1. Krull Dimension

Recall the definition of dimension of a ring. In this lecture we shall call it the Krull dimension and denote it by dim.

Definition 1: Let A be a ring and $\mathfrak{p}$ a prime ideal of A. We define the height of $\mathfrak{p}$, denoted by $\operatorname{ht}\mathfrak{p}$, to be the supremum of lengths n of all chains $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n = \mathfrak{p}$ of prime ideals $\mathfrak{p}_i \in A$ such that $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$.

The Krull dimension of A is $\dim A = \sup\{\operatorname{ht}\mathfrak{p} \mid \mathfrak{p} \subset A \text{ prime ideal}\}$.

Clearly, $\operatorname{ht}\mathfrak{p} = 0 \iff \mathfrak{p}$ is a minimal prime ideal. Moreover, the Krull dimension is in fact the supremum over the heights of all maximal ideals of a ring.

Example 1. We have already seen that artinian rings and fields have Krull dimension 0. Moreover, Dedekind domains by definition have Krull dimension 1.

Exercise (i). Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal and $A_\mathfrak{p}$ the localisation of A at $\mathfrak{p}$, then show that $\operatorname{ht}\mathfrak{p} = \dim A_\mathfrak{p}$. Infer that $\dim A = \sup\{\dim A_\mathfrak{m} \mid \mathfrak{m} \subset A, \text{maximal ideal}\}$.

Example 2 (Integral extension has same Krull dimension)**.** If $A \subset B$ are rings and B is integral over A then $\dim A = \dim B$. Recall that if $\mathfrak{q} \subset \mathfrak{q}'$ are prime ideals of B such that $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$, then $\mathfrak{q} = \mathfrak{q}'$. Hence if $\mathfrak{q}_0 \subset \ldots \subset \mathfrak{q}_n$ is a chain of prime ideals in B, and $\mathfrak{p}_i = \mathfrak{q}_i \cap A$, then $\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_n$ is a chain in A, thus $\dim A \geq \dim B$. On the other hand the going up theorem shows that $\dim B \geq \dim A$.

Exercise (ii). Show that a principal ideal domain has Krull dimension 1.

The following result is outlined in Exercise 6 of Chapter 11 in Atiyah-MacDonald.

Theorem 2 (Krull dimension of Polynomial ring)**.** If A is a ring with finite Krull dimension and $A[x]$ is the polynomial ring over A in one variable then $\dim A + 1 \leq \dim A[x] \leq 2\dim A + 1$.

Proof. First we shall show that if $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_n$ is a chain of prime ideals $\mathfrak{q}_i \in A[x]$ with $\mathfrak{q}_i \neq \mathfrak{q}_{i+1}$, such that $\mathfrak{q}_i \cap A = \mathfrak{p}$ for all $i = 0, \ldots, n$ then the chain has length $n \leq 1$.

Let $S = A - \mathfrak{p}$, then S is a multiplicatively closed subset of A since $\mathfrak{p}$ is prime and $\mathfrak{q}_i \cap S = \emptyset$. Hence $S^{-1}\mathfrak{q}_i$ are prime ideals of $S^{-1}(A[x]) = A_\mathfrak{p}[x]$. Let $\mathfrak{m} = S^{-1}\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$ be the maximal ideal of $A_\mathfrak{p}$, and let $\mathfrak{m}[x]$ denote the ideal of polynomials in $A_\mathfrak{p}[x]$ with coefficients in $\mathfrak{m}$. Then we have an isomorphism

$$A_\mathfrak{p}[x]/\mathfrak{m}[x] \cong (A_p/\mathfrak{m})[x] \quad \text{where } k = A_p/\mathfrak{m} \text{ is a field.}$$

Moreover $\mathfrak{q}_i \cap A = \mathfrak{p} \Rightarrow S^{-1}\mathfrak{q} \cap A_\mathfrak{q} = \mathfrak{m} \Rightarrow S^{-1}\mathfrak{q} \supset \mathfrak{m}[x]$. Hence we have a chain of prime ideals

$$S^{-1}\mathfrak{q}_0/\mathfrak{m}[x] \subset \ldots \subset S^{-1}\mathfrak{q}_n/\mathfrak{m}[x] \quad \text{(strict inclusions)}$$

of length n in $k[x]$. By Exercise (ii) $k[x]$ has Krull dimension 1, hence $n \leq 1$.

Now suppose $\mathfrak{p} \subset A$ is a prime ideal. As before we denote by $\mathfrak{p}[x] = \mathfrak{p}A[x]$ the ideal consisting of polynomials with coefficients in $\mathfrak{p}$. We also have the ideal $(\mathfrak{p}, x)$ generated by $\mathfrak{p}$ and x, this ideal consists of all polynomials which the constant term in $\mathfrak{p}$. Both of these ideals are prime since

$$A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x] \quad \text{and} \quad A[x]/(\mathfrak{p}, x) \cong A/\mathfrak{p}.$$

Moreover $\mathfrak{p}[x] \subset (\mathfrak{p}, x)$ is a strict inclusion.

Let $d = \dim A$ and $\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_d$ be a maximal chain of prime ideals in A, then

$$\mathfrak{p}_0[x] \subset \ldots \subset \mathfrak{p}_d[x] \subset (\mathfrak{p}_d, x)$$

is a chain of length $d+1$ in $A[x]$, hence $\dim A[x] \geq d+1$.

Finally if $\mathfrak{q}_0 \subset \ldots \subset \mathfrak{q}_n$ is a chain of prime ideals of length n in $A[x]$, let $\mathfrak{p}_i = A \cap \mathfrak{q}_i$. The fact that for any i the three primes $\mathfrak{p}_{i-1}, \mathfrak{p}_i, \mathfrak{p}_{i+1}$ can not all be the same gives the desired upper bound. We may argue as follows. Let

$$S = \{i \mid i = 0 \text{ or } 1 \leq i \leq n \text{ and } \mathfrak{p}_{i-1} \neq \mathfrak{p}_i\}.$$

Let $i_0 < \ldots < i_k$ be the elements of S then $\mathfrak{p}_{i_0} \subset \ldots \subset \mathfrak{p}_{i_k}$ is a chain of prime ideals in A, hence $k \leq d$ and $i_{s+1} - i_s \leq 2$. Since $i_0 = 0$ this gives $i_k \leq 2k$ and $n \leq i_k + 1 \leq 2k+1 \leq 2d+1$. $\square$

Exercise (iii). Search literature to find out whether the upper bound of Theorem 2 is achieved by some ring.

2. Dimension of a Noetherian Local Rings

As we saw in Exercise (i), the height of a prime ideal of a ring is the same as the Krull dimension of the local ring obtained by localising at that prime so it is important to be able to calculate the dimension of local rings. We shall demonstrate that three different definitions all match up for noetherian local rings, so we restrict our attention to noetherian local rings.

In this section let us assume that A is a noetherian local ring and $\mathfrak{m}$ is its maximal ideal.

We defined the Hilbert dimension of A to be $\mathrm{d}_{\mathrm{Hilb}}(G_{\mathfrak{m}}(A)$ which is in fact the same as the degree of the characteristic polynomial for $\mathfrak{m}$, that is

$$\mathrm{d}_{\mathrm{Hilb}}(A) = \mathrm{d}_{\mathrm{Hilb}}(G_{\mathfrak{m}}(A)) = \deg \chi^A_{\mathfrak{m}},$$

where $\chi^A_{\mathfrak{m}}(n) = l(A/\mathfrak{m}^n)$ for all sufficiently large n.

Definition 3: We denote by $\delta(A)$ the minimal number of generators for any $\mathfrak{m}$-primary ideal $\mathfrak{q} \subset A$.

Remark. Our goal for this lecture is to show that $\delta(A) = \mathrm{d}_{\mathrm{Hilb}}(A) = \dim A$. In particular the Krull dimension is the same as the Hilbert dimension. We shall achieve this by showing $\delta(A) \geq \mathrm{d}_{\mathrm{Hilb}}(A) \geq \dim A \geq \delta(A)$.

Proposition 4. $\delta(A) \geq \mathrm{d}_{\mathrm{Hilb}}(A)$.

Proof. This immediately follows from Corollary 13 and Proposition 14 of the lecture on Hilbert polynomials. $\square$

Next up we shall show that $d_{\mathrm{Hilb}}(A) \geq \dim A$. For this we need the following lemma.

Lemma 5. If $x \in A$ is not a unit or a zero-divisor then $d_{\mathrm{Hilb}} A/(x) \leq d_{\mathrm{Hilb}} A - 1$.

Proof. Let $N = (x)$ and $M = A/(x)$ which are both A modules. Let $N_n = \mathfrak{m}^n \cap N$, then we have an exact sequence

$$0 \to N/N_n \to A/\mathfrak{m}^n \to M/\mathfrak{m}^n M \to 0.$$

Hence we have

$$l(M/\mathfrak{m}^n M) = l(A/\mathfrak{m}^n) - l(N/N_n) \quad \text{for all } n > 0.$$

Since $l(A/\mathfrak{m}^n) = \chi^A_{\mathfrak{m}}(n)$ and $l(M/\mathfrak{m}^n M) = \chi^M_{\mathfrak{m}}(n)$ are both polynomials, hence so is $g(n) = l(N/N_n)$.

By the Artin-Rees lemma, Lemma 5 in the lecture on Graded rings and its corollary, Corollary 6 there is an integer n_0 such that

$$N_{n+n_0} = \mathfrak{m}^n N_{n_0} \subset \mathfrak{m}^n N \quad \text{for all } n \geq 0.$$

On the other since N is an ideal in A we have

$$\mathfrak{m}^n N \subset \mathfrak{m}^n \cap N = N_n.$$

Hence for all $n \geq 0$

$$g(n + n_0) \geq l(N/\mathfrak{m}^n N) = \chi^N_{\mathfrak{m}}(n) \geq g(n).$$

Since g and $\chi^N_{\mathfrak{m}}$ are both polynomials this means

$$\lim_{n\to\infty} \frac{g(n)}{\chi^N_{\mathfrak{m}}(n)} = 1.$$

Thus g and $\chi^N_{\mathfrak{m}}$ have the same degree and same leading coefficient. Moreover, since x is not a zero divisor we have $N \cong A$ as an A module. Hence $\chi^N_{\mathfrak{m}} = \chi^A_{\mathfrak{m}}$, and since g and $\chi^A_{\mathfrak{m}}$ have the same leading terms and $\chi^M_{\mathfrak{m}} = \chi^A_{\mathfrak{m}} - g$ we have $\deg \chi^M_{\mathfrak{m}} < \deg \chi^A_{\mathfrak{m}}$. $\square$

Proposition 6. $d_{\mathrm{Hilb}}(A) \geq \dim A$ and in particular $\dim A$ is finite.

Proof. We shall prove this by induction on $d = d_{\mathrm{Hilb}}(A)$. If $d = 0$, then $l(A/\mathfrak{m}^n)$, (length as A module), is constant for all n sufficiently large. This means $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for large n, so by Nakayama's lemma $\mathfrak{m}^n = 0$. Hence, A is artinian and $\dim A = 0$.

Now assume $d > 0$. We shall show that for any chain $\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals in A of length r, we must have $r \leq d$ which will prove the proposition. Let $x \in \mathfrak{p}_1$ such that $x \notin \mathfrak{p}_0$. Consider the integral domain $A' = A/\mathfrak{p}_0$. Let $\mathfrak{m}' = \mathfrak{m}/\mathfrak{p}_0$, then since $A'/(\mathfrak{m}')^n$ is a quotient of $A/\mathfrak{m}^n$, so $l(A'/(\mathfrak{m}')^n) \leq l(A/\mathfrak{m}^n)$ for all $n > 0$. Thus $d_{\mathrm{Hilb}}(A') \leq d_{\mathrm{Hilb}}(A) = d$.

The image x' of x in A' is not a zero divisor or a unit. Hence

$$d_{\mathrm{Hilb}}(A'/(x')) \leq d_{\mathrm{Hilb}}(A') - 1 \leq d - 1.$$

Hence by the inductive hypothesis $\dim A'/(x') \leq d-1$. But the chain $\mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ descends to a chain of prime ideals in $A'/(x')$ of length $r-1$, hence $r-1 \leq d-1$. $\square$

Remark. The height of any prime ideal in a noetherian ring is finite. If R is a noetherian ring and $\mathfrak{p} \subset R$ a prime ideal, then $\operatorname{ht} \mathfrak{p} = \dim R_{\mathfrak{p}}$. Since $R_{\mathfrak{p}}$ is a noetherian ring the result follows from the previous proposition.

Proposition 7. Let $\dim A = d$ then there is an $\mathfrak{m}$ primary ideal generated by d elements, thus $\dim A \geq \delta(A)$.

Proof. First note that $\dim A = \operatorname{ht} \mathfrak{m} = d$. Moreover for any prime ideal $\mathfrak{p} \subset A$, we have $\operatorname{ht} \mathfrak{p} \leq \operatorname{ht} \mathfrak{m}$ and equality holds only if $\mathfrak{p} = \mathfrak{m}$. We shall choose elements $x_1, \ldots, x_d \in A$ such that for $1 \leq i \leq d$ any prime ideal containing $(x_1, \ldots, x_i)$ has height at least i.

We shall do this by induction. So assume $i > 0$ and that we have already chosen $x_1, \ldots x_{i-1}$ satisfying this property. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals containing $\mathfrak{a}_{i-1} = (x_1, \ldots, x_{i-1})$ such that $\operatorname{ht} p_j = i - 1$. Since $i - 1 < d$ we must have $\mathfrak{m} \neq \mathfrak{p}_j$. Thus $\mathfrak{m} \neq \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_s$. Choose $x_i \in \mathfrak{m}$ such that $x_i \notin \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_s$. Note that if $i = 1$ we take $\mathfrak{a}_0 = (0)$.

Now consider the ideal $\mathfrak{a}_i = (x_1, \ldots, x_i)$. If $\mathfrak{q}$ is a prime ideal containing $\mathfrak{a}_i$ then it must contain a minimal prime ideal $\mathfrak{p}$ containing $\mathfrak{a}_{i-1}$. Then by the inductive hypothesis $\operatorname{ht} \mathfrak{p} \geq i-1$. If $\operatorname{ht} \mathfrak{p} \geq i$ then clearly $\operatorname{ht} \mathfrak{q} \geq i$. So assume $\operatorname{ht} \mathfrak{p} = i - 1$, which means $\mathfrak{p} = \mathfrak{p}_j$ for some $j \in \{1, \ldots, s\}$. But $x_i \notin \mathfrak{p}_j$ so $\mathfrak{q} \neq \mathfrak{p}_j$, thus $\operatorname{ht} \mathfrak{q} > \operatorname{ht} \mathfrak{p}_j = i - 1$.

In this way me choose $x_1, \ldots, x_d$ and claim that $\mathfrak{a}_d = (x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary. To see this note that is $\mathfrak{p}$ is any prime ideal containing $\mathfrak{a}_d$ then $\operatorname{ht} \mathfrak{p} = d$, thus $\mathfrak{p} = \mathfrak{m}$. Hence $\mathfrak{m}$ is the only prime ideal containing $\mathfrak{a}_d$ thus $\mathfrak{m} = \sqrt{\mathfrak{a}_d}$. □

We have now proved the following theorem.

Theorem 8 (Dimension theorem)**.** Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$, then the following integers are all same:

(a) the maximal length of chains of prime ideals in A, that is the Krull dimension of A,
(b) degree of the characteristic polynomial of $\mathfrak{m}$, that is the degree of $\chi^A_{\mathfrak{m}}(n) = l(A/\mathfrak{m}^n)$,
(c) the least number of generators of an $\mathfrak{m}$-primary ideal of A.

Remark. If the local ring A is not noetherian, the three numbers of Theorem 8 may not necessarily be the same.

Example 3. Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring over a field k and $\mathfrak{m} = (x_1, \ldots, x_m)$. We denote the localisation of A at the maximal ideal $\mathfrak{m}$ by $A_{\mathfrak{m}}$ and let $\mathfrak{n} = \mathfrak{m}A_{\mathfrak{m}}$. It is easy to see that $G_{\mathfrak{n}}(A_{\mathfrak{m}}) \cong k[x_1, \ldots, x_n]$, which has Poincaré series $\dfrac{1}{(1-t)^n}$. Thus

$$\operatorname{ht} \mathfrak{m} = \dim A_{\mathfrak{m}} = \mathrm{d}_{\mathrm{Hilb}}(A_{\mathfrak{m}}) = n.$$

Remark. Let k be a field then we denote as usual the dimension of a k vector space by $\dim_k$. This is not to be confused with the Krull dimension of a ring.

Corollary 9. *Let $k = A/\mathfrak{m}$ be the residue field then $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.*

Proof. If $x_1, \ldots, x_d \in \mathfrak{m}$ are such that their images $\overline{x_1}, \ldots, \overline{x_d}$ form a basis of $\mathfrak{m}/\mathfrak{m}^2$ as a k vector space, then it is easy to see that $x_1, \ldots, x_d$ generate the ideal $\mathfrak{m}$. Thus $\dim_k \mathfrak{m}/\mathfrak{m}^2 = d \geq \dim A$. □

Proposition 10. *If $\widehat{A}$ is the $\mathfrak{m}$-adic completion of A, then $\dim \widehat{A} = \dim A$.*

Proof. We have $A/\mathfrak{m}^n \cong \widehat{A}/\widehat{\mathfrak{m}}^n$ for all $n > 0$ by (see proof of Proposition 11 in the lecture on graded rings). Hence $\chi_{\mathfrak{m}}^{A} = \chi_{\widehat{\mathfrak{m}}}^{\widehat{A}}$. □

3. Dimension of Noetherian Rings

Now that we have three different techniques for calculating the height of a prime ideal in a noetherian ring, we shall use them to prove some useful results about dimension of noetherian rings.

Remark. A noetherian local ring always has finite dimension. However in general a noetherian ring may not have finite dimension. An example due to Nagata is given in Exercise 4 of Chapter 11 in Atiyah-MacDonald.

Theorem 11. *Let R be a noetherian ring and $x_1, \ldots, x_n \in R$. If $\mathfrak{p} \subset R$ is a minimal prime ideal containing $\mathfrak{a} = (x_1, \ldots, x_n)$, then $\operatorname{ht} \mathfrak{p} \leq n$.*

Proof. Consider the ring $A = R_{\mathfrak{p}}$ and let $\mathfrak{m} = \mathfrak{p}R_{\mathfrak{p}}$ be its maximal ideal. Since $\mathfrak{m}$ is the only prime ideal in A containing $\mathfrak{a}A$ we infer that $\mathfrak{a}A$ is an $\mathfrak{m}$-primary ideal. Moreover $\operatorname{ht} \mathfrak{p} = \dim R_{\mathfrak{p}}$ and since $\mathfrak{a}A$ is an $\mathfrak{m}$-primary ideal generated by n elements, the images of $x_1, \ldots, x_n$ in A, we have $\dim A \leq n$. □

We have a strong converse to the previous theorem.

Proposition 12. *Let R be a noetherian ring and $\mathfrak{p}$ be a prime ideal in R of height h, then there are elements $x_1, \ldots, x_h \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime ideal containing $(x_1, \ldots, x_h)$.*

Proof. We shall inductively choose elements $x_1, \ldots, x_i \in \mathfrak{p}$ for $i \leq h$ such that the minimal prime ideals containing $\mathfrak{a}_i = (x_1, \ldots, x_i)$ and contained in $\mathfrak{p}$ have height i.

For $i = 1$, let $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_s$ be the minimal prime ideals of R contained in $\mathfrak{p}$. Then $\operatorname{ht} \mathfrak{q}_i = 0$, so $\mathfrak{p} \neq \mathfrak{q}_i \Rightarrow \mathfrak{p} \neq \cup_{i=1}^{s} \mathfrak{q}_i$. Choose $x_1 \in \mathfrak{p} - \cup_{i=1}^{s} \mathfrak{q}_i$. Then any minimal prime ideal containing (x_1) and contained in $\mathfrak{p}$ has height 1 by the previous theorem.

Suppose we have already chosen $x_1, \ldots, x_i$ and $i < h$. Let the minimal prime ideals containing $\mathfrak{a}_i$ and contained in $\mathfrak{p}$ be $\mathfrak{q}'_1, \ldots, \mathfrak{q}'_t$. Then $\operatorname{ht} \mathfrak{q}'_i = i < \operatorname{ht} \mathfrak{p}$ so $\mathfrak{p} \neq \mathfrak{q}'_i \Rightarrow \mathfrak{p} \neq \cup_{i=1}^t \mathfrak{q}'_i$. Hence we may choose $x_{i+1} \in \mathfrak{p} - \cup_{i=1}^t \mathfrak{q}'_i$. Then any minimal prime ideal containing $\mathfrak{a}_{i+1} = (x_1, \ldots, x_{i+1})$ and contained in $\mathfrak{p}$ must have height $i+1$ because it strictly contains a minimal prime ideal containing $\mathfrak{a}_i$.

Continuing in this way we can choose $x_1, \ldots, x_h$. If $\mathfrak{q}$ is a minimal prime ideal containing $\mathfrak{a}_h$ and contained in $\mathfrak{p}$, then $\operatorname{ht} \mathfrak{q} = \operatorname{ht} \mathfrak{p} = h$ so $\mathfrak{p} = \mathfrak{q}$. □

Corollary 13 (Krull's Principal Ideal Theorem)**.** Let R be a noetherian ring and $x \in R$ be an element which is not a unit or a zero-divisor. Then every minimal prime ideal $\mathfrak{p}$ containing (x) has height 1 and $\dim R/(x) \leq \dim R - 1$.

Proof. By the previous corollary $\operatorname{ht} \mathfrak{p} \leq 1$. If $\operatorname{ht} \mathfrak{p} = 0$, then $\mathfrak{p}$ is a minimal prime ideal of R. Since R is noetherian it has only finitely many minimal prime ideals (since the ideal (0) has primary decomposition). Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be the other minimal prime ideals of R then the nilradical is the intersection

$$\sqrt{(0)} = \mathfrak{p} \cap \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s \supset \mathfrak{p}\mathfrak{q}_1 \cdots \mathfrak{q}_s.$$

This shows that xy is nilpotent for some $y \in \mathfrak{q}_1 \cdots \mathfrak{q}_s$ which implies that x is a zero-divisor. Thus $\operatorname{ht} \mathfrak{p} = 1$. The second statement now follows from the correspondence between prime ideals of $R/(x)$ and those of R containing (x). □

Corollary 14. Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and let $x \in \mathfrak{m}$ be an element which is not a zero-divisor. Then $\dim A/(x) = \dim A - 1$.

Proof. From Lemma 5 it follows that $\dim A/(x) \leq \dim A - 1$. Let $d = \dim \dim A/(x)$ and $\mathfrak{n} = \mathfrak{m}/(x)$ be the maximal ideal of $A/(x)$. Let $x_1, \ldots, x_d \in \mathfrak{m}$ be such that their images generate an $\mathfrak{n}$-primary ideal $\mathfrak{a}$ in $A/(x)$. The $\mathfrak{n}^r \subset \mathfrak{a} \subset \mathfrak{n}$ for some positive integer r. It is easy to see that $\mathfrak{m}^r \subset (x, x_1, \ldots, x_d) \subset \mathfrak{m}$. Thus $(x, x_1, \ldots, x_d)$ is an $\mathfrak{m}$-primary ideal. Hence $\dim A/(x) + 1 \geq \dim A$ completing the proof. □

Remark. The next proposition shows that if A is a noetherian ring of finite dimension and $A[x_1, \ldots, x_n]$ is the polynomial algebra in n variables over A then $\dim A[x_1, \ldots, x_n] = \dim A + n$. Hence, in particular if $A = k$ is a field then $\dim k[x_1, \ldots, x_n] = n$. Using Corollary 9 and Proposition 10 we also see that $k[[x_1, \ldots, x_n]]$ has dimension n.

Theorem 15. If A is a noetherian ring of finite dimension and $A[x]$ is a polynomial ring over A in one variable, then $\dim A[x] = \dim A + 1$.

Proof. By Theorem 2 $\dim A[x] \geq \dim A + 1$. If $\mathfrak{p}$ is a prime ideal of A, then it follows from the proof of Theorem 2 that $\mathfrak{p}[x]$ is a prime ideal of $A[x]$ and $\operatorname{ht} \mathfrak{p}[x] \geq \operatorname{ht} \mathfrak{p}$.

Let $h = \operatorname{ht} \mathfrak{p}$ then by Proposition 12 there are $a_1, \ldots, a_h \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime ideal containing $\mathfrak{a} = (a_1, \ldots, a_h)$. Let $\mathfrak{q} \subset A[x]$ be a prime ideal such that $\mathfrak{a}[x] \subset \mathfrak{q} \subset \mathfrak{p}[x]$, then

$\mathfrak{a} \subset \mathfrak{q} \cap A \subset \mathfrak{p}$, and $\mathfrak{q} \cap A$ is prime so $\mathfrak{q} \cap A = \mathfrak{p}$. This shows the reverse inclusion $\mathfrak{q} \supset \mathfrak{p}[x]$. Thus $\mathfrak{p}[x]$ is a minimal prime ideal containing $\mathfrak{a}[x]$.

Now $\mathfrak{a}[x]$ is also generated in $A[x]$ by $a_1, \ldots, a_h$ hence $\operatorname{ht} \mathfrak{p}[x] = h$. Moreover from the proof of Theorem 2 we see that if $\mathfrak{q}_0 \subset \ldots \mathfrak{p}_n$ is a chain of prime ideals in $A[x]$ with $\mathfrak{q}_i \cap A = \mathfrak{p}$, then $n \leq 1$. Moreover $\mathfrak{q}_i \subset \mathfrak{p}[x]$. Since $\mathfrak{p}[x]$ is prime the only possibility for the length of the chain to be 1 is when $\mathfrak{q}_0 = \mathfrak{p}[x]$.

Now let $\mathfrak{q}_0 \subset \ldots \subset \mathfrak{q}_n$ be a chain of prime ideals in $A[x]$. Let $\mathfrak{p}_i = \mathfrak{q}_i \cap A$. If $\mathfrak{p}_{i-1} \neq \mathfrak{p}_i$ for all i then $n \leq d$. Otherwise, let k be the largest integer between 1 and n such that $\mathfrak{p}_{k-1} = \mathfrak{p}_k$. Then $\mathfrak{p}_k, \ldots, \mathfrak{p}_n$ are distinct prime ideals of A, so $n - k \leq \dim A - \operatorname{ht} \mathfrak{p}_k$. Moreover, we must have $\mathfrak{q}_{k-1} = \mathfrak{p}_k[x]$, so $\operatorname{ht} \mathfrak{q}_{k-1} = \operatorname{ht} \mathfrak{p}_k$, hence $k - 1 \leq \operatorname{ht} \mathfrak{p}_k$. Thus $n \leq \dim A + 1$. □

4. Dimension of finitely generated algebras over a field

Definition 16: Let A be an integral domain which is a finitely generated algebra over a field k. Let $L = Q(A)$ be the field of fractions of A. Note that L is a finitely generated field extension of k, hence L has finite transcendence degree over k which we denote by $\mathrm{Tr}\ \deg_k L$. The transcendental dimension of A is defined to be this transcendence degree

$$\mathrm{d_{tr}} A = \mathrm{Tr}\ \deg_k L.$$

Remark. In this section we shall show that if A is an integral domains which is finitely generated over a field k, then Krull dimension is equal to transcendental dimension. In fact in this case all maximal ideals of A have the same heigh equal to the Krull dimension of A.

In general all maximal ideals of a ring may not have the same height even if the ring is an integral domain. See the following thread for examples of such rings as well as some criteria for all maximal ideals of a ring to have the same height:
https://math.stackexchange.com/questions/161937/what-conditions-guarantee-that-all-maximal-ideals-have-the-same-height

Exercise (iv). Let $A \subset B$ be rings such that B is integral over A. If $\mathfrak{m} \subset B$ is a maximal ideal then $\mathfrak{n} = A \cap \mathfrak{m}$ is also a maximal ideal of A. Show that

$$\dim A_{\mathfrak{n}} = \dim B_{\mathfrak{m}}.$$

Theorem 17. If A is an integral domain which is a finitely generated algebra over a field k, then for any maximal ideal $\mathfrak{m} \subset A$ we have,

$$\operatorname{ht} \mathfrak{m} = \dim A = \mathrm{d_{tr}} A.$$

Proof. Let L be the field of fractions of A. By Noether normalisation theorem there are $x_1, \ldots, x_d \in A$ which are algebraically independent over k that is $A' = k[x_1, \ldots, x_d]$ is isomorphic to the polynomial algebra over k in d variables, and A is integral over A'. Clearly L contains $k(x_1, \ldots, x_d)$ the field of fractions of A'.

Recall that: if $C \subset D$ is are rings such that D is integral over A, $T \subset D$ is a multiplicatively closed subset and $S = T \cap C$ then $T^{-1}D$ is integral over $S^{-1}C$. Hence L is an algebraic extension of $k(x_1, \ldots, x_d)$. The upshot of this is that

$$\mathrm{d_{tr}}A = \mathrm{Tr}\ \deg_k L = d.$$

On the other hand since A is integral over $k[x_1, \ldots, x_d]$ so by 2

$$\dim A = \dim k[x_1, \ldots, x_d] = d.$$

Finally let $\mathfrak{n} = \mathfrak{m} \cap A'$ then by Exercise (iv), $\mathrm{ht}\,\mathfrak{m} = \mathrm{ht}\,\mathfrak{n}$. We thus need to find the height of a maximal ideal of $k[x_1, \ldots, x_d]$. Let $\overline{k}$ be the algebraic closure of k. Then $\overline{k}[x_1, \ldots, x_d]$ is integral over $k[x_1, \ldots, x_d]$ by Exercise (v). Hence there is a maximal ideal $\mathfrak{m}' \subset \overline{k}[x_1, \ldots, x_d]$ such that $\mathfrak{m}' \cap k[x_1, \ldots, x_d] = \mathfrak{n}$. Again $\mathrm{ht}\,\mathfrak{n} = \mathrm{ht}\,\mathfrak{m}'$. By Hilbert's-Nullstellensatz

$$\mathfrak{m}' = (x_1 - a_1, \ldots, x_d - a_d) \quad \text{for some } a_1, \ldots a_d \in \overline{k}.$$

Hence

$$0 \subset (x_1 - a_1) \subset \ldots \subset (x_1 - a_1, \ldots, x_i - a_i) \subset \ldots \subset (x_1 - a_1, \ldots, x_d - a_d) = \mathfrak{m}'$$

is a chain of prime ideals of length d. Thus $\mathrm{ht}\,\mathfrak{m}' = d$. □

Exercise (v). Let A be a finitely generated k algebra for a field k and $\overline{k}$ be the algebraic closure of k, then show that $B = \overline{k} \otimes_k A$ is integral over A.

Example 4. In Theorem 17 if we drop the assumption that A is an integral domain then it may not be true that $\mathrm{ht}\,\mathfrak{m} = \dim A$ for every maximal ideal of A. Consider

$$A = \frac{\mathbb{C}[x, y, z]}{(xz, yz)} \quad \text{and} \quad k = \mathbb{C}.$$

For any ideal $\mathfrak{a} \subset \mathbb{C}[x, y, z]$ we write the corresponding image in A as $\overline{\mathfrak{a}}$. Then $\mathrm{ht}\,\overline{(x, y, z)} = 2$ since $\overline{(z)} \subset \overline{(x, z)} \subset \overline{(x, y, z)}$ is a chain of length 2. However $\mathrm{ht}\,\overline{(x, y, z-1)} = 1$. Because the only prime ideal of $\mathbb{C}[x, y, z]$ $\mathfrak{p}$ such that $(xz, yz) \subset \mathfrak{p} \subset (x, y, z-1)$ such that $\mathfrak{p} \neq (x, y, z-1)$ is $\mathfrak{p} = (x, y)$. What is Spec$A$?

Remark. See Chapter 18 of Milne's notes for more results on the dimension of finitely generated algebras over a field.