0 = A$.

Proposition 3. The following statements are equivalent for a graded ring R:

(a) R is noetherian;

(b) R_0 is noetherian and R is a finitely generated R_0 algebra.

Proof. For (a) $\Rightarrow$ (b) first note that $R_0 \cong R/R_+$, hence R_0 is noetherian. Moreover R_+ is an ideal of R, hence finitely generated over R. Let $R_+ = (x_1, \ldots, x_k)$, then we may take x_i to be homogeneous of some degree $m_i > 0$ (why?).

Let $S = R_0[x_1, \ldots, x_n]$ be the subring of R generated over R_0 by $x_1, \ldots, x_k$. Clearly $R_0 \subset S$. Assume that $R_m \subset S$ for all $m < n$. If $r \in R_n$, then $r = a_1x_1 + \cdots + a_kx_k$ for some $a_1, \ldots, a_k \in R$. Since $x_k \in R_{m_i}$ we must have $a_i \in R_{n-m_i} \subset S$. Thus $r \in S$ and by induction $R = S$.

The converse (b) $\Rightarrow$ (a) follows from the Hilbert basis theorem. $\square$

Example 2. If A is any ring (not graded) and $\mathfrak{a} \subset A$ is an ideal we can for a graded ring

$$A^* = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n$$

since if $x \in \mathfrak{a}^n$ and $y \in \mathfrak{a}^m$ then $xy \in \mathfrak{a}^{m+n}$. Clearly $A = (A^*)_0$ is a subring. If $\mathfrak{a}$ is finitely generated by $x_1, \ldots, x_n$ then A^* is a finitely generated A algebra generated by $x_1, \ldots, x_n$. It may not be a polynomial algebra over those generators, for example when $\mathfrak{a}$ is nilpotent $\mathfrak{a}^n = 0$ for all large n. If A is noetherian then by the previous proposition A^* is also noetherian.

Similarly if M is an A module and $(M_n)_{n\geq 0}$ is an $\mathfrak{a}$-filtration of M, that is $\mathfrak{a}^m M_n \subset M_{n+m}$, then we may form a graded A^* module $M^* = \bigoplus_{n=0} M_n$.

Proposition 4. Let A be a noetherian ring, $\mathfrak{a} \subset R$ an ideal. Let M be a finitely generated R module and $(M_n)_{n\geq 0}$ be an $\mathfrak{a}$-filtration of M. Then the following are equivalent:

(a) $(M_n)_{n\geq 0}$ is a stable $\mathfrak{a}$-filtration.
(b) M^* is a finitely generated graded A^* module.

Proof. For (a) $\Rightarrow$ (b) note that there is $n > 0$ such that $\mathfrak{a}^m M_n = M_{m+n}$. Thus

$$M^* = M_0 \oplus \cdots \oplus M_n \oplus \mathfrak{a}M_n \oplus \mathfrak{a}^2 M_n \oplus \cdots \oplus \mathfrak{a}^m M_n \oplus \cdots$$

and since $M_0, \ldots, M_n$ are finitely generated over A we can choose generators $x_{1,i}, \ldots, x_{k_i,i}$ of M_i for $0 \leq i \leq n$. These will generate M^* over A^*.

Now for the converse let $Q_n = M_0 \oplus \cdots \oplus M_n$. Then Q_n is a finitely generated A module but not in general an A^* module. The A^* submodule of M^* generated by Q_n is

$$M_n^* = M_0 \oplus \cdots \oplus M_n \oplus \mathfrak{a}M_n \oplus \mathfrak{a}^2 M_n \oplus \cdots \oplus \mathfrak{a}^m M_n \oplus \cdots$$

Clearly this is finitely generated. Now $M_0^* \subset M_1^* \subset \cdots \subset M_n^* \subset \cdots$ is an ascending chain of submodules of M^* and $M^* = \cup_n M_n^*$. Since A^* is noetherian and M^* is finitely generated this chain must stabilise, that is there exists $n > 0$ such that

$$M^* = M_n^* = M_{n+1}^* = \ldots.$$

This precisely means $M_{n+m} = \mathfrak{a}^m M_n$. $\square$

Using this we get the Artin-Rees lemma which will be the most useful result of this lecture.

Lemma 5 (Artin-Rees)**.** Let A be a noetherian ring, $\mathfrak{a} \subset A$ an ideal, M an A module and $(M_n)_{n \geq n}$ a stable $\mathfrak{a}$-filtration of M. If $K \subset M$ is a submodule, then $(K \cap M_n)_{n \geq 0}$ is a stable $\mathfrak{a}$-filtration of K.

Proof. Let $K_n = K \cap M_n$, then $\mathfrak{a}K_n \subset (\mathfrak{a}K) \cap (\mathfrak{a}M_{n+1}) \subset K \cap M_{n+1} = K_{n+1}$, hence $(K_n)_{n \geq 0}$ is an $\mathfrak{a}$-filtration of K. Thus $K^* = \bigoplus_{n=0}^{\infty} K_n$ is a submodule of M^*. Since A^* is noetherian and M^* is finitely generated so is K^*. Hence we obtain the result using the previous proposition. □

As an immediate corollary we have the following result which is usually called the Artin-Rees lemma.

Corollary 6. Let A be a noetherian ring, $\mathfrak{a} \subset A$ an ideal, M an A module. If $K \subset M$ is a submodule, then there exists an integer $n_0 > 0$ such that

$$(\mathfrak{a}^{n+n_0}M) \cap K = \mathfrak{a}^n((\mathfrak{a}^{n_0}M) \cap K) \text{ for all } n > 0.$$

Proof. Just take $M_n = \mathfrak{a}^n M$ then $(M_n)_{n \geq 0}$ is a stable a-filtration of M. Now use Lemma 5 to infer that $((\mathfrak{a}^n M) \cap K)_{n \geq 0}$ is a stable $\mathfrak{a}$-filtration of K. □

Theorem 7. Let A be a noetherian ring, $\mathfrak{a} \subset A$ an ideal, M an A module. Let $K \subset M$ be a submodule, then the $\mathfrak{a}$-adic topology on K is the same as the subspace topology induced by the $\mathfrak{a}$-adic topology on M.

Proof. The induced topology on K of the $\mathfrak{a}$-adic topology on M is generated by the the filtration $((\mathfrak{a}^n M) \cap K)_{n \geq 0}$. This is an stable $\mathfrak{a}$-filtration of K thus by Proposition 15 of Completions lecture the result follows. □

Using Theorem 7 we obtain an exactness property of completion of modules.

Proposition 8. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated modules over a noetherian ring A. Let $\mathfrak{a} \subset A$ be an ideal then the sequence of a-adic completions

$$0 \to \widehat{M'} \to \widehat{M} \to \widehat{M''} \to 0$$

is also exact.

Proof. The induced topology on M' by the $\mathfrak{a}$-adic topology on M is the same as the $\mathfrak{a}$-adic topology on M'. Similarly the quotient topology on M'' given by the $\mathfrak{a}$-adic topology on M is precisely the $\mathfrak{a}$-adic topology on M''. Thus the result follows from the exactness properties of the inverse limit. □

2. Completion of Rings continued

We shall first show that for a noetherian ring A, an ideal $\mathfrak{a} \subset A$ and a finitely generated A module M the $\widehat{A}$ modules $\widehat{M}$ and $\widehat{A} \otimes_A M$ are isomorphic. Here $\widehat{A}$ and $\widehat{M}$ are the $\mathfrak{a}$-adic completions.

As a consequence we shall show that $\widehat{A}$ is a flat A algebra. These results are not true in general without the noetherian assumption on A.

Remark. Let A be a ring, $\mathfrak{a} \subset A$ an ideal and M be an A module. Let $\widehat{A}$ and $\widehat{M}$ be the $\mathfrak{a}$-adic completions, then we have a natural morphism $\widehat{A}$ module homomorphism $\widehat{A} \otimes_A M \to \widehat{M}$ which is given as follows: There is a natural map $\imath : M \to \widehat{M}$, and the map $\widehat{A} \times M \to \widehat{M}$ given by $(a, m) \mapsto a\imath(m)$ is clearly A-bilinear, hence it induces a linear map from the tensor product.

Theorem 9. Let Let A be a ring, $\mathfrak{a} \subset A$ an ideal and M be a finitely generated A module. Let $\widehat{A}$ and $\widehat{M}$ be the $\mathfrak{a}$-adic completion. Then the map $\widehat{A} \otimes_A \widehat{M} \to \widehat{M}$ is surjective. Moreover, if A is noetherian then $\widehat{A} \otimes_A \widehat{M} \to \widehat{M}$ is an isomorphism.

Proof. Using the inverse limit description it follows that if $F \cong A^n$ then $\widehat{A} \otimes_A F \cong \widehat{F} \cong \widehat{A}^n$. Since M is finitely generated, we have an exact sequence of A modules

$$0 \to N \to A^n \to M \to 0.$$

This gives rise to a commutative diagram

$$\begin{tikzcd}
 & \widehat{A} \otimes_A N \arrow[r, "a"] \arrow[d, "f"] & \widehat{A}^n \arrow[r, "b"] \arrow[d, "g"] & \widehat{A} \otimes_A M \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \widehat{N} \arrow[r, "c"] & \widehat{A}^n \arrow[r, "d"] & \widehat{M} \arrow[r] & 0
\end{tikzcd}$$

The top line is exact since tensor product is right exact. The bottom line is exact because of Proposition 8. Since d is surjective and g is an isomorphism, so h must be surjective.

If A is noetherian, f is also surjective by the same reasoning. Some diagram chasing shows that h is injective in this case. Suppose $x'' \in \ker(h)$ and let $x \in b^{-1}\{x''\}$. Since $d(g(x)) = h(x'') = 0$, there exists $y' \in \widehat{N}$ such that $c(y') = g(x)$. Pick $x' \in f^{-1}\{y'\}$. Then $g(a(x')) = c(y') = g(x)$, but g is an isomorphism, hence $x = a(x')$. Therefore, $x'' = b(a(x')) = 0$. □

Hence as a corollary we have the following result. Recall to check that a module is flat it is enough to check that tensor product with it preserves injectivity of a morphism of finitely generated modules.

Corollary 10. If A is noetherian and $\mathfrak{a} \subset A$ an ideal, the $\mathfrak{a}$-adic completion $\widehat{A}$ is a flat A algebra.

Remark. Note that if M is not finitely generated $M \mapsto \widehat{M}$ may not be an exact functor. However $M \mapsto \widehat{A} \otimes_A M$ is exact, and the two functors coincide on finitely generated modules.

Exercise (iii). Let A be a noetherian ring and $\mathfrak{a} \subset A$ an ideal. Suppose $\widehat{A}$ is the $\mathfrak{a}$-adic completion of A then show that:

(a) $\widehat{\mathfrak{a}} = \widehat{A}a \cong \widehat{A} \otimes_A \mathfrak{a}$,

(b) $\widehat{(\mathfrak{a}^n)} = (\widehat{\mathfrak{a}})^n$, and
(c) $\mathfrak{a}^n/\mathfrak{a}^{n+1} \cong \widehat{\mathfrak{a}}^n/\widehat{\mathfrak{a}}^{n+1}$.

Proposition 11. If A is noetherian and $\mathfrak{a} \subset A$ an ideal, then $\widehat{\mathfrak{a}}$ is contained in the Jacobson radical of $\widehat{A}$ the $\mathfrak{a}$-adic completion.

Proof. There is an exact sequence $0 \to \mathfrak{a}^n \to A \to A/\mathfrak{a}^n \to 0$. Thus taking completions and using Exercise (iii) we get an exact sequence

$$0 \to \widehat{\mathfrak{a}}^n \to \widehat{A} \to \widehat{A/\mathfrak{a}^n} \to 0.$$

Since $A/\mathfrak{a}^n$ has discrete topology $\widehat{A/\mathfrak{a}^n} \cong A/\mathfrak{a}^n$, thus $A/\mathfrak{a}^n \cong \widehat{A}/\widehat{\mathfrak{a}}^n$. Taking inverse limits we see that $\widehat{(\widehat{A})} \cong \widehat{A}$, where $\widehat{(\widehat{A})}$ is the $\widehat{\mathfrak{a}}$-adic completion of $\widehat{A}$, showing that $\widehat{A}$ is complete in the $\widehat{\mathfrak{a}}$-adic topology. For any $x \in \widehat{\mathfrak{a}}$ the sequence $y_n = 1 + x + \ldots + x^n$ is clearly Cauchy hence it converges to an element y in $\widehat{A}$. We also have

$$(1 - x)y_n = 1 - x^{n-1},$$

hence by continuity of multiplication we must have $y(1 - x) = 1$. Thus for any $x \in \widehat{\mathfrak{a}}$, $1 - x$ is a unit. This proves that $\widehat{\mathfrak{a}}$ is conained in the Jacobson radical. □

Exercise (iv). If A is noetherian, $\mathfrak{a} \subset A$ an ideal and $\widehat{A}$ the $\mathfrak{a}$-adic completion of A then show that there is a bijection between the maximal ideals of $\widehat{A}$ and the maximal ideals of $A/\mathfrak{a}$. If $\mathfrak{a}$ is a maximal ideal then show that $\widehat{A}$ is local with maximal ideal $\widehat{\mathfrak{a}}$.

Exercise (v). Let $p \in \mathbb{Z}$ be a prime, denote by $\mathbb{Z}_{(p)}$ the localisation of $\mathbb{Z}$ at the prime ideal (p) and by $\widehat{\mathbb{Z}}_p$ the (p)-adic completion of $\mathbb{Z}$ or the ring of p-adic integers. Then show that $\widehat{\mathbb{Z}}_p$ is isomorphic to the $p\mathbb{Z}_{(p)}$-adic completion of $\mathbb{Z}_{(p)}$.

Corollary 12. If A is a noetherian local ring with maximal ideal $\mathfrak{m}$, then the $\mathfrak{m}$-adic completion $\widehat{A}$ is a local ring with maximal ideal $\widehat{\mathfrak{m}}$.

Proof. Since $\widehat{A}/\widehat{\mathfrak{m}} \cong A/m$ which is a field the ideal $\widehat{\mathfrak{m}}$ is maximal. Moreover by the previous proposition any maximal ideal of $\widehat{A}$ must contain $\widehat{A}$. Thus $\widehat{A}$ is local with the only maximal ideal $\mathfrak{m}$. □

Remark. It is often the case that the completion of a local ring (with respect to its maximal ideal) is easier to deal with than the local ring itself. This is especially useful in Algebraic Geometry. Many of the properties of the local ring are preserved by the completion. To study a noetherian ring R one often localises it at some prime ideal $\mathfrak{p}$, to get a local ring $R_{\mathfrak{p}}$, and then completes with respect to the maximal ideal $\mathfrak{m} = \mathfrak{p}R_{\mathfrak{p}}$ to get a complete local ring $\widehat{R_{\mathfrak{p}}}$. If we start with $R = \mathbb{Z}$ and a prime ideal $\mathfrak{p} = (p)$ we arrive at the p-adic integers.

The next theorem by Krull determines the kernel of $M \to \widehat{M}$ for an R module.

Theorem 13 (Krull's theorem). Let A be a noetherian ring, $\mathfrak{a} \subset A$ an ideal and M a finitely generated A module. If $\widehat{M}$ denotes the $\mathfrak{a}$-adic completion of M and

$$E = \bigcap_{n=0}^{\infty} \mathfrak{a}^n M = \ker(M \to \widehat{M})$$

then

$$E = \{x \in M \mid (1-\alpha)x = 0 \text{ for some } \alpha \in \mathfrak{a}\}.$$

That is $\ker(M \to \widehat{M})$ consists precisely of elements of M annihilated by some element of $1 + \mathfrak{a}$.

Proof. One side inclusion is trivial. If $(1-\alpha)x = 0$ then $x = \alpha x = \alpha^2 x = \dots$, so $x \in \mathfrak{a}^n M$ for all $n > 0$ which means $x \in E$.

On the other hand $\mathfrak{a}E = E$ and being a submodule of a finitely generated module over a noetherian ring, E is itself finitely generated. Let $a_1, \dots, x_n$ generate E then $x_i = \alpha_i x_i$ for some $\alpha_i \in \mathfrak{a}$. Consider the diagonal matrix D whose entries are $\alpha_1, \dots, \alpha_n$ and I be the $n \times n$ identity matrix then

$$\det(I_n - D) = 1 + \alpha \text{ for some } \alpha \in \mathfrak{a}$$

and $(1+\alpha)x = 0$ for all $x \in E$ (consequence of Cayley-Hamilton theorem). □

Remark. In the setting of Krull's theorem let $S = 1 + \mathfrak{a}$. Then S is a multiplicatively closed subset of A. Consider the morphism $\phi : A \to S^{-1}A$. If $x \in \ker(\phi)$ then $(1+\alpha)x = 0$ for some $\alpha \in \mathfrak{a}$. Hence by Krull's theorem

$$\ker(\phi) = \bigcap_{n=0}^{\infty} \mathfrak{a}^n = \ker(A \to \widehat{A}).$$

Corollary 14. Let A be a noetherian ring, $\mathfrak{a} \subset A$ an ideal and M a finitely generated A module. If $\widehat{A}$ denotes the $\mathfrak{a}$-adic completion of A and $S = 1 + \mathfrak{a}$ then there is an injective ring homomorphism

$$S^{-1}A \to \widehat{A}.$$

Hence $S^{-1}A$ can be regarded as a subring of $\widehat{A}$.

Proof. The morphism $A \to \widehat{A}$ extends to $S^{-1}A \to \widehat{A}$ since any element of $1 + \mathfrak{a}$ is a unit in $\widehat{A}$. For any $\alpha \in \mathfrak{a}$, $1 - \alpha + \alpha^2 + \dots$ is the inverse of $(1+\alpha)$ (see proof of Proposition 11.) This morphism is injective by Krull's theorem. □

Exercise (vi) (In what generality is this true?). Let A be a noetherian ring and $\mathfrak{m} \subset A$ be a maximal ideal. We denote by $A_{\mathfrak{m}}$ the localisation of A at $\mathfrak{m}$ and by $\mathfrak{n} = \mathfrak{m}A_{\mathfrak{m}}$ the maximal ideal of $A_{\mathfrak{m}}$ then

$$\widehat{A} \cong \widehat{A_{\mathfrak{m}}}$$

where $\widehat{A}$ is the $\mathfrak{m}$-adic completion of A and $\widehat{A_{\mathfrak{m}}}$ is the $\mathfrak{n}$-adic completion of $\widehat{A_{\mathfrak{m}}}$.

Exercise (vii). Let $\widehat{\mathbb{Z}}_p$ be the ring of p-adic integers for some prime $p \in \mathbb{Z}$. Show that the power series ring $\widehat{\mathbb{Z}}_p[[x]]$ in one variable is isomorphic to the (p, x)-adic completion of $\mathbb{Z}[x]$.

We shall end this lecture with some more easy but useful corollaries of Krull's theorem.

Corollary 15. Let A be a noetherian integral domain and $\mathfrak{a} \neq (1)$ an ideal then $\cap_n \mathfrak{a}^n = (0)$. In particular the map $A \to \widehat{A}$ the $\mathfrak{a}$-adic completion is injective, hence A is a subring of $\widehat{A}$.

Proof. There are no zero-divisors in $1 + \mathfrak{a}$ so the result follows by Krull's theorem. $\square$

Corollary 16. Let A be a noetherian ring, $\mathfrak{a}$ an ideal of A contained in the Jacobson radical and M be a finitely generated A module. Then $\cap_n \mathfrak{a}^n M = 0$ and the morphism $M \to \widehat{M}$ into the $\mathfrak{a}$-adic completion is injective.

Proof. The elements of $1 + \mathfrak{a}$ are units hence they do not annihilate any element. $\square$

As a special case we have.

Corollary 17. For a noetherian local ring A with maximal ideal $\mathfrak{m}$ and a finitely generated A module M, we have $\cap_{n=1}^{\infty} \mathfrak{m}^n M = 0$, hence the map $M \to \widehat{M}$ the $\mathfrak{a}$-adic completion is injective. In particular $\cap_{n=1}^{\infty} \mathfrak{m}^n = (0)$ hence $A \to \widehat{A}$ is injective.