1. Topological Groups

We shall need a few results about topological groups which we compile here.

Definition 1: A topological group is a group G with a topology such that the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are continuous. A homomorphism $\phi : G \to H$ of topological groups is a group homomorphism which is also continuous.

We shall only consider abelian topological groups and we denote the group operation always by $+$ and the group identity by 0. Note that a topological space X is hausdorff if and only if the diagonal in $X \times X$ is a closed subset.

Proposition 2. A topological group G is hausdorff if and only if 0 is a closed point in G.

Proof. The map $\alpha : G \times G \to G$ given by $\alpha(x, y) = x - y$ is continuous and $\alpha^{-1}(0)$ is precisely the diagonal. Hence if $\{0\}$ is closed in G, G is hausdorff. The other side is standard. □

By our assumption, for any $x \in G$, the map $y \mapsto y + x$ is continuous with continuous inverse $y \mapsto y - x$, hence it is a homeomorphism. Similarly $x \mapsto -x$ is also a homeomorphism of G. Thus $U \subset G$ is open if and only if $U + x$ is open. Similarly U is open if and only if $-U$ is open.

Exercise (i). Let G be a topological group. Show that any open neighbourhood V of $x \in G$ is of the form $U + x$ where U is an open neighbourhood of 0.

Hence by this exercise the open neighbourhoods of 0 determine the topology of G.

Proposition 3. Let H be the intersection of all the open neighbourhoods of 0 in a topological group G. Then

(a) the set H is a subgroup of G,
(b) the subgroup H is the closure of $\{0\}$ in G,
(c) the group G/H with the quotient topology is hausdorff.
(d) the group G is hausdorff if and only if $H = \{0\}$.

Proof. For part (a) it is enough to show that for any $x, y \in H$, $x - y \in H$. Let U be any open neighbourhood of 0, then $V = U \cap (-U)$ is also an open neighbourhood of 0. Thus $y \in V$, which means $-y \in U$. Thus $0 \in U + y$ and $U + y$ is of course open. Thus $x \in U + y \Rightarrow x - y \in U$. Hence $x - y \in H$.

Let $x \in H$, and $K \subset G$ be a closed set containing 0. Suppose $x \notin K$, then $x \in U = G - K$. Thus $x - U$ is an open neighbourhood of 0 that does not contain x which contradicts the fact that $x \in H$. Similarly if $x \in \overline{\{0\}}$, suppose there is an open neighbourhood U of 0 not containing

x. Then $x \in K = G - U$ and $x - K$ is a closed set containing 0 but not x, contradicting $x \in \overline{\{0\}}$. This completes the proof of (b).

Parts (c) and (d) follow immediately from Proposition 2. □

Example 1. The groups $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are all hausdorff topological groups, $\mathbb{Z}$ is discrete. The quotient group $\mathbb{R}/\mathbb{Q}$ is not hausdorff (what is the topology here?).

Example 2. Let $(G_i, g_{j,i})$ be an inverse system of finite groups with discrete topology indexed by some infinite set I. Then $\varprojlim G_i$ can be realised as a subgroup of $\prod_{i \in I} G_i$. It thus inherits the subspace topology of the product topology. Such a group is called a profinite group. We may take $I = \mathbb{N}$ and $G_n = \mathbb{Z}/(p^n)$ for some prime p. Then $g_{n,m}$ is the natural quotient map. The inverse limit $\varprojlim \mathbb{Z}/(p^n)$ is usually denoted by $\mathbb{Z}_p$. This is a compact hausdorff group.

2. Completion

From here on we shall assume our topological groups are all first countable. Let us recall the definition of Cauchy sequences.

Definition 4: A sequence $(x_n) = x_1, x_2, \ldots$ of elements of G converges to $x \in G$ if for any open neighbourhood U of x there is an integer $N(U)$ such that $x_n - x \in U$ for all $n \geq N(U)$. Such a sequence is called convergent and we write $x = \lim x_n$.

A sequence (x_n) is called Cauchy if for any neighbourhood U of 0, there is an integer $N(U)$ such that $x_n - x_m \in U$ for all $m, n \geq N(U)$.

Exercise (ii). Show that the sequence (x_n) converges to x if an only if $(x_n - x)$ converges to 0. Show that any convergent sequence is Cauchy.

Theorem 5 (Completion). The set of all Cauchy sequences of G forms a group $\widetilde{G}$ under the addition $(x_n) + (y_n) = (x_n + y_n)$. The subset $K \subset \widetilde{G}$ consisting of sequences that converge to 0 is a subgroup and we define the completion of G to be $\widehat{G} = \widetilde{G}/K$. We have a group homomorphism $\imath : G \to \widehat{G}$ which maps any $x \in G$ to the class of the constant sequence (x).

Exercise (iii). Prove Theorem 5.

Definition 6: The group G is called complete if $\imath : G \to \widehat{G}$ is an isomorphism.

The homomorphism $\imath$ is not injective in general. In fact $\ker(\imath) = \overline{\{0\}}$, thus $\imath$ is injective if and only if G is hausdorff.

Proposition 7. An abelian topological group G is complete if and only if G is a hausdorff and all Cauchy sequences in G converge.

Proof. If G is complete $\imath : G \to \widehat{G}$ is an isomorphism hence in particular injective so G is hausdorff. Moreover $\imath$ is surjective so for any Cauchy sequence (x_n) there is $x \in G$ such that $\lim x_n - x = 0$. Hence $\lim x_n = x$. This proves the forward implication.

Now for the converse. Since G is hausdorff a convergent sequence converges to a unique element. Thus we define a map $\phi : \widetilde{G} \to G$ given by $\phi((x_n)) = \lim x_n$. This map is clearly a group homomorphism and $K = \ker(\phi)$ so it induces an isomorphism $\widehat{G} \to G$ with inverse $\imath$. □

Remark. Completion is a functor. That is if $f : G \to H$ is a homomoprhism of topological groups then since it is continuous it takes Cauchy sequences to Cauchy sequences. Thus it induces a group homomorphism $\widehat{f} : \widehat{G} \to \widehat{H}$. Moreover is $g : H \to K$ is another homomorphism of topological groups then $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.

Example 3. We may take $G = \mathbb{Q}$ then $\widehat{G} = \mathbb{R}$. Of course $\mathbb{R}$ and $\mathbb{Z}$ are complete.

Exercise (iv). Phrase and prove the universal property of completion.

Example 4. We have already seen that with the usual topology on $\mathbb{Q}$ the completion $\widehat{\mathbb{Q}} \cong \mathbb{R}$. However for any prime $p \in \mathbb{Z}$ there is the p-adic norm on $\mathbb{Q}$ defined as follow: For any rational number $r \neq 0$ we can write $r = p^n \dfrac{a}{b}$ where $n \in \mathbb{Z}$ and p does not divide a, b then let

$$||r||_p = p^{-n},$$

and let $||0||_p = 0$. This is a norm on $\mathbb{Q}$ and thus defines a metric. In this metric $\mathbb{Q}$ is not complete. For instance it is not trivial but can be checked that the sequence

$$a_n = 1 + p + p^2 + \ldots + p^n$$

is Cauchy but it does not converge to any rational number. The completion $\mathbb{Q}_p = \widehat{Q}$ with respect to this metric is called the field of p-adic numbers and is of much interest in number theory.

3. Completion as Inverse limit

For this part we need the topology on the group G to be a bit special.

Definition 8: An abelian topological group G is said to have linear topology if a countable sequence of subgroups $G = G_0 \supset G_1 \supset G_2 \supset G_3 \ldots$ form a system of open neighbourhoods of 0.

Thus for a group G with linear topology a set $U \subset G$ is open for any $x \in U$, $x + G_n \subset U$ for some n. The subgroups G_n are both open and closed. In fact any coset of G_x is also open and $G - G_n$ is a union of cosets of G_n hence also open.

For abelian groups with a linear topology the completion can be obtained purely algebraically as an inverse limit. Note that the groups G/G_n form a surjective inverse system of $\mathbb{Z}$ modules indexed by $\mathbb{N}$. The map $g_{n,m} : G/G_n \to G/G_m$ is just the quotient map since $G_m \supset G_n$ for $m \leq n$.

Proposition 9. We have $\varprojlim G/G_n \cong \widehat{G}$.

Proof. Since $G_n \subset G$ is both open and closed G/G_n has discrete topology hence by Proposition 7 it is complete. Thus the quotient map $G \to G/G_n$ induces a group homomorphism $\gamma_n : \widehat{G} \to G/G_n$. Basically any Cauchy sequence in G/G_n must be eventually constant, so a Cauchy sequence in $\widehat{G}$ becomes eventually constant after reducing mod G_n and γ_n is the limit of that sequence.

Moreover it is clear that $g_{n,m} \circ \gamma_n = \gamma_m$. Hence we get a induced homomorphism

$$\gamma : \widehat{G} \to \varprojlim G/G_n.$$

This map is injective because $\gamma((x_k)) = \gamma((y_k)) \Rightarrow \gamma_n((x_k)) = \gamma_n((y_k))$ for all n. Then $\lim[x_k] = \lim[y_k] \in G/G_n$ for each n which means $\lim(x_k - y_k) = 0 \in G$. Thus $[(x_k)] = [(y_k)] \in \widehat{G}$.

Now recall that $\varprojlim G/G_n$ is the subgroup of the product $\prod_{n\in\mathbb{N}} G/G_n$. In fact a tuple $(\xi_n) \in \varprojlim G/G_n$ if for each $n > m$ we have $g_{n,m}(\xi_n) = \xi_m$. Let $(\xi_n) \in \varprojlim G/G_n$ and pick $x_n \in G$ such that $[x_n] = \xi_n \in G/G_n$. Then (x_n) is a Cauchy sequence in G and $\gamma((x_n)) = (\xi_n)$. Hence γ is also surjective. □

Proposition 10. Let

$$0 \to G' \to G \xrightarrow{q} G'' \to 0$$

be an exact sequence of abelian topological groups with each group having a linear topology. Moreover assume that the topologies on G' and G'' are induced by G. That is if $G_n \subset G$ are the open neighbourhoods of 0 in G then those in G' are $G' \cap G_n$ and those in G'' are $q(G_n)$. Then the induced sequence

$$0 \to \widehat{G'} \to \widehat{G} \xrightarrow{\hat{q}} \widehat{G''} \to 0$$

is also exact.

Proof. We have an exact sequence of inverse systems

$$0 \to G'/(G' \cap G_n) \to G/G_n to G''/q(G_n) \to 0.$$

and since $G'/(G' \cap G_n)$ is a surjective system the proposition follows from the exactness properties of the inverse limit. □

Remark. Note that if G has linear topology then the homomorphism $\imath : G \to \widehat{G}$ induces inclusions $\widehat{G_n} \to \widehat{G}$. Thus $\widehat{G}$ can be given a linear topology where the open neighbourhoods of 0 are the subgroups $\widehat{G_n}$. The next corollary says that in case of an abelian group with linear topology the completion is complete.

Corollary 11. Let G be an abelian group with linear topology then $\widehat{\widehat{G}} \cong \widehat{G}$.

Proof. The exact sequences $0 \to G_n \to G \to G/G_n \to 0$ induce an exact sequence

$$0 \to \widehat{G_n} \to \widehat{G} \to \widehat{G/G_n} \to 0.$$

However G/G_n has discrete topology so $G/G_n \cong \widehat{G/G_n}$. Hence $\widehat{G}/\widehat{G_n} \cong G/G_n$. Now taking inverse limits we get the desired isomorphism. □

4. Completion of Rings and Modules

Definition 12: A ring R with a topology is called a topological ring if the ring operations are continuous in that topology. So in particular it is also a topological abelian group.

Definition 13: Let R be a ring then it is of course an abelian group with respect to addition. Let $\mathfrak{a} \subset R$. be an ideal. The $\mathfrak{a}$-adic topology on R is the linear topology where the open neighbourhoods of 0 are $\mathfrak{a}^n$.

In this topology the ring multiplication $\mu : R \times R \to R$ is also continuous. Let $U \subset R$ be open and consider $(r, s) \in \mu^{-1}(U)$. Then $rs \in U$ and there is an n such that $rs + \mathfrak{a}^n \subset U$. Then clearly $(r + \mathfrak{a}^n) \times (s + \mathfrak{a}^n) \subset \mu^{-1}(U)$ hence $\mu^{-1}(U)$ is open. Hence in this topology R is a topological ring. The topology is hausdorff if and only if $\bigcap_n \mathfrak{a}^n = \{0\}$.

We can then form the completion $\widehat{R}$ of R by the construction using Cauchy sequences or as the direct limit $\widehat{R} = \varprojlim R/\mathfrak{a}^n$. Both the constructions are isomorphic and $\widehat{R}$ has a natural ring structure. The product of Cauchy sequences is again a Cauchy sequence since multiplication is continuous. We have a continuous ring homomorphism $\imath : R \to \widehat{R}$. This ring $\widehat{R}$ is called the $\mathfrak{a}$-adic completion of R.

Similarly when M is an R module and $\mathfrak{a} \subset R$ is an ideal there is a an a-adic topology on M, where the open neighbourhoods or 0 are $\mathfrak{a}^n M$. This makes M into a topological R module where R and M both have $\mathfrak{a}$-adic topology. The completion $\widehat{M}$ with respect this topology is called the $\mathfrak{a}$-adic completion of M.

Exercise (v). Show that $\widehat{M}$ is a topological $\widehat{R}$ module. Moreover if $f : M \to N$ is an R module homomorphism then f is continuous in the $\mathfrak{a}$-adic topologies on M and N and $\widehat{f} : \widehat{M} \to \widehat{N}$ is a continuous $\widehat{R}$ module homomorphism.

Example 5. If $R = k[x]$ the polynomial ring over a field k and $\mathfrak{a} = (x)$ then as we have already seen that $\mathfrak{a}$-adic completion of R is $\widehat{R} = k[[x]]$.

Example 6. If $R = \mathbb{Z}$ and $\mathfrak{a} = (p)$ for some prime $p \in \mathbb{Z}$, then $\mathbb{Z}_p = \widehat{R}$ is the ring of p-adic integers. In fact if we have the p-adic norm on $\mathbb{Q}$ then the induced topology on $\mathbb{Z}$ is the p-adic topology and $\mathbb{Z}_p \subset \mathbb{Q}_p$. In fact $\mathbb{Q}_p$ is the field of fractions of $\mathbb{Z}_p$.

5. Filtrations

Definition 14: Let R be a ring and M an R module. A filtration of M is a sequence of submodules

$$M = M_0 \supset M_1 \supset M_2 \supset \dots$$

Such a filtration will be denoted by M_n and defines a linear topology on M.

If $\mathfrak{a} \subset R$ is an ideal, the filtration (M_n) is called an $\mathfrak{a}$-filtration if $\mathfrak{a}M_n \subset M_{n+1}$ and called stable $\mathfrak{a}$-filtration if further there is an integer $n_0 > 0$ such that $\mathfrak{a}M_n = M_{n+1}$ for all $n \geq n_0$.

Proposition 15. The linear topology defined by any stable $\mathfrak{a}$-filtration is the $\mathfrak{a}$-adic topology on M.

Proof. If is enough to show that the open neighbourhoods of 0 are the same in both the topologies. Since $\mathfrak{a}M_n \subset M_{n+1}$ we have $\mathfrak{a}^n M \subset M_n$ for all n. Which shows M_n are open in the $\mathfrak{a}$-adic topology. Conversely $\mathfrak{a}M_n = M_{n+1}$ for $n \geq n_0$, thus $M_{n_0+n} = \mathfrak{a}^n M_{n_0} \subset \mathfrak{a}^n M$. This completes the proof. □