On the number of representations of certain quadratic forms in 8 variables

B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh

Abstract. In this paper, we find the number of representations of integers by certain quadratic forms in 8 variables by using the theory of modular forms. By expressing these formulas in terms of certain convolution sums of the divisor function and using our formulas, we deduce formulas for the convolution sums $W_{j,7}(n)$ for $j = 1, 2, 3, 4$.

1. Introduction

For positive integers a, b, n, define the convolution sum $W_{a,b}(n)$ by

$$W_{a,b}(n) := \sum_{\substack{l,m \in \mathbb{N} \\ al+bm=n}} \sigma(l)\sigma(m), \tag{1}$$

where $\sigma(n)$ is the divisor function. We note that $W_{a,1}(n) = W_{1,a}(n)$, which is denoted by $W_a(n)$. These type of sums were evaluated as early as the 19th century. For example, the sum $W_1(n)$ was evaluated by M. Besge, J. W. L. Glaisher and S. Ramanujan [**3, 6, 12**]. Some of the convlotution sums of the above type have been obtained by several authors (see for example [**7, 13, 11**] and also the works of K. S. Williams and his co-authors ([**16**] and the references therein)).

Let Q be the quadratic form in four variables defined by

$$Q : x_1^2 + x_1x_2 + 2x_2^2 + x_3^2 + x_3x_4 + 2x_4^2,$$

and

$$R_Q(n) = \text{card}\left\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : Q(x_1, x_2, x_3, x_4) = n\right\}$$

be the number of representations of a positive integer n by the quadratic form Q. A formula for $R_Q(n)$ was obtained in [**15**] by using an elementary method, which is given by

$$R_Q(n) = 4\sigma(n) - 28\sigma\left(\frac{n}{7}\right), \tag{2}$$

where we define $\sigma(x) = 0$, when x is not a positive integer. In this article we find a formula for $R_{Q\oplus jQ}(n)$ where

$$Q \oplus jQ := x_1^2 + x_1x_2 + 2x_2^2 + x_3^2 + x_3x_4 + 2x_4^2 + j(x_5^2 + x_5x_6 + 2x_6^2 + x_7^2 + x_7x_8 + 2x_8^2),$$

2010 *Mathematics Subject Classification.* Primary 11E25, 11F11, 11F99; Secondary 11F20, 11F30.

Key words and phrases. convolution sums of the divisor functions, representation numbers of quadratic forms, modular forms of one variable.

for $j = 1, 2, 3, 4$ using theory of modular forms. Next, by using the above formula for $R_Q(n)$, we express $R_{Q\oplus jQ}(n)$ in terms of the convolution sums $W_j(n)$, $W_{7j}(n)$ and $W_{j,7}(n)$. Since the convolutions $W_j(n)$, $1 \le j \le 4$ and $W_{7j}(n)$, $j = 1, 2, 4$ are already known, we use them along with $W_{21}(n)$ (which we prove using the theory of quasimodular forms), to give formulas for $W_{j,7}(n)$ for $1 \le j \le 4$. We note that $R_{Q\oplus Q}(n)$ has also been evaluated by K. S. Williams in [**15**], using the convolution sums method.

2. Preliminaries and Statement of Results

Let $M_k(N)$ be the space of modular forms of weight k for the congruence subgroup $\Gamma_0(N)$ and $S_k(N)$ be the subspace of cusp forms of weight k for the congruence subgroup $\Gamma_0(N)$.

Let $\mathscr{Q} : \mathbb{Z}^{2m} \to \mathbb{Z}$ be a positive definite quadratic form given by

$$\mathscr{Q}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^t \mathcal{A}\mathbf{x},$$

where $\mathbf{x} \in \mathbb{Z}^{2m}$. Then the associated theta series, denoted by $\Theta_{\mathscr{Q}}(z)$, is defined as

$$\Theta_{\mathscr{Q}}(z) = \sum_{\mathbf{x}\in\mathbb{Z}^{2m}} e^{2\pi i \mathscr{Q}(\mathbf{x})z}.$$

Let M be the smallest positive integer such that $M\mathcal{A}^{-1}$ is an even integral matrix (i.e., a matrix with diagonal entries as even integers and off-diagonal entries as integers) and $\chi_{m,M}$ be the quadratic character defined by $\left(\frac{(-1)^m M}{\cdot}\right)$. Then, it is known that $\Theta_{\mathscr{Q}}(z)$ is a modular form of weight m on $\Gamma_0(M)$ with character $\chi_{m,M}$. For more details on theta series associated to integral quadratic forms we refer to Chapter IX of [**14**] and [**4**, p.32].

For $k \ge 4$, let E_k denote the normalized Eisenstein series of weight k in $M_k(1)$ given by

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n\ge1}\sigma_{k-1}(n)q^n \quad (q = e^{2\pi i z},\ \mathrm{Im}(z) > 0),$$

where B_k is the k-th Bernoulli number defined by $\frac{x}{e^x - 1} = \sum_{m=0}^{\infty}\frac{B_m}{m!}x^m$. The Eisenstein series $E_4(z) = 1+240\sum_{n\ge1}\sigma_3(n)q^n$ is used in our proof. For $k = 2$, the Eisenstein series $E_2(z)$ is a quasimodular form of weight 2, depth 1 on $SL_2(\mathbb{Z})$, which has the following Fourier expansion:

$$E_2(z) = 1 - 24\sum_{n\ge1}\sigma(n)q^n$$

and it is fundamental in the theory of quasimodular forms. In order to evaluate the convolution sum $W_{21}(n)$, we use the following structure theorem on $\tilde{M}_k^{\le k/2}(N)$, the space of quasimodular forms of weight k, depth $\le k/2$ on $\Gamma_0(N)$ (see [**8, 10**]).

Theorem A (Kaneko-Zagier): *For an even integer k with $k \ge 2$, we have*

$$\tilde{M}_k^{\le k/2}(N) = \bigoplus_{j=0}^{k/2-1} D^j M_{k-2j}(N) \oplus \mathbb{C}D^{k/2-1}E_2, \tag{3}$$

where the differential operator D is defined by $D := \frac{1}{2\pi i}\frac{d}{dz}$.

Using this theorem, one can express each quasimodular form of weight k and depth $\leq k/2$ as a linear combination of j-th derivatives of modular forms of weight $k-2j$ on $\Gamma_0(N)$, $0 \leq j \leq k/2-1$ and the $(k/2-1)$-th derivate of the quasimodular form E_2. For details on basic theory of modular forms and quasimodular forms, we refer the reader to [**4, 8, 10**].

Let d be the dimension of the space $S_k^{new}(N)$, which is spanned by the normalised Hecke eigenforms in the space $S_k(N)$. We denote these basis elements as $\{\Delta_{k,N;j}(z) : 1 \leq j \leq d\}$, where we denote their Fourier expansions by

$$\Delta_{k,N;j}(z) = \sum_{n\geq 1} \tau_{k,N;j}(n) q^n.$$

If $d=1$, then we write the basis element as $\Delta_{k,N}$ and its Fourier coefficients are denoted as $\tau_{k,N}(n)$. For more details on the theory of newforms, we refer to [**2**]. We also need the following 9 cusp forms which are basis elements of the space $S_4(28)$. Here we use the same notation for their Fourier coefficients as in [**1**]. For $1 \leq j \leq 9$, the nine cusp forms $f_j(z) = \sum_{n\geq 1} c_j(n) q^n$ are defined by the following eta products/quotients.

$$\begin{array}{rclrcl}
f_1(z) &=& \frac{\eta^5(z)\eta^5(7z)}{\eta(2z)\eta(14z)}; & f_2(z) &=& \eta^2(z)\eta^2(2z)\eta^2(7z)\eta^2(14z); \\
f_3(z) &=& \frac{\eta^6(z)\eta^6(14z)}{\eta^2(2z)\eta^2(7z)}; & f_4(z) &=& \frac{\eta^6(2z)\eta^6(7z)}{\eta^2(z)\eta^2(14z)}; \\
f_5(z) &=& \eta^2(4z)\eta^4(14z)\eta^2(28z); & f_6(z) &=& \frac{\eta^6(2z)\eta^6(28z)}{\eta^2(4z)\eta^2(14z)}; \\
f_7(z) &=& \frac{\eta^4(2z)\eta^6(28z)}{\eta^2(4z)}; & f_8(z) &=& \frac{\eta(z)\eta(2z)\eta(7z)\eta^8(28z)}{\eta^3(14z)}; \\
f_9(z) &=& \frac{\eta(2z)\eta(4z)\eta^9(28z)}{\eta^3(14z)}, & & &
\end{array}$$

where $\eta(z)$ is the Dedekind eta function given by $\eta(z) = q^{1/24} \prod_{n\geq 1} (1-q^n)$.

We need the following convolution sums $W_N(n)$, $N = 1,2,3,4,14,28$. Among these, $W_{28}(n)$ is recently evaluated by A. Alaca, S. Alaca and E. Ntienjem [**1**] and the rest were evaluated by E. Royer [**13**].

Theorem 2.1. *(See* [**1, 13**]*) For a natural number n,*

$$W_1(n) = \frac{5}{12}\sigma_3(n) + \left(\frac{1}{12} - \frac{n}{2}\right)\sigma(n),$$

$$W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{n}{8}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{2}\right),$$

$$W_3(n) = \frac{1}{24}\sigma_3(n) + \frac{3}{8}\sigma_3\left(\frac{n}{3}\right) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{3}\right),$$

$$W_4(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) + \left(\frac{1}{24} - \frac{n}{16}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{4}\right),$$

$$\begin{aligned}
W_{14}(n) = {} & \frac{1}{600}\sigma_3(n) + \frac{1}{150}\sigma_3\left(\frac{n}{2}\right) + \frac{49}{600}\sigma_3\left(\frac{n}{7}\right) + \frac{49}{150}\sigma_3\left(\frac{n}{14}\right) \\
& + \left(\frac{1}{24} - \frac{n}{56}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{14}\right) - \frac{3}{350}\tau_{4,7}(n) - \frac{6}{175}\tau_{4,7}\left(\frac{n}{2}\right) \\
& - \frac{1}{84}\tau_{4,14;1}(n) - \frac{1}{200}\tau_{4,14;2}(n),
\end{aligned}$$

$$W_{28}(n) = \frac{1}{2400}\sigma_3(n) + \frac{1}{800}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{150}\sigma_3\left(\frac{n}{4}\right) + \frac{49}{2400}\sigma_3\left(\frac{n}{7}\right) + \frac{49}{800}\sigma_3\left(\frac{n}{14}\right)$$
$$+\frac{49}{150}\sigma_3\left(\frac{n}{28}\right) + \left(\frac{1}{24} - \frac{n}{112}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{28}\right)$$
$$+\frac{1121}{67200}c_1(n) + \frac{2389}{22400}c_2(n) - \frac{1}{128}c_3(n) - \frac{3349}{67200}c_4(n)$$
$$-\frac{101}{200}c_5(n) - \frac{17}{40}c_6(n) + \frac{13}{200}c_7(n) - \frac{433}{150}c_8(n) - \frac{254}{75}c_9(n).$$

Following the method of Royer [**13**], we evaluate the following convolution sum.

THEOREM 2.2. *For a natural number n,*

$$\begin{aligned} W_{21}(n) &= \frac{1}{24}\sigma(n) + \frac{1}{24}\sigma\left(\frac{n}{21}\right) - \frac{1}{4}n\sigma\left(\frac{n}{21}\right) - \frac{1}{84}n\sigma(n) + \frac{1}{1200}\sigma_3(n) \\ &\quad +\frac{3}{400}\sigma_3\left(\frac{n}{3}\right) + \frac{49}{1200}\sigma_3\left(\frac{n}{7}\right) + \frac{147}{400}\sigma_3\left(\frac{n}{21}\right) - \frac{1}{175}\tau_{4,7}(n) \\ &\quad -\frac{9}{175}\tau_{4,7}\left(\frac{n}{3}\right) - \frac{1}{100}\tau_{4,21;2}(n) - \frac{5}{672}(\tau_{4,21;3}(n) + \tau_{4,21;4}(n)) \\ &\quad -\frac{11}{12768}\sqrt{57}(\tau_{4,21;3}(n) - \tau_{4,21;4}(n)). \end{aligned}$$

Note: The dimension of $S_4^{new}(21)$ is 4 and out of which two newforms have rational Fourier coefficients and the other two newforms do not have rational Fourier coefficients. In the latter case, the eigenvalues of the Hecke operators $T(p)$ for $p \neq 3, 7$ satisfy the quadratic polynomial $x^2 + 3x - 12$, and therefore the eigenvalues (and hence the Fourier coefficients) of these two newforms belong to the number field $\mathbb{Q}(\sqrt{57})$ (57 is the discriminant of the quadratic polynomial). More precisely, the Fourier coefficients $\tau_{4,21;3}(n)$ and $\tau_{4,21;4}(n)$ belong to the number field $\mathbb{Q}(\sqrt{57})$. However, we note that both $\tau_{4,21;3}(n) + \tau_{4,21;4}(n)$ and $\sqrt{57}(\tau_{4,21;3}(n) - \tau_{4,21;4}(n))$ are rational numbers.

The following is the main theorem of this section.

THEOREM 2.3. *For a natural number n,*

$$\begin{aligned} R_{Q\oplus Q}(n) &= \frac{24}{5}\sigma_3(n) + \frac{1176}{5}\sigma_3\left(\frac{n}{7}\right) + \frac{16}{5}\tau_{4,7}(n), \\ R_{Q\oplus 2Q}(n) &= \frac{24}{25}\sigma_3(n) + \frac{96}{25}\sigma_3\left(\frac{n}{2}\right) + \frac{1176}{25}\sigma_3\left(\frac{n}{7}\right) + \frac{4704}{25}\sigma_3\left(\frac{n}{14}\right) \\ &\quad +\frac{48}{25}\tau_{4,7}(n) + \frac{192}{25}\tau_{4,7}\left(\frac{n}{2}\right) + \frac{28}{25}\tau_{4,14;1}(n), \\ R_{Q\oplus 3Q}(n) &= \frac{12}{25}\sigma_3(n) + \frac{108}{25}\sigma_3\left(\frac{n}{3}\right) + \frac{588}{25}\sigma_3\left(\frac{n}{7}\right) + \frac{5292}{25}\sigma_3\left(\frac{n}{21}\right) \\ &\quad +\frac{32}{25}\tau_{4,7}(n) + \frac{288}{25}\tau_{4,7}\left(\frac{n}{3}\right) + \frac{56}{25}\tau_{4,21;2}(n), \end{aligned}$$

$$\begin{aligned} R_{Q\oplus 4Q}(n) &= \frac{6}{25}\sigma_3(n)+\frac{18}{25}\sigma_3\left(\frac{n}{2}\right)+\frac{96}{25}\sigma_3\left(\frac{n}{4}\right)+\frac{294}{25}\sigma_3\left(\frac{n}{7}\right)+\frac{882}{25}\sigma_3\left(\frac{n}{14}\right)\\ &\quad+\frac{4704}{25}\sigma_3\left(\frac{n}{28}\right)-\frac{356}{175}c_1(n)-\frac{316}{25}c_2(n)+2c_3(n)+\frac{1014}{175}c_4(n)\\ &\quad+\frac{1328}{25}c_5(n)+\frac{336}{5}c_6(n)-\frac{664}{25}c_7(n)+\frac{10432}{25}c_8(n)+\frac{10432}{25}c_9(n). \end{aligned}$$

REMARK 2.1. As mentioned in the introduction K. S. Williams [**15**] evaluated the formula for $R_{Q\oplus Q}(n)$ by using the convolution sum $W_7(n)$. In this paper we have computed this formula using the theory of modular forms. Therefore, as a consequence to our formula for $R_{Q\oplus Q}(n)$, we obtain the convolution sum $W_7(n)$ (see Corollary 2.4 below). The proof is demonstrated in §3.3.

REMARK 2.2. By the Atkin-Lehner theory of newforms [**2**], we see that the normalized newforms $\Delta_{4,7}(z)$ and $\Delta_{4,14;j}(z)$, $j=1,2$ are eigenforms under the Hecke operator for the prime 7 dividing the level with eigenvalues ± 7. Using this, we have for $a\geq 1$,

$$\tau_{4,7}(7^a) = (-1)^a 7^a;\ \tau_{4,14;1}(7^a) = (-1)^a 7^a;\ \tau_{4,14;2}(7^a) = 7^a.$$

Therefore, in Theorem 2.3, the formulas for $R_{Q\oplus jQ}(n)$ for $1\leq j\leq 3$ have elementary evaluations when $n=7^a$, $a\geq 1$.

In the following corollary, we use the formulas of Theorem 2.3 and the convolution sums $W_N(n)$, $N=1,2,3,4,14,21,28$ (given by Theorems 2.1 and 2.2), to obtain the convolution sums $W_7(n)$, $W_{2,7}(n)$, $W_{3,7}(n)$ and $W_{4,7}(n)$. We note that in [**5**], Chan and Cooper also used a different method to evaluate the convolution sums $W_p(n)$, $p=3,7,11,23$.

COROLLARY 2.4. *For a natural number n, we have*

$$\begin{aligned} W_7(n) &= \frac{1}{120}\sigma_3(n)+\frac{49}{120}\sigma_3\left(\frac{n}{7}\right)+\left(\frac{1}{24}-\frac{n}{28}\right)\sigma(n)+\left(\frac{1}{24}-\frac{n}{4}\right)\sigma\left(\frac{n}{7}\right)\\ &\quad-\frac{1}{70}\tau_{4,7}(n),\\ W_{2,7}(n) &= \frac{1}{600}\sigma_3(n)+\frac{1}{150}\sigma_3\left(\frac{n}{2}\right)+\frac{49}{600}\sigma_3\left(\frac{n}{7}\right)+\frac{49}{150}\sigma_3\left(\frac{n}{14}\right)\\ &\quad+\left(\frac{1}{24}-\frac{n}{28}\right)\sigma\left(\frac{n}{2}\right)+\left(\frac{1}{24}-\frac{n}{8}\right)\sigma\left(\frac{n}{7}\right)-\frac{3}{350}\tau_{4,7}(n)\\ &\quad-\frac{6}{175}\tau_{4,7}\left(\frac{n}{2}\right)+\frac{1}{600}\tau_{4,14;1}(n)+\frac{1}{200}\tau_{4,14;2}(n),\\ W_{3,7}(n) &= \frac{1}{1200}\sigma_3(n)+\frac{3}{400}\sigma_3\left(\frac{n}{3}\right)+\frac{49}{1200}\sigma_3\left(\frac{n}{7}\right)+\frac{147}{400}\sigma_3\left(\frac{n}{21}\right)\\ &\quad+\left(\frac{1}{24}-\frac{n}{28}\right)\sigma\left(\frac{n}{3}\right)+\left(\frac{1}{24}-\frac{n}{12}\right)\sigma\left(\frac{n}{7}\right)-\frac{1}{175}\tau_{4,7}(n)\\ &\quad-\frac{9}{175}\tau_{4,7}\left(\frac{n}{3}\right)-\frac{1}{100}\tau_{4,21;2}(n)+\frac{5}{672}(\tau_{4,21;3}(n)+\tau_{4,21;4}(n))\\ &\quad+\frac{11}{12768}\sqrt{57}(\tau_{4,21;3}(n)-\tau_{4,21;4}(n)), \end{aligned}$$

$$
\begin{aligned}
W_{4,7}(n) \;=\;& \frac{1}{2400}\sigma_3(n) + \frac{1}{800}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{150}\sigma_3\left(\frac{n}{4}\right) + \frac{49}{2400}\sigma_3\left(\frac{n}{7}\right) + \frac{49}{800}\sigma_3\left(\frac{n}{14}\right)\\
&+\frac{49}{150}\sigma_3\left(\frac{n}{28}\right) + \left(\frac{1}{24} - \frac{n}{28}\right)\sigma\left(\frac{n}{4}\right) + \left(\frac{1}{24} - \frac{n}{16}\right)\sigma\left(\frac{n}{7}\right)\\
&+\frac{697}{470400}c_1(n) + \frac{139}{22400}c_2(n) - \frac{9}{896}c_3(n) - \frac{893}{470400}c_4(n)\\
&+\frac{43}{1400}c_5(n) - \frac{7}{40}c_6(n) + \frac{241}{1400}c_7(n) - \frac{881}{1050}c_8(n) - \frac{178}{525}c_9(n).
\end{aligned}
$$

3. Proofs

For the proofs of our theorems, we need the following newforms $\Delta_{k,N}(z)$, $(k,N) \in \{(2,14),(2,21),(4,7)\}$, $\Delta_{4,14;1}(z)$, $\Delta_{4,14;2}(z)$ and $\Delta_{4,21;j}(z)$, $1 \le j \le 4$. Below we give their expression in terms of Eisenstein series and eta products. We have used the L-functions and modular forms database [**9**] to get some of these expressions.

$$
\begin{aligned}
\Delta_{2,14}(z) \;=\;& \eta(z)\eta(2z)\eta(7z)\eta(14z),\\
\Delta_{2,21}(z) \;=\;& \frac{\eta(7z)}{2\eta^2(z)\eta(3z)\eta(9z)\eta(21z)}\big(3\eta^2(z)\eta^2(7z)\eta^4(9z) + 3\eta^4(z)\eta^2(9z)\eta^2(63z)\\
&-\eta^5(3z)\eta(7z)\eta(9z)\eta(21z) + 7\eta(z)\eta^2(3z)\eta(9z)\eta^4(21z)\\
&+3\eta^3(z)\eta(7z)\eta^3(9z)\eta(63z) - 3\eta(z)\eta^5(3z)\eta(21z)\eta(63z)\big),
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{4,7}(z) \;=\;& \frac{(\eta^3(z)\eta^3(7z) + 4\eta^3(2z)\eta^3(14z))\eta^2(z)\eta^2(7z)}{\eta(2z)\eta(14z)},\\
\Delta_{4,14;1}(z) \;=\;& -\frac{9}{4}\Delta_{4,7}(z) - 9\Delta_{4,7}(2z) + \frac{\Delta_{2,14}(z)}{4}\big(24\Delta_{2,14}(z) + 14E_2(14z) - E_2(z)\big),\\
\Delta_{4,14;2}(z) \;=\;& \Delta_{4,7}(z) - 4\Delta_{4,7}(2z) - 5\Delta^2_{2,14}(z),\\
\Delta_{4,21;1}(z) \;=\;& \Delta_{2,21}(z)\left(-\frac{1}{24}E_2(z) - \frac{1}{8}E_2(3z) + \frac{7}{24}E_2(7z) + \frac{7}{8}E_2(21z)\right),\\
\Delta_{4,21;2}(z) \;=\;& \Delta_{2,21}(z)\left(\frac{1}{12}E_2(z) - \frac{1}{4}E_2(3z) - \frac{7}{12}E_2(7z) + \frac{7}{4}E_2(21z)\right).
\end{aligned}
$$

We have not found expressions for the remaining two newforms $\Delta_{4,21;3}(z)$ and $\Delta_{4,21;4}(z)$ (of weight 4 and level 21), whose Fourier coefficients lie in the real quadratic field $\mathbb{Q}(\sqrt{57})$. (Refer to the Note before Theorem 2.3.) Below, we give their first few Fourier coefficients using the database [**9**].

$$
\begin{aligned}
\Delta_{4,21;3}(z) \;=\;& q + \frac{-3+\sqrt{57}}{2}q^2 + 3q^3 + \frac{17-3\sqrt{57}}{2}q^4 + (3-\sqrt{57})q^5\\
&+\frac{-9+\sqrt{57}}{2}q^6 + 7q^7 + \cdots,\\
\Delta_{4,21;4}(z) \;=\;& q + \frac{-3-\sqrt{57}}{2}q^2 + 3q^3 + \frac{17+3\sqrt{57}}{2}q^4 + (3+\sqrt{57})q^5\\
&-\frac{9+\sqrt{57}}{2}q^6 + 7q^7 + \cdots.
\end{aligned}
$$

We also need the following weight 2 Eisenstein series defined (for two positive integers $a, b, a < b$) by

$$\Phi_{a,b}(z) = \frac{1}{b-a}\left(bE_2(bz) - aE_2(az)\right).$$

3.1. Proof of Theorem 2.2. The vector space $M_2(21)$ is of dimension 4 with a basis

$$\{\Phi_{1,3}(z), \Phi_{1,7}(z), \Phi_{1,21}(z), \Delta_{2,21}(z)\},$$

and $M_4(21)$ is of dimension 10 with a basis

$$\{E_4(dz), d|21; \Delta_{4,7}(z), \Delta_{4,7}(3z), \Delta_{4,21;j}(z), 1 \leq j \leq 4\}.$$

We note that $E_2(z)E_2(21z) \in \tilde{M}_4^{\leq 2}(21)$. Using Theorem A and the above basis, we have

$$\begin{aligned} E_2(z)E_2(21z) = & \frac{1}{500}E_4(z) + \frac{9}{500}E_4(3z) + \frac{49}{500}E_4(7z) + \frac{441}{500}E_4(21z) - \frac{576}{175}\Delta_{4,7}(z) \\ & - \frac{5184}{175}\Delta_{4,7}(3z) - \frac{144}{25}\Delta_{4,21;2}(z) + \left(-\frac{66}{133}\sqrt{57} - \frac{30}{7}\right)\Delta_{4,21;3}(z) \\ & + \left(\frac{66}{133}\sqrt{57} - \frac{30}{7}\right)\Delta_{4,21;4}(z) + \frac{40}{7}D\Phi_{1,21}(z) + \frac{4}{7}DE_2(z). \end{aligned}$$

By comparing the n-th Fourier coefficients of both the sides of the above identity, we get the required formula for $W_{21}(n)$.

Remark 3.1. In this remark we derive an expression for the newform $\Delta_{4,14;1}(z)$ in terms of the Eisenstein series $E_2(z)$ and its derivative. We note that (see [**13**])

$$\begin{aligned} E_2(z)E_2(14z) = & \frac{1}{250}E_4(z) + \frac{2}{125}E_4(2z) + \frac{49}{250}E_4(7z) + \frac{98}{125}E_4(14z) \\ & - \frac{864}{175}\Delta_{4,7}(z) - \frac{3456}{175}\Delta_{4,7}(2z) - \frac{48}{7}\Delta_{4,14;1}(z) - \frac{72}{25}\Delta_{4,14;2}(z) \\ & + \frac{39}{7}D\Phi_{1,14}(z) + \frac{6}{7}DE_2(z). \end{aligned}$$

Now, consider the quasimodular form $E_2(2z)E_2(7z)$ in $\tilde{M}_4^{\leq 2}(14)$, and using Theorem A, we have the following expression.

$$\begin{aligned} E_2(2z)E_2(7z) = & \frac{1}{250}E_4(z) + \frac{2}{125}E_4(2z) + \frac{49}{250}E_4(7z) + \frac{98}{125}E_4(14z) \\ & - \frac{864}{175}\Delta_{4,7}(z) - \frac{3456}{175}\Delta_{4,7}(2z) + \frac{48}{7}\Delta_{4,14;1}(z) - \frac{72}{25}\Delta_{4,14;2}(z) \\ & + \frac{18}{7}D\Phi_{1,7}(z) + \frac{39}{7}D\Phi_{1,14}(z) - \frac{36}{7}D\Phi_{2,14}(z) + \frac{6}{7}DE_2(z). \end{aligned}$$

Therefore, from the above two expressions, we get

$$E_2(2z)E_2(7z) - E_2(z)E_2(14z) = \frac{96}{7}\Delta_{4,14;1}(z) + \frac{18}{7}D\Phi_{1,7}(z) - \frac{36}{7}D\Phi_{2,14}(z).$$

This gives an expression for the newform $\Delta_{4,14;1}(z)$ in terms of Eisenstein series. More precisely, we have

$$\begin{aligned} \Delta_{4,14;1}(z) = & \frac{7}{96}E_2(2z)E_2(7z) - \frac{7}{96}E_2(z)E_2(14z) - \frac{7}{32}DE_2(7z) \\ & + \frac{1}{32}DE_2(z) + \frac{7}{16}DE_2(14z) - \frac{1}{16}DE_2(2z). \end{aligned} \tag{4}$$

3.2. Proof of Theorem 2.3. In this section, we shall obtain the formulas for $R_{Q\oplus jQ}(n)$ as given in Theorem 2.3. Let

$$\Theta_{Q\oplus jQ}(z) = \sum_{x_1,\cdots,x_8\in\mathbb{Z}} q^{(Q\oplus jQ)(x_1,\cdots,x_8)} \tag{5}$$

be the theta series associated with the quadratic forms $Q \oplus jQ$ for $j = 1, 2, 3, 4$. Now using the description mentioned in the Preliminaries section, we see that the theta series $\Theta_{Q\oplus jQ}(z)$ belongs to the space $M_4(7j)$ and has the Fourier expansion

$$\Theta_{Q\oplus jQ}(z) = \sum_{n=0}^{\infty} R_{Q\oplus jQ}(n)q^n.$$

Therefore, it is sufficient to obtain explicit bases for the spaces of modular forms $M_4(7j)$. The required formulas will follow by expressing each theta series as a linear combination of the corresponding modular form basis and comparing the nth Fourier coefficients. We shall give below explicit bases for the spaces $M_4(7j)$ for $j = 1, 2, 3, 4$ in tabular form.

Space	Dimension	Basis
$M_4(7)$	3	$\{E_4(az), a\vert 7; \Delta_{4,7}(z)\}$
$M_4(14)$	8	$\{E_4(az), a\vert 14; \Delta_{4,7}(bz), b\vert 2; \Delta_{4,14;i}(z), 1 \le i \le 2\}$
$M_4(21)$	10	$\{E_4(az), a\vert 21; \Delta_{4,7}(bz), b\vert 3; \Delta_{4,21;i}(z), 1 \le i \le 4\}$
$M_4(28)$	15	$\{E_4(az), a\vert 28; f_i(z), 1 \le i \le 9\}$

(Note that the newforms $\Delta_{k,N}(z)$, $\Delta_{k,N;i}(z)$ appearing in the above table are defined in §3 and the forms $f_i(z)$ that apper for level 28 case are defined in §2.)

Using the above bases, we have the following:

$$\Theta_{Q\oplus Q}(z) = \frac{1}{50}E_4(z) + \frac{49}{50}E_4(7z) + \frac{16}{5}\Delta_{4,7}(z), \tag{6}$$

$$\begin{aligned}\Theta_{Q\oplus 2Q}(z) &= \frac{1}{250}E_4(z) + \frac{2}{125}E_4(2z) + \frac{49}{250}E_4(7z) + \frac{98}{125}E_4(14z)\\ &+\frac{48}{25}\Delta_{4,7}(z) + \frac{192}{25}\Delta_{4,7}(2z) + \frac{28}{25}\Delta_{4,14;1}(z),\end{aligned} \tag{7}$$

$$\begin{aligned}\Theta_{Q\oplus 3Q}(z) &= \frac{1}{500}E_4(z) + \frac{9}{500}E_4(3z) + \frac{49}{500}E_4(7z) + \frac{441}{500}E_4(21z)\\ &+\frac{32}{25}\Delta_{4,7}(z) + \frac{288}{25}\Delta_{4,7}(3z) + \frac{56}{25}\Delta_{4,21;2}(z),\end{aligned} \tag{8}$$

$$\begin{aligned}\Theta_{Q\oplus 4Q}(z) &= \frac{1}{1000}E_4(z) + \frac{3}{1000}E_4(2z) + \frac{2}{125}E_4(4z) + \frac{49}{1000}E_4(7z)\\ &+\frac{147}{1000}E_4(14z) + \frac{98}{125}E_4(28z) - \frac{356}{175}f_1(z) - \frac{316}{25}f_2(z) + 2f_3(z)\\ &+\frac{1014}{175}f_4(z) + \frac{1328}{25}f_5(z) + \frac{336}{5}f_6(z) - \frac{664}{25}f_7(z)\\ &+\frac{10432}{25}f_8(z) + \frac{10432}{25}f_9(z).\end{aligned} \tag{9}$$

The theorem now follows by comparing the nth Fourier coefficients from the above expressions.

3.3. Proof of Corollary 2.4. In order to get the convolution sums $W_{a,b}(n)$ for $(a,b)=(1,7),(2,7),(3,7),(4,7)$ in the corollary, we first compute the formulas $R_{Q\oplus jQ}(n)$ $(1\le j\le 4)$ using the convolution sums method.
Let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $n\in N$ we know that (see [**15**])

$$R_Q(n)=4\sigma(n)-28\sigma\left(\frac{n}{7}\right). \tag{10}$$

Then for $1\le j\le 4$, $R_{Q\oplus jQ}(n)$ is given by

$$\begin{aligned} R_{Q\oplus jQ}(n) &= \sum_{\substack{a,b\in\mathbb{N}_0\\ a+jb=n}}\Big(\sum_{Q(x_1,\cdots,x_4)=a}1\Big)\Big(\sum_{Q(x_5,\cdots,x_8)=b}1\Big)\\ &= R_Q(n)+R_Q(n/j)+\sum_{\substack{a,b\in\mathbb{N}\\ a+jb=n}}R_Q(a)R_Q(b)\\ &= 4\sigma(n)-28\sigma\left(\frac{n}{7}\right)+4\sigma\left(\frac{n}{j}\right)-28\sigma\left(\frac{n}{7j}\right)+\\ &\quad 16W_j(n)-4\times 28W_{7j}(n)-4\times 28W_{j,7}(n)+28^2W_j\left(\frac{n}{7}\right). \end{aligned}$$

Comparing these formulas with the formulas given in Theorem 2.3 and using the convolution sums given in Theorem 2.1 and Theorem 2.2, we get the convolution sums $W_{j,7}(n)$, for $1\le j\le 4$. This completes the proof.

Acknowledgements

We have used the open-source mathematics software SAGE (www.sagemath.org) to do our calculations. The second author is partially funded by SERB grant SR/FTP/MS-053/2012. He would like to thank HRI, Allahabad for the warm hospitality where this work has been carried out. The SAGE worksheets are available in the last author's website[1]. Finally, we would like to thank the referee for making helpful suggestions.

References

[1] A. Alaca, S. Alaca and E. Ntienjem, *The convolution sum* $\sum_{al+bm=n}\sigma(l)\sigma(m)$ *for* $(a,b)=(1,28),(4,7),(1,14),(2,7),(1,7)$, arXiv:1607.06039v1 [math.NT]
[2] A. O. L. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(m)$, Math. Ann. **185** (1970), 134–160.
[3] M. Besge, *Extrait d'une lettre de M. Besge á M. Liouville,* J. Math. Pures Appl. **7** (1862), 256.
[4] J. H. Bruinier, G. van der Geer, G. Harder and D. Zagier, *The 1-2-3 of modular forms*, Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004. Edited by Kristian Ranestad. Universitext. Springer-Verlag, Berlin, 2008. 266 pp.
[5] H. H. Chan and S. Cooper, *Powers of theta functions*, Pacific J. Math. **235** (2008) No.1, 1-14.
[6] J. W. L. Glaisher, *On the squares of the series in which the coefficients are the sums of the divisor of the exponents*, Mess. Math. **15** (1885), 1–20.
[7] J. G. Huard, Z. M. Ou, B. K. Spearman and K. S. Williams, *Elementary evaluation of certain convolution sums involving divisor functions,* in Number Theory for the Millennium, II (Urbana, IL, 2000) (A. K. Peters, Natick, MA, 2002), 229–274.
[8] M. Kaneko and D. Zagier, *A generalized Jacobi theta function and quasimodular forms.* In 'The moduli space of curves (Texel Island, 1994)', 165–172, Progr. Math., **129**, Birkhäuser Boston, Boston, MA, 1995.

[1]URL: https://sites.google.com/site/theanupkumarsingh/sage-worksheets

[9] LMFDB, The database of L-functions, modular forms, and related objects, http://www.lmfdb.org/

[10] F. Martin and E. Royer, *Formes modulaires et périodes*, In 'Formes modulaires et transcendance', 1–117, Sémin. Congr., **12**, Soc. Math. France, Paris, 2005.

[11] B. Ramakrishnan and Brundaban Sahu, *Evaluation of the convolution sums* $\sum_{l+15m=n} \sigma(l)\sigma(m)$ *and* $\sum_{3l+5m=n} \sigma(l)\sigma(m)$ *and an application*, Int. J. Number Theory **9** (2013), no. 3, 1–11.

[12] S. Ramanujan, *On certain arithmetical functions,* Trans. Cambridge Philos. Soc. 22 (1916) 159–184.

[13] E. Royer, *Evaluating convolution sums of the divisor function by quasimodular forms*, Int. J. Number Theory **3** (2007), no. 2, 231–261.

[14] B. Schoeneberg, Elliptic Modular functions: An introduction, Die Grundlehren der Mathematischen Wissenschaften, Vol **203** Springer, New York, 1974

[15] K. S. Williams, *On a double series of Chan and Ong*, Georgian Math. J. **13** (2006) no.4, 793-805.

[16] K. S. Williams, Number Theory in the spirit of Liouville, *London Mathematical Student Texts* **76**, Cambridge Univ. Press, 2011.

(B. Ramakrishnan) Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad - 211 019, India.

(Brundaban Sahu) School of Mathematical Sciences, National Institute of Science Education and Research, HBNI, Bhubaneswar PO: Jatni, Khurda, Odisha - 752 050, India.

(Anup Kumar Singh) Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad - 211 019, India.

Email address, B. Ramakrishnan: ramki@hri.res.in

Email address, Brundaban Sahu: brundaban.sahu@niser.ac.in

Email address, Anup Kumar Singh: anupsingh@hri.res.in