In this lecture we shall finally prove that if A is noetherian the $\mathfrak{a}$-adic completion $\widehat{A}$ is also noetherian for any ideal $\mathfrak{a}$. First we need to make some more definitions.

Definition 1: Let A be a ring and $\mathfrak{a} \subset A$ an ideal then the associated graded ring $G_{\mathfrak{a}}(A)$ is defined as

$$G_{\mathfrak{a}}(A) = \bigoplus_{n=0}^{\infty} \frac{\mathfrak{a}^n}{\mathfrak{a}^{n+1}}$$

with the notation that $\mathfrak{a}^0 = A$. On the face, it is just an abelian group, but it has a natural multiplication which makes it a graded ring. Let $x_i \in \mathfrak{a}^i$ for $i = m, n$ and let $\overline{x}_i$ denote their images in $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ then we define

$$\overline{x}_n \overline{x}_m = \overline{x_n x_m} \in \mathfrak{a}^{m+n}/\mathfrak{a}^{m+n+1}.$$

It is easy to check that this is well defined.

Similarly if M is an A module and $(M_n)_{n\geq 0}$ an $\mathfrak{a}$-filtration, we can define the associated graded module

$$G(M) = \bigoplus_{n=0}^{\infty} \frac{M_n}{M_{n+1}}.$$

This has a natural graded $G_a(A)$ module structure, if $r \in \mathfrak{a}^n$ and $x \in M_k$ let $\overline{r} \in \mathfrak{a}^n/\mathfrak{a}^{n+1}$ and $\overline{x} \in M_k/M_{k+1}$ then we define

$$\overline{r}\ \overline{x} = \overline{rx} \in M_{n+k}/M_{n+k+1}.$$

It is easy to check that this is well defined.

We need to establish a few easy results about associated graded rings.

Proposition 2. Let A be a noetherian ring $\mathfrak{a}$ an ideal. Then

(a) $G_{\mathfrak{a}}(A)$ is noetherian;
(b) if $\widehat{A}$ is the $\mathfrak{a}$-adic completion then, $G_{\mathfrak{a}}(A)$ and $G_{\widehat{\mathfrak{a}}}(\widehat{A})$ are isomorphic as graded rings;
(c) if M is a finitely generated A module and $(M_n)_{n\geq 0}$ a stable $\mathfrak{a}$-filtration of M, then $G(M)$ is a finitely generated graded $G_{\mathfrak{a}}(A)$ module.

Proof. For part (a) note that A is noetherian, $\mathfrak{a}$ is fintitely generated, hence let

$$\mathfrak{a} = (x_1, \ldots, x_n)$$

and let $\overline{x}_i \in \mathfrak{a}/\mathfrak{a}^2$, then $G_{\mathfrak{a}}(A)$ is generated by $\overline{x}_1, \ldots, \overline{x}_n$ as an algebra over $A/\mathfrak{a}$. Now since $G_{\mathfrak{a}}(A)$ is a finitely generated algebra over the noetherian ring $A/\mathfrak{a}$, hence $G_{\mathfrak{a}}(A)$ is also noetherian by the Hilbert basis theorem.

Part (b) follows from the isomorphisms $\mathfrak{a}^n/\mathfrak{a}^{n+1} = \widehat{\mathfrak{a}}^n/\widehat{\mathfrak{a}}^{n+1}$ (see Exercise (iii) of the lecture on graded rings).

Since $(M_n)_{n\geq 0}$ is a stable $\mathfrak{a}$-filtration there is an j such that $\mathfrak{a}^k M_i = M_{j+k}$. Hence $G(M)$ is clearly generated by the subgroup

$$\bigoplus_{j}^{n_0} \frac{M_n}{M_{n+1}}$$

as a $G_{\mathfrak{a}}(A)$ module. Each M_n/M_{n+1} is finitely generated over A and annihilated by $\mathfrak{a}$, hence it is finitely generated over $A/\mathfrak{a}$. Choosing generators $\overline{x}_{1,n}, \ldots, \overline{x}_{k_n,n}$ it is easy to see that $G(M)$ is generated by $\overline{x}_{1,1}, \ldots, \overline{x}_{k_1,1}, \ldots, \overline{x}_{1,j}, \ldots, \overline{x}_{k_j,j}$ as a $G_{\mathfrak{a}}(A)$ module. □

Remark. Let A be a ring and $\mathfrak{a} \subset A$ an ideal. Let M and N be A modules with $\mathfrak{a}$-filtrations $(M_k)_{k\geq 0}$ and $(N_k)_{k\geq 0}$.

Suppose $\phi : M \to N$ is an A module homomorphism such that $\phi(M_k) \subset N_k$, then we have the homomorphisms $\phi_n : M_k/M_{k+1} \to N_k/N_{k+1}$ thus producing an induced $G_{\mathfrak{a}}(A)$ module homomorphism

$$G(\phi) : G(M) \to G(N).$$

Similarly by passing to the inverse limit of the homomorphisms $M/M_k \to N/N_k$ we obtain an induced $\widehat{A}$ module homomorphism of the completions

$$\widehat{\phi} : \widehat{M} \to \widehat{N}.$$

Proposition 3. With the notations as in the preceding remark, we have the following results relating $G(\phi)$ and $\widehat{\phi}$:

(a) $G(\phi)$ is injective $\Rightarrow \widehat{\phi}$ is also injective,
(b) $G(\phi)$ is surjective $\Rightarrow \widehat{\phi}$ is also surjective.

Proof. Consider the following commutative diagram where the rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_k/M_{k+1} & \longrightarrow & M/M_{k+1} & \longrightarrow & M/M_k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle G(\phi)} & & \downarrow{\scriptstyle \alpha_{k+1}} & & \downarrow{\scriptstyle \alpha_k} & & \\ 0 & \longrightarrow & N_k/N_{k+1} & \longrightarrow & N/N_{k+1} & \longrightarrow & N/N_k & \longrightarrow & 0 \end{array}$$

hence, using snake's lemma we get an exact sequence

$$0 \to \ker(G(\phi)) \to \ker(\alpha_{k+1}) \to \ker(\alpha_k) \to \operatorname{coker}(G(\phi)) \to \operatorname{coker}(\alpha_{k+1}) \to \operatorname{coker}(\alpha_k) \to 0.$$

By induction it is clear that if $\ker(G(\phi)) = 0$ then $\ker(\alpha_k) = 0$ for all k. Similarly if $\operatorname{coker}(G(\phi)) = 0$ then $\operatorname{coker}(\alpha_k) = 0$ for all k. Taking inverse limits we get the result. □

As usual let A be an ring $\mathfrak{a} \subset A$ an ideal, M an A module and $(M_n)_{n\geq 0}$ an $\mathfrak{a}$-filtration.

Proposition 4. Suppose A is complete in the $\mathfrak{a}$-adic topology and $\cap_n M_n = 0$.

(1) If $G(M)$ is a finitely generated $G_{\mathfrak{a}}(A)$ module, then M is a finitely generated A module.
(2) If $G(M)$ is a noetherian $G_{\mathfrak{a}}(A)$ module, then M is a noetherian A module.

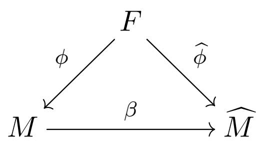
Proof. For part (a) choose homogeneous generators $\xi_1, \ldots, \xi_k$ of $G(M)$ with the degree of ξ_i being $n(i)$. Hence $\xi_i \in M_{n(i)}/M_{n(i)+1}$, and we can choose $x_i \in M_{n(i)}$ whose image under the quotient map is ξ_i. We assert that $x_1, \dots, x_k$ generate M. Let $F^{(i)} = A$ with the stable $\mathfrak{a}$-filtration

$$F_k^{(i)} = \begin{cases} A, & k = 0; \\ \mathfrak{a}^{n(i)+k}, & k > 0. \end{cases}$$

Let $\phi_i : F^{(i)} \to M$ be given by $\phi_i(1) = x_i$, then clearly $\phi_i(F_k^{(i)}) \subset M_k$. We define $F = F^{(1)} \oplus \cdots \oplus F^{(n)}$ and $\phi : F \to M$ by $\phi = \phi_1 \oplus \cdots \oplus \phi_n$. Then $F_k = \oplus_{i=1}^n F_k^{(i)}$ is a stable $\mathfrak{a}$-filtration of F and $\phi(F_k) \subset M_k$. Hence by the previous proposition we get a map of graded $G_{\mathfrak{a}}(A)$ modules

$$G(\phi) : G(F) \to G(M).$$

Moreover, since $A = \widehat{A}$ we also have $\widehat{F} = F$ and there is a map of completions $\widehat{\phi} : F \to \widehat{M}$. Since $G(\phi)$ is surjective by construction, the previous proposition says $\widehat{\phi}$ is also surjective. The following diagram clearly commutes



The kernel of β is $\cap_n M_n = 0$ hence β is injective $\widehat{\phi}$ is surjective, therefore ϕ is also surjective. This proves the assertion.

For part (b) we have to show that any submodule $M' \subset M$ is finitely generated. We have a $\mathfrak{a}$-filtration on M' given by $M'_k = M_k \cap M'$. Since M'_n / M'_{n+1} is injective the homomorphism of the associated graded modules $G(M') \to G(M)$ is also injective. Thus since $G(M)$ is noetherian $G(M')$ is finitely generated. Moreover, $\cap_n M'_n \subset \cap_n M_n = 0$, hence by part (a), M' is also finitely generated. □

Finally the coveted result of this lecture turns out to be a simple corollary of this proposition.

Theorem 5. Let A be an ring $\mathfrak{a} \subset A$ an ideal and $\widehat{A}$ the $\mathfrak{a}$-adic completion. If A is noetherian then $\widehat{A}$ is also noetherian.

Proof. Consider $\widehat{A}$ as a module over itself. We want to show that it is noetherian. The ideal $\widehat{\mathfrak{a}}$ defines a topology on $\widehat{A}$ and since $\widehat{A}$ is complete with respect to this topology it must be hausdorff (Proposition 7 and Corollary 11 of the lecture on completions). Hence

$$\bigcap_{n=0}^{\infty} \widehat{\mathfrak{a}}^n = 0.$$

Hence by Proposition 4 it is enough to prove that $G_{\widehat{\mathfrak{a}}}(\widehat{A})$ is noetherian. However, by Proposition 2 $G_{\widehat{\mathfrak{a}}}(\widehat{A})$ is isomorphic to $G_{\mathfrak{a}}(A)$ which is noetherian since A is. □

Exercise (i). Show that if A is a noetherian ring then the powerseries ring over A is fintely many variables $A[[x_1, \dots, x_n]]$ is also noetherian.

A local ring A is called complete if it is $\mathfrak{m}$-adically complete for its maximal ideal $\mathfrak{m}$. The following result is called Hensel's lemma.

Exercise (ii). Let A be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. For any polynomial $f(x) \in A[x]$ denote by $\overline{f}(x) \in k[x]$ the reduction mod $\mathfrak{m}$. If $f \in A[x]$ is a monic polynomial and if there are coprime polynomials $p, q \in k[x]$ such that

$$\overline{f} = pq$$

then show that there exist monic polynomials $g, h \in A[x]$ such that $p = \overline{g}$, $q = \overline{h}$ and $f = gh$.

Exercise (iii). With the notation of Exercise (ii) show that if $f \in A[x]$ is monic and $\overline{f} \in k[x]$ has a simple root $\alpha \in k$, then f has a simple root $a \in A$ such that $\alpha = a \pmod{\mathfrak{m}}$.

Exercise (iv). Show that the ring of p-adic integers have all the $p - 1$-th roots of unity.