REPRESENTATIONS OF SQUARES BY CERTAIN DIAGONAL QUADRATIC FORMS IN ODD NUMBER OF VARIABLES

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ABSTRACT. In this paper, we consider the following diagonal quadratic forms
$$a_1 x_1^2 + a_2 x_2^2 + \cdots + a_\ell x_\ell^2,$$
where $\ell \geq 5$ is an odd integer and $a_i \geq 1$ are integers. By using the extended Shimura correspondence, we obtain explicit formulas for the number of representations of $|D|n^2$ by the above type of quadratic forms, where D is either a square-free integer or a fundamental discriminant such that $(-1)^{(\ell-1)/2} D > 0$. We demonstrate our method with many examples, in particular, we obtain all the formulas (when $\ell = 5$) obtained in the work of Cooper-Lam-Ye [6] and all the representation formulas for n^2 obtained in [7] when n is even. The works of Cooper et. al make use of certain theta function identities combined with a method of Hurwitz to derive these formulas. It is to be noted that our method works in general with arbitrary coefficients a_i. As a consequence to some of our formulas, we obtain certain identities among the representation numbers and also some congruences involving Fourier coefficients of certain newforms of weights $6, 8$ and the divisor functions.

1. INTRODUCTION

Let $r_k(n)$ denote the number of representations of a positive integer n as a sum of k integer squares. Finding formulas for $r_k(n)$ is a classical problem in number theory. A general formula for $r_k(n)$ is not known so far when k is odd. However, some explicit formulas of $r_k(n)$ or $r_k(n^2)$ for $k = 1, 3, 5, 7, 9, 11$ and 13 are known. Here we give some references [1, 3, 4, 10, 17, 23, 24]. For a comprehensive list we refer to [4]. In [9], the first author in collaboration with Sanoli Gun obtained a general formula for $r_{2m+1}(n^2)$ $(2 \leq m \in \mathbb{N})$ by using the extended Shimura correspondence obtained by A. G. van Asch [27]. The problem becomes difficult if one considers diagonal quadratic forms with integer coefficients (and one of the coefficients is greater than 1). In this direction, formulas for the number of representations of n^2 by the quadratic forms $x^2 + by^2 + cz^2$ for $b, c \in \{1, 2, 3\}$ were obtained by S. Cooper and Y. Lam in [5]. Similar works to find formulas for the number of representations of n^2 by sums of 5 squares and 7 squares with coefficients (i.e., by the quadratic forms $a_1 x_1^2 + \cdots + a_\ell x_\ell^2$, with $\ell = 5, 7$, $a_i \in \{1, 2, 3, 6\}$, when $\ell = 5$ and $a_i \in \{1, 2, 4\}$, when $\ell = 7$) were carried out by S. Cooper, Y. Lam and D. Ye in [6, 7] and they refer these quadratic forms as quinary ($\ell = 5$) and septenary ($\ell = 7$) forms. In these works, Cooper-Lam-Ye used a method of Hurwitz as the main tool to get explicit formulas for the number of representations.

The purpose of this paper is to use the Shimura correspondence to deduce a more general formula. More precisely, we find formulas for $r_\ell(a_1, \ldots, a_\ell; |D|n^2) = r_\ell(\mathbf{a}; |D|n^2)$, the number of representations of $|D|n^2$ by the quadratic form $a_1 x_1^2 + \cdots + a_\ell x_\ell^2$, where $\mathbf{a} = (a_1, \ldots, a_\ell)$, a_i's are positive integers, $\ell \geq 5$ is an odd integer and D is either a square-free integer t or a fundamental discriminant (which depends on ℓ, $N_{\mathbf{a}}$). Here $N_{\mathbf{a}}$ is the least common multiple of all the coefficients a_i, $1 \leq i \leq \ell$ and we assume that $(-1)^{(\ell-1)/2}D > 0$. This is achieved by observing the fact that the generating function corresponding to the quadratic form is a modular form of weight $\ell/2$ on $\Gamma_0(4N_{\mathbf{a}})$ with certain quadratic character depending on $\mathbf{a}$. So, we can apply the extended Shimura map $\mathcal{S}_D$ obtained by Jagathesan and Manickam [15, 16] and express it as a linear combination of basis elements of modular forms of weight $\ell - 1$ and level $2N_{\mathbf{a}}$. Now the required formula follows by comparing the Fourier coefficients and taking Möbius inversion. Thus, our approach gives a more general formula for the number of representations.

We now give the details of results proved in this paper. In §2, we give some preliminaries and state our main theorem and in §3 we give a proof. In §4, explicit examples are discussed for the cases $\ell = 5, 7, 9$. In §4.1 we consider the quinary forms ($\ell = 5$) and provide 54 examples by taking $a_i \in \{1, 2, 3, 4, 6\}$. All the 18 formulas obtained in the work of Cooper-Lam-Ye [6] are presented as Corollary 4.1. Necessary details for the remaining 36 formulas are given in Tables 4 and 5. Some explicit examples (among these 36 cases) are presented in Corollaries 4.2 and 4.3. The septenary case ($\ell = 7$) is presented in §4.2. Here we take $a_i \in \{1, 2, 4\}$, which gives 27 forms, out of which 18 cases correspond to the formulas obtained in [7]. Our formulas coincide with the formulas of Cooper-Lam-Ye when n is even. We remark that the existing results on the mapping property of Shimura maps allow us to use only the Shimura-Kohnen map when $N_{\mathbf{a}}$ is even. However, by assuming the mapping property of the extended Shimura maps $\mathcal{S}_t$ in the case when $\ell \equiv 3 \pmod 4$ and $N_{\mathbf{a}}$ is even, one could obtain formulas for the number of representations of n^2. In particular, the remaining formulas obtained in [7] could be proved under this assumption. This is explained in §4.2.2 and the conjectural formulas are given in this section. In §4.3, we consider the case $\ell = 9$ and obtain 52 formulas by taking a_i in the three sets $\{1, b\}$, $b = 2, 3$ and $\{1, 2, 4\}$. Complete data for obtaining the formulas are given in Tables 7 to 10 and sample formulas are provided in Corollary 4.7. As a consequence to these formulas, we derive certain congruences between newform Fourier coefficients and the divisor function, which is presented in Corollary 4.8. We also obtain certain relations among the representation numbers $r_9(\mathbf{a}; n^2)$, which is presented as a proposition in §4.3. So far the known results for the number of representations are mainly for perfect squares. The method of Shimura correspondence gives such formulas for the number of representations of integers which are not necessarily perfect squares and we could get formulas for tn^2, where t is a positive square-free integers. In §5 we discuss about such formulas (when $\ell = 7$) and list some of them which are not proved earlier. We also mention about additional formulas that can be obtained using our method which are not presented here due to the large size of the data.

2. Preliminaries and Main Results

We consider the following quadratic form in ℓ variables with (positive) integer coefficients a_i:

$$a_1 x_1^2 + a_2 x_2^2 + \cdots + a_\ell x_\ell^2. \tag{1}$$

Let $r_\ell(a_1, a_2, \ldots, a_\ell; n)$ denote the number of representations of a natural number n by the above quadratic form. i.e., writing $\mathbf{a} = (a_1, a_2, \ldots, a_\ell)$,

$$r_\ell(\mathbf{a}; n) := r_\ell(a_1, a_2, \ldots, a_\ell; n) = \# \left\{ (x_1, x_2, \ldots, x_\ell) \in \mathbb{Z}^\ell; \sum_{i=1}^{\ell} a_i x_i^2 = n \right\}. \tag{2}$$

When all the a_i's are equal to 1, it is denoted by $r_\ell(n)$. If a_j appears with multiplicity i_j, then we represent $\mathbf{a}$ by $(a_1^{i_1}, a_2^{i_2}, \cdots, a_r^{i_r})$, where $i_1 + i_2 + \cdots + i_r = \ell$. We drop the power i_j, if it is equal to 1.

Before stating our main theorem, we shall fix some notations. For natural numbers k and M, the vector spaces of modular forms of weight $k + 1/2$ (resp. weight $2k$) on $\Gamma_0(4M)$ with character χ modulo $4M$ (resp. on $\Gamma_0(M)$ with character χ modulo M) as $M_{k+1/2}(4M, \chi)$ (resp. $M_{2k}(M, \chi)$). The respective subspaces of cusp forms are denoted by $S_{k+1/2}(4M, \chi)$ and $S_{2k}(M, \chi)$. In the case of principal character, we omit χ from the notation. For a vector $\mathbf{a} = (a_1, a_2, \ldots, a_\ell)$, let $N_\mathbf{a}$ be as defined in the introduction (lcm of all the integers $a_1, \ldots, a_\ell$). Corresponding to $\mathbf{a}$, let us also associate a quadratic Dirichlet character $\prod_{j=1}^{\ell} \left(\frac{a_j}{\cdot} \right)$ and denote it by $\psi_\mathbf{a}$. Note that $\psi_\mathbf{a}$ is the principal Dirichlet character modulo $4N_\mathbf{a}$, if for each j, $1 \leq j \leq \ell$, either a_j is a perfect square or it appears even number of times in (1).

Theorem 2.1. *For a vector $\mathbf{a} = (a_1, a_2, \ldots, a_\ell)$, with $a_i \in \mathbb{N}$, let $N_\mathbf{a}$ be the positive integer and $\psi_\mathbf{a}$ be the Dirichlet character associated to $\mathbf{a}$ as defined above. Let $\nu(\ell, N_\mathbf{a})$ denote the dimension of the vector space $M_{\ell-1}(2N_\mathbf{a})$. Let D be a square-free integer or a fundamental discriminant depending on $N_\mathbf{a}$ is odd or even such that $(-1)^{(\ell-1)/2} D > 0$. Then for $n \in \mathbb{N}$, we have*

$$r_\ell(\mathbf{a}; |D|n^2) = \sum_{\substack{d|n \\ (d, 2N_\mathbf{a})=1}} \mu(d) \psi_\mathbf{a}(d) \left(\frac{D}{d} \right) d^{(\ell-3)/2} \sum_{j=1}^{\nu(\ell, N_\mathbf{a})} \lambda_{\ell, D, j}(\mathbf{a}) A_{\ell, \mathbf{a}; j}(n/d), \tag{3}$$

where $\lambda_{\ell, D, j}(\mathbf{a})$ are constants which depend on $\ell, D, \mathbf{a}$ and also on the choice of a basis of $M_{\ell-1}(2N_\mathbf{a})$. The n-th Fourier coefficients of these basis elements are denoted by $A_{\ell, \mathbf{a}; j}(n)$, $1 \leq j \leq \nu(\ell, N_\mathbf{a})$.

3. Proof of the Main Theorem

In [15, 16], Jagathesan and Manickam have obtained the Shimura correspondence for non-cusp forms of half-integral weight on $\Gamma_0(4N)$, $N \in \mathbb{N}$. When $N = 1$, the work was carried out by A. G. van Asch [27]. For more details on Shimura and Shimura-Kohnen maps, we refer to the works of Shimura and Kohnen [25, 12, 13]. Below we mention a result of Jagathesan and Manickam, which is the main ingredient in our proof.

Theorem 3.1. (*Jagathesan and Manickam* [15, 16])
Let $k \geq 2$ be an integer and t be a square-free integer with $(-1)^k t > 0$. Write $D = t$ or $4t$ according as $t \equiv 1$ or $2,3 \pmod 4$. Then D is a fundamental discriminant with $(-1)^k D > 0$. For a modular form $f(z) = \sum_{n \geq 0} a(n) q^n \in M_{k+1/2}(4N, \chi)$, where χ is an even quadratic Dirichlet character modulo $4N$, define the t-th Shimura map and D-th Shimura-Kohnen map as follows:

$$\mathcal{S}_t(f)(z) = a(0) H_k(t) + \sum_{n \geq 1} \left(\sum_{\substack{d|n \\ (d,2N)=1}} \chi(d) \left(\frac{t}{d}\right) d^{k-1} a(|t|n^2/d^2) \right) q^n, \qquad (4)$$

$$\mathcal{S}_D(f)(z) = a(0) H_k(D) + \sum_{n \geq 1} \left(\sum_{\substack{d|n \\ (d,N)=1}} \chi(d) \left(\frac{D}{d}\right) d^{k-1} a(|D|n^2/d^2) \right) q^n, \qquad (5)$$

where $z \in \mathbb{C}$, $\mathrm{Im}\, z > 0$, $q = e^{2\pi i z}$ and

$$H_k(t) = \frac{1}{2} L\left(1 - k, \left(\frac{D}{\cdot}\right)\right) \prod_{p|2N} \left(1 - \left(\frac{D}{p}\right) p^{k-1}\right),$$

$$\qquad\qquad (6)$$

$$H_k(D) = \frac{1}{2} L\left(1 - k, \left(\frac{D}{\cdot}\right)\right) \prod_{p|N} \left(1 - \left(\frac{D}{p}\right) p^{k-1}\right).$$

Then, we have $\mathcal{S}_t(f) \in M_{2k}(2N)$, if N is odd and $\mathcal{S}_D(f) \in M_{2k}(2N)$, if N is even.

Remark 3.1. In the above theorem, extended Shimura map is obtained for the modular forms space with trivial character. However, Professor Manickam informed us that in his forthcoming work the results are proved for the spaces with real quadratic character. So, we make use of this fact also in our work.

Proof of Theorem 2.1.

We are now ready to prove our main theorem. Let $\ell \geq 5$ be an odd integer. Let $\Theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$ be the classical theta series of weight $1/2$ for the group $\Gamma_0(4)$. Then the generating function for the quadratic form $a_1 x_1^2 + \cdots + a_\ell x_\ell^2$ is the following product of theta functions:

$$\Theta_{\mathbf{a}}(z) = \prod_{j=1}^{\ell} \Theta(a_j z),$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_\ell)$ and it is a modular form of weight $\ell/2$ on $\Gamma_0(4N_{\mathbf{a}})$, with character $\psi_{\mathbf{a}}$ where $N_{\mathbf{a}}$ and $\psi_{\mathbf{a}}$ are defined as before. This follows from the basic theory of modular forms. For a proof of this fact, we refer to [19, Fact II, p.758]. Now we make use of Theorem 3.1. When $N_{\mathbf{a}}$ is odd, we apply the Shimura map $\mathcal{S}_t$ with square-free t such that $(-1)^{(\ell-1)/2} t > 0$ on $\Theta_{\mathbf{a}}(z)$ and when $N_{\mathbf{a}}$ is even, then we apply the Shimura-Kohnen map $\mathcal{S}_D$, where D is a fundamental discriminant such the $(-1)^{(\ell-1)/2} D > 0$ on the theta function $\Theta_{\mathbf{a}}(z)$. For the sake of uniformity in the theorem, we used the symbol D to denote a square-free integer or a fundamental discriminant depending on the parity

(odd or even) of $N_\mathbf{a}$. Thus, $\mathcal{S}_D(\Theta_\mathbf{a})(z)$ is a modular form in $M_{\ell-1}(2N_\mathbf{a})$. Let $f_{\ell,j}(z)$, $1 \leq j \leq \nu(\ell, N_\mathbf{a})$ be a basis for the space $M_{\ell-1}(2N_\mathbf{a})$, whose n-th Fourier coefficients are denoted by $A_{\ell,\mathbf{a};j}(n)$, $1 \leq j \leq \nu(\ell, N_\mathbf{a})$. Therefore, we can write $\mathcal{S}_D(\Theta_\mathbf{a})(z)$ as a linear combination of these basis elements and then comparing their n-th Fourier coefficients, we get the following identity for any integer $n \geq 1$:

$$\sum_{\substack{d|n \\ (d,2N_\mathbf{a})=1}} \psi_\mathbf{a}(d) \left(\frac{D}{d}\right) d^{(\ell-3)/2} r_\ell(\mathbf{a}; |D|n^2/d^2) = \sum_{j=1}^{\nu(\ell,N_\mathbf{a})} \lambda_{\ell,D,j}(\mathbf{a}) A_{\ell,\mathbf{a};j}(n), \qquad (7)$$

where $\lambda_{\ell,D,j}(\mathbf{a})$ are constants that depend on ℓ, D, $\mathbf{a}$ and the basis chosen for the space $M_{\ell-1}(2N_\mathbf{a})$. Taking Möbius inversion, the above identity gives the required expression for $r_\ell(\mathbf{a}; |D|n^2)$ as stated in the theorem, namely:

$$r_\ell(\mathbf{a}; |D|n^2) = \sum_{\substack{d|n \\ (d,2N_\mathbf{a})=1}} \mu(d)\psi_\mathbf{a}(d) \left(\frac{D}{d}\right) d^{(\ell-3)/2} \sum_{j=1}^{\nu(\ell,N_\mathbf{a})} \lambda_{\ell,D,j}(\mathbf{a}) A_{\ell,\mathbf{a};j}(n/d). \qquad (8)$$

This completes the proof. $\qquad \square$

In the following sections, we make the above formulas more explicit.

4. Sample formulas

In this section we shall apply our main theorem to derive explicit formulas for $r_\ell(\mathbf{a}; n^2)$ for $\ell = 5, 7, 9$, with certain coefficients a_i. In the case of quinary forms ($\ell = 5$), we prove all the identities obtained in [6] and in the case of septenary forms ($\ell = 7$), we prove most of the formulas obtained in [7] apart from proving new identities. We add a section which contains some conjectural formulas, which are derived under the assumption of the mapping property of the t-th Shimura maps. When $\ell = 9$, we get the formulas when a_i belong to the sets $\{1, 2\}$, $\{1, 3\}, \{1, 2, 4\}$. All the formulas obtained in our work are listed as Corollaries to our main theorem.

We use the following notations in our formulas. For an odd positive integer $m = \prod_{p \geq 3} p^{\lambda_p}$, we define the following functions.

$$s_1(m) = \prod_{p \geq 3} \left(\frac{p^{3\lambda_p+3} - 1}{p^3 - 1} - p\frac{p^{3\lambda_p} - 1}{p^3 - 1}\right), \qquad (9)$$

$$s_2(m) = \prod_{p \geq 3} \left(\frac{p^{3\lambda_p+3} - 1}{p^3 - 1} - p\left(\frac{2}{p}\right)\frac{p^{3\lambda_p} - 1}{p^3 - 1}\right), \qquad (10)$$

$$s_3(m) = \prod_{p \geq 3} \left(\frac{p^{3\lambda_p+3} - 1}{p^3 - 1} - p\left(\frac{3}{p}\right)\frac{p^{3\lambda_p} - 1}{p^3 - 1}\right), \qquad (11)$$

$$s_4(m) = \prod_{p \geq 3} \left(\frac{p^{3\lambda_p+3} - 1}{p^3 - 1} - p\left(\frac{6}{p}\right)\frac{p^{3\lambda_p} - 1}{p^3 - 1}\right), \qquad (12)$$

$$s_5(m) = \prod_{p \geq 3} \left(\frac{p^{5\lambda_p+5} - 1}{p^5 - 1} - p^2 \left(\frac{-2}{p} \right) \frac{p^{5\lambda_p} - 1}{p^5 - 1} \right), \tag{13}$$

$$s_6(m) = \prod_{p \geq 3} \left(\frac{p^{5\lambda_p+5} - 1}{p^5 - 1} - p^2 \left(\frac{-4}{p} \right) \frac{p^{5\lambda_p} - 1}{p^5 - 1} \right). \tag{14}$$

$$s_7(m) = \prod_{p \geq 3} \left(\frac{p^{7\lambda_p+7} - 1}{p^7 - 1} - p^3 \frac{p^{7\lambda_p} - 1}{p^7 - 1} \right). \tag{15}$$

Remark 4.1. The functions (9)-(12) appear in the formulas of Theorems 1.1 to 1.4 of [6]. We note that the functions (13) and (14) are denoted as $s(m)$ and $t(m)$ respectively in [7, Eqs. (2.1), (2.2)].

4.1. **Quinary forms.** In this section, we assume that $\ell = 5$ and consider the form $a_1 x_1^2 + \cdots + a_5 x_5^2$, where a_i belong to the set $\{1, 2, 3, 4, 6\}$. Specifically, we take them in the two sets $\{1, 2, 4\}$ and $\{1, 2, 3, 6\}$. There are 14 forms corresponding to the first set and the functions corresponding to them belong to the space of modular forms of weight $5/2$ on $\Gamma_0(8)$ (if $a_i \neq 4$) or $\Gamma_0(16)$ (if some $a_i = 4$) with character depending on $\mathbf{a}$. There are 40 quinary forms when a_i's belong to the set $\{1, 2, 3, 6\}$. The corresponding generating functions belong to either $M_{5/2}(12, \chi)$, $M_{5/2}(16, \chi')$ or $M_{5/2}(24, \chi'')$, where χ, χ', χ'' are real quadratic characters modulo 12, 16 or 24 depending on the coefficients a_i's. To apply our method, we need to find explicit bases for the following modular form spaces: $M_4(N)$, where $N = 4, 6, 8, 12$. In this section, we consider certain bases for these modular form spaces and apply our theorem to get formulas for $r_5(\mathbf{a}; n^2)$ for the above 54 quinary forms. In particular, we derive all the 18 formulas proved in [6] and the remaining 36 formulas are new. In the table below, we shall give the list of vectors and the corresponding modular forms space. In Table 2, we shall describe the required basis elements for the evaluation of our formulas. As mentioned earlier, we use the notation $\mathbf{a} = (a_1^{i_1}, a_2^{i_2}, \cdots, a_r^{i_r})$, $i_1 + i_2 + \cdots + i_r = \ell$.

Table 1 ($\ell = 5$, $a_i \in \{1, 2, 4\}, \{1, 2, 3, 6\}$)

$\mathbf{a} = (a_1^{i_1}, a_2^{i_2}, \cdots, a_r^{i_r})$ with $i_1 + i_2 + \cdots + i_r = 5$	$4N_{\mathbf{a}}, \psi_{\mathbf{a}}$
$(1^3, 2^2), (1, 2^4)$	8, χ_0
$(1^4, 2), (1^2, 2^3)$	8, χ_2
$(1^4, 4), (1^3, 4^2), (1^2, 4^3), (1, 4^4), (1, 2^2, 4^2), (1^2, 2^2, 4)$	16, χ_0
$(1^3, 2, 4), (1^2, 2, 4^2), (1, 2^3, 4), (1, 2, 4^3),$	16, χ_2
$(1, 3^4), (1^3, 3^2)$	12, χ_0
$(1^4, 3), (1^2, 3^3)$	12, χ_3
$(1, 2^2, 3^2), (1, 3^2, 6^2), (1^3, 6^2), (1, 6^4), (1, 2^2, 6^2), (2^3, 3, 6), (2, 3^3, 6), (2, 3, 6^3), (1^2, 2, 3, 6)$	24, χ_0
$(2^3, 3^2), (2, 3^4), (1^2, 2, 3^2), (1^3, 3, 6), (1, 3^3, 6), (1, 3, 6^3), (1^2, 2, 6^2), (2, 3^2, 6^2), (1, 2^2, 3, 6)$	24, χ_2
$(2^4, 3), (2^2, 3^3), (1^2, 2^2, 3), (1^2, 3, 6^2), (1^3, 2, 6), (1, 2^3, 6), (1, 2, 6^3), (2^2, 3, 6^2), (1, 2, 3^2, 6)$	24, χ_3
$(1^3, 2, 3), (1, 2^3, 3), (1, 2, 3^3), (1^4, 6), (1^2, 6^3), (1^2, 3^2, 6), (1^2, 2^2, 6), (2^2, 3^2, 6), (1, 2, 3, 6^2)$	24, χ_6

In the above, χ_0 denotes the principal character modulo $4N_{\mathbf{a}}$ and χ_m is the quadratic Dirichlet character $\left(\frac{m}{\cdot} \right)$ modulo $4m$, where $m | N_{\mathbf{a}}$.

In the following table, we describe the basis elements for the vector spaces $M_4(N)$, $N = 4, 6, 8$ and 12. We use the following notation for these basis elements. The function $E_k(z)$ denotes the normalised Eisenstein series of an even integer weight $k \geq 4$ for the full modular group whose Fourier expansion is given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n)q^n, \tag{16}$$

where B_k is the k-th Bernoulli number, $q = e^{2\pi i z}$ and $\sigma_{k-1}(n)$ is the divisor function. For a given weight $k \geq 2$ and level N, if the normalised newform is unique, then we denote it by $\Delta_{k,N}(z)$, and denote by $\tau_{k,N}(n)$ its n-th Fourier coefficient. If there are more than one newform of weight k, level N, then we denote them by $\Delta_{k,N;j}(z)$, where j runs over the number of such newforms. Their n-th Fourier coefficients are denoted by $\tau_{k,N;j}(n)$.

Table 2

Level $2N_{\mathbf{a}}$	Basis elements	Dimension $\nu(5, N_{\mathbf{a}})$
4	$E_4(z), E_4(2z), E_4(4z)$	3
6	$E_4(z), E_4(2z), E_4(3z), E_4(6z), \Delta_{4,6}(z)$	5
8	$E_4(z), E_4(2z), E_4(4z), E_4(8z), \Delta_{4,8}(z)$	5
12	$E_4(tz); t|12, \Delta_{4,6}(z), \Delta_{4,6}(2z), \Delta_{4,12}(z)$	9

Let $\eta(z) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$ be the Dedekind eta-function, which is a modular form of weight $1/2$. Then the newforms $\Delta_{k,N}(z)$ appearing in the above table are given by the following eta-products or linear combination of eta-quotients.

$$\Delta_{4,6}(z) = \eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z); \qquad \Delta_{4,8}(z) = \eta^4(2z)\eta^4(4z);$$
$$\Delta_{4,12}(z) = \frac{\eta^2(2z)\eta^3(3z)\eta^3(4z)\eta^2(6z)}{\eta(z)\eta(12z)} - \frac{\eta^3(z)\eta^2(2z)\eta^2(6z)\eta^3(12z)}{\eta(3z)\eta(4z)}.$$

In the following, we derive the 18 formulas for the quinary forms obtained in the work of Cooper-Lam-Ye [6]. As $\ell = 5$, $(\ell - 1)/2$ is even, we may take $t = D = 1$. In this case $\mathcal{S}_t$ and $\mathcal{S}_D$ are the same map and the image belongs to one of the spaces $M_4(N)$, where $N = 4, 6, 8$ or 12. By considering the 18 vectors $\mathbf{a}$ and taking $t = D = 1$ in (3) we express $r_5(\mathbf{a}; n^2)$ explicitly. In the following table, we first give the images of $\Theta_{\mathbf{a}}(z)$ under $\mathcal{S}_1$ with respect to the bases given in the previous table for $\mathbf{a} \in \{(1^4, 2), (1^4, 3), (1^3, 2, 3), (1^3, 2^2), (1^4, 4), (1^3, 4^2), (1^2, 2^2, 4), (1^2, 4^3), (1, 2^4), (1, 2^2, 4^2), (1, 4^4), (1, 2^2, 3^2), (1, 3^4), (1^2, 2^3), (1, 2^3, 4), (1^2, 2, 3^2), (1, 3^3, 6), (1, 2^3, 6)\}$. Using the explicit images given in the table below (Table 3) and comparing the n-th Fourier coefficients and finally taking Möbius inversion we obtain the 18 formulas corresponding to the vectors $\mathbf{a}$ listed above. We present these formulas in the form of a corollary after the table.

Table 3

$\mathbf{a}$	$\mathcal{S}_1(\Theta_{\mathbf{a}})(z)$
$(1^4, 2)$	$\frac{1}{30}E_4(z) - \frac{8}{15}E_4(4z)$
$(1^4, 3)$	$\frac{1}{30}E_4(z) - \frac{2}{15}E_4(2z) + \frac{3}{10}E_4(3z) - \frac{6}{5}E_4(6z) - \frac{18}{5}E_4(12z)$
$(1^3, 2, 3)$	$\frac{1}{40}E_4(z) - \frac{1}{20}E_4(2z) - \frac{9}{40}E_4(3z) + \frac{2}{5}E_4(4z) + \frac{9}{20}E_4(6z)$
$(1^3, 2^2)$	$\frac{1}{40}E_4(z) - \frac{1}{60}E_4(2z)$
$(1^4, 4)$	$\frac{1}{30}E_4(z) - \frac{23}{120}E_4(2z) + \frac{1}{5}E_4(4z)$
$(1^3, 4^2)$	$\frac{1}{40}E_4(z) - \frac{11}{60}E_4(2z) + \frac{1}{5}E_4(4z)$
$(1^2, 2^2, 4)$	$\frac{1}{60}E_4(z) - \frac{1}{24}E_4(2z) + \frac{1}{15}E_4(4z)$
$(1^2, 4^3)$	$\frac{1}{60}E_4(z) - \frac{13}{120}E_4(2z) + \frac{2}{15}E_4(4z)$
$(1, 2^4)$	$\frac{1}{120}E_4(z) + \frac{1}{30}E_4(2z)$
$(1, 2^2, 4^2)$	$\frac{1}{120}E_4(z) - \frac{1}{30}E_4(2z) + \frac{1}{15}E_4(4z)$
$(1, 4^4)$	$\frac{1}{120}E_4(z) - \frac{1}{60}E_4(2z) + \frac{1}{15}E_4(4z)$
$(1, 2^2, 3^2)$	$\frac{1}{120}E_4(z) - \frac{1}{30}E_4(2z) + \frac{3}{40}E_4(3z) - \frac{3}{20}E_4(6z)$
$(1, 3^4)$	$\frac{1}{120}E_4(z) - \frac{11}{120}E_4(3z)$
$(1^2, 2^3)$	$\frac{1}{60}E_4(z) + \frac{1}{60}E_4(2z) - \frac{16}{30}E_4(4z)$
$(1, 2^3, 4)$	$\frac{1}{120}E_4(z) - \frac{1}{120}E_4(2z) + \frac{1}{30}E_4(4z) - \frac{16}{30}E_4(8z)$
$(1^2, 2, 3^2)$	$\frac{1}{60}E_4(z) - \frac{1}{30}E_4(2z) - \frac{3}{20}E_4(3z) + \frac{4}{15}E_4(4z) + \frac{3}{10}E_4(6z) - \frac{12}{5}E_4(12z)$
$(1, 3^3, 6)$	$\frac{1}{120}E_4(z) - \frac{1}{60}E_4(2z) - \frac{17}{120}E_4(3z) + \frac{2}{15}E_4(4z) + \frac{17}{60}E_4(6z) - \frac{34}{15}E_4(12z)$
$(1, 2^3, 6)$	$\frac{1}{120}E_4(z) - \frac{1}{60}E_4(2z) - \frac{3}{40}E_4(3z) + \frac{2}{15}E_4(4z) + \frac{3}{20}E_4(6z) - \frac{6}{5}E_4(12z)$

Corollary 4.1. *For a natural number n, we have*

$$r_5(1^4, 2; n^2) = \sum_{d|n} \mu(d)d\left(\frac{2}{d}\right)\left\{8\sigma_3(n/d) - 128\sigma_3(n/4d)\right\},$$

$$r_5(1^4, 3; n^2) = \sum_{d|n} \mu(d)d\left(\frac{3}{d}\right)\left\{8\sigma_3(n/d) - 32\sigma_3(n/2d) + 72\sigma_3(n/3d) - 288\sigma_3(n/6d)\right\},$$

$$r_5(1^3, 2, 3; n^2) = \sum_{d|n} \mu(d)d\left(\frac{6}{d}\right)\left\{6\sigma_3(n/d) - 12\sigma_3(n/2d) - 54\sigma_3(n/3d) + 96\sigma_3(n/4d)\right.$$
$$\left. + 108\sigma_3(n/6d) - 864\sigma_3(n/12d)\right\},$$

$$r_5(1^3, 2^2; n^2) = \sum_{d|n} \mu(d)d\left(\frac{4}{d}\right)\left\{6\sigma_3(n/d) - 4\sigma_3(n/2d)\right\},$$

$$r_5(1^4, 4; n^2) = \sum_{d|n} \mu(d)d\left(\frac{4}{d}\right)\left\{8\sigma_3(n/d) - 46\sigma_3(n/2d) + 48\sigma_3(n/4d)\right\},$$

$$r_5(1^3, 4^2; n^2) = \sum_{d|n} \mu(d)d\left(\frac{4}{d}\right)\left\{6\sigma_3(n/d) - 44\sigma_3(n/2d) + 48\sigma_3(n/4d)\right\},$$

$$r_5(1^2, 2^2, 4; n^2) = \sum_{d|n} \mu(d)d\left(\frac{4}{d}\right)\left\{4\sigma_3(n/d) - 10\sigma_3(n/2d) + 16\sigma_3(n/4d)\right\},$$

$$r_5(1^2, 4^3; n^2) = \sum_{d|n} \mu(d) d \left(\frac{4}{d}\right) \{4\sigma_3(n/d) - 26\sigma_3(n/2d) + 32\sigma_3(n/4d)\},$$

$$r_5(1, 2^4; n^2) = \sum_{d|n} \mu(d) d \left(\frac{4}{d}\right) \{2\sigma_3(n/d) + 8\sigma_3(n/2d)\},$$

$$r_5(1, 2^2, 4^2; n^2) = \sum_{d|n} \mu(d) d \left(\frac{4}{d}\right) \{2\sigma_3(n/d) - 8\sigma_3(n/2d) + 16\sigma_3(n/4d)\},$$

$$r_5(1, 4^4; n^2) = \sum_{d|n} \mu(d) d \left(\frac{4}{d}\right) \{2\sigma_3(n/d) - 4\sigma_3(n/2d) + 16\sigma_3(n/4d)\},$$

$$r_5(1, 2^2, 3^2; n^2) = \sum_{d|n} \mu(d) d \left(\frac{4}{d}\right) \{2\sigma_3(n/d) - 4\sigma_3(n/2d) + 18\sigma_3(n/3d) - 36\sigma_3(n/6d)\},$$

$$r_5(1, 3^4; n^2) = \sum_{d|n} \mu(d) d \left(\frac{4}{d}\right) \{2\sigma_3(n/d) - 22\sigma_3(n/3d)\},$$

$$r_5(1^2, 2^3; n^2) = \sum_{d|n} \mu(d) d \left(\frac{2}{d}\right) \{4\sigma_3(n/d) + 4\sigma_3(n/2d) - 128\sigma_3(n/4d)\},$$

$$r_5(1, 2^3, 4; n^2) = \sum_{d|n} \mu(d) d \left(\frac{2}{d}\right) \{2\sigma_3(n/d) - 2\sigma_3(n/2d) + 8\sigma_3(n/4d) - 128\sigma_3(n/8d)\},$$

$$r_5(1^2, 2, 3^2; n^2) = \sum_{d|n} \mu(d) d \left(\frac{2}{d}\right) \{4\sigma_3(n/d) - 8\sigma_3(n/2d) - 36\sigma_3(n/3d) + 64\sigma_3(n/4d)$$
$$+ 72\sigma_3(n/6d) - 576\sigma_3(n/12d)\},$$

$$r_5(1, 3^3, 6; n^2) = \sum_{d|n} \mu(d) d \left(\frac{2}{d}\right) \{2\sigma_3(n/d) - 4\sigma_3(n/2d) - 34\sigma_3(n/3d) + 32\sigma_3(n/4d)$$
$$+ 68\sigma_3(n/6d) - 544\sigma_3(n/12d)\},$$

$$r_5(1, 2^3, 6; n^2) = \sum_{d|n} \mu(d) d \left(\frac{3}{d}\right) \{2\sigma_3(n/d) - 4\sigma_3(n/2d) - 18\sigma_3(n/3d) + 32\sigma_3(n/4d)$$
$$+ 36\sigma_3(n/6d) - 288\sigma_3(n/12d)\}.$$

Remark 4.2. All the above 18 formulas give rise to the same formulas obtained in the work of Cooper-Lam-Ye [6]. We demonstrate this in the next section.

4.1.1. **Equivalence of formulas.** As mentioned in the above remark, all the above formulas are the same as given in [6]. We will demonstrate the equivalence in three cases. The rest of them follow using similar computations. When $\mathbf{a} = (1^4, 2)$, we have

$$r_5(1^4, 2; n^2) = \sum_{d|n} \mu(d) d \left(\frac{2}{d}\right) \{8\sigma_3(n/d) - 128\sigma_3(n/4d)\}.$$

Writing $n = 2^{\lambda_2} m$, $m = \prod_{\substack{p|m \\ p \geq 3}} p^{\lambda_p}$, the above formula becomes, when $\lambda_2 \geq 1$,

$$r_5(1^4, 2; n^2) = \sum_{d|n} \mu(d) d \left(\frac{2}{d}\right) \{8\sigma_3(n/d) - 128\sigma_3(n/4d)\}$$

$$= \left(8\sigma_3(2^{\lambda}) - 128\sigma_2(2^{\lambda-2})\right) \sum_{d|m} \mu(d) \left(\frac{2}{d}\right) d\sigma_3(m/d),$$

$$= 24 \left(\frac{2^{3\lambda_2+1} + 5}{2^3 - 1}\right) s_2(m),$$

where $s_2(m)$ is given by (10). When $\lambda_2 = 0$, i.e., when n is odd, we only get the factor 8 (outside the product) in the last equation. This is exactly the formula obtained in [6, Theorem 1.2]. Next we consider the case $\mathbf{a} = (1^4, , 3)$. Writing $n = 2^{\lambda_2} 3^{\lambda_3} m$, with $\gcd(m, 6) = 1$, we have

$$r_5(1^4, 3; n^2) = \sum_{d|n} \mu(d) d \left(\frac{3}{d}\right) \{8\sigma_3(n/d) - 32\sigma_3(n/2d) + 72\sigma_3(n/3d) - 288\sigma_3(n/6d)\}$$

$$= 8\left(\sigma_3(2^{\lambda_2}) - 4\sigma_3(2^{\lambda_2-1})\right) \left(\sigma_3(3^{\lambda_3}) + 9\sigma_3(3^{\lambda_3-1})\right) \sum_{d|m} \mu(d) \left(\frac{3}{d}\right) d\sigma_3(m/d)$$

$$= 8 \left(\frac{2^{3\lambda_2+2} + 3}{2^3 - 1}\right) \left(\frac{36 \times 3^{3\lambda_3} - 10}{3^3 - 1}\right) s_3(m),$$

where $s_3(m)$ is given by (11). The last expression is nothing but Theorem 1.3 of [6]. Now we take $\mathbf{a} = (1^3, 2, 3)$. Writing n as above with $\gcd(m, 6) = 1$, we get

$$r_5(1^3, 2, 3; n^2) = \sum_{d|n} \mu(d) d \left(\frac{6}{d}\right) \{6\sigma_3(n/d) - 12\sigma_3(n/2d) - 54\sigma_3(n/3d) + 96\sigma_3(n/4d)$$

$$+ 108\sigma_3(n/6d) - 864\sigma_3(n/12d)\}$$

$$= 6 \left(\sigma_3(2^{\lambda_2}) - 2\sigma_3(2^{\lambda_2-1}) + 16\sigma_3(2^{\lambda_2-2})\right) \left(\sigma_3(3^{\lambda_3}) - 9\sigma_3(3^{\lambda_3-1})\right)$$

$$\times \sum_{d|m} \mu(d) \left(\frac{6}{d}\right) d\, \sigma_3(m/d).$$

Simplifying the above expression, we get

$$r_5(1^3, 2, 3; n^2) = 12 \left|\frac{2^{3\lambda_2+3} - 15}{2^3 - 1}\right| \left(\frac{3^{3\lambda_3+2} + 4}{3^3 - 1}\right) s_4(m),$$

where $s_4(m)$ is given by (12). The above formula is nothing but Theorem 1.4 of [6].

4.1.2. **Data for the remaining cases in Table 1.** In Table 1 there are 54 cases mentioned, out of which we have given explicit formulas for the 18 cases in Corollary 4.1. In this section, we obtain the remaining 36 formulas. In the next table, we give the explicit constants $\lambda_{5,1;j}(\mathbf{a})$ that appear in (3). These constants are with respect to the corresponding basis elements attached to each vector $\mathbf{a}$. Using these constants, one gets explicit formulas for the remaining 36 cases.

The first table gives the linear combination coefficients for the three forms $(1^3, 2, 4)$, $(1^2, 2, 4)$, $(1, 2, 4^3)$. We use the basis for $M_4(8)$ given in Table 2, which has dimension 5. The constants are denoted as $\lambda_{4,8;j}$, $1 \leq j \leq 5$.

Table 4

$\mathbf{a}$	$\lambda_{4,8;1}$	$\lambda_{4,8;2}$	$\lambda_{4,8;3}$	$\lambda_{4,8;4}$	$\lambda_{4,8;5}$
$(1^3, 2, 4)$	$\frac{1}{60}$	$\frac{-1}{60}$	$\frac{1}{30}$	$\frac{-8}{15}$	2
$(1^2, 2, 4^2)$	$\frac{1}{120}$	$\frac{-1}{120}$	$\frac{1}{30}$	$\frac{-8}{15}$	2
$(1, 2, 4^3)$	$\frac{1}{240}$	$\frac{-1}{240}$	$\frac{1}{30}$	$\frac{-8}{15}$	1

Formulas for $r_5(\mathbf{a}; n^2)$ using the above table are given in the following corollary.

Corollary 4.2. *For a natural number* $n = 2^{\lambda_2} m$, *where* m *is an odd positive integer, we have*

$$r_5(1^3, 2, 4; n^2) = c_{\lambda_2} s_2(m) + 2C_1(m), \tag{17}$$

$$r_5(1^2, 2, 4^2; n^2) = c'_{\lambda_2} s_2(m) + 2C_1(m), \tag{18}$$

$$r_5(1, 2, 4^3; n^2) = c''_{\lambda_2} s_2(m) + 2C_1(m), \tag{19}$$

where $s_2(m)$ *is given by* (10) *and* $C_1(m)$ *is defined as follows:*

$$C_1(m) = \tau_{4,8}(m) \prod_{p \geq 3} \left(1 - p\left(\frac{2}{p}\right) \frac{\tau_{4,8}(m/p)}{\tau_{4,8}(m)} \right), \tag{20}$$

with $\tau_{4,8}(m)$ *being the m-th Fourier coefficient of the newform* $\Delta_{4,8}(z)$. *The constants* c_{λ_2}, c'_{λ_2} *and* c''_{λ_2} *are given in the following table:*

	$\lambda_2 = 0$	$\lambda_2 = 1$	$\lambda_2 \geq 2$
c_{λ_2}	4	32	$12\left(\dfrac{9 \times 2^{3\lambda_2 - 2} + 10}{2^3 - 1} \right)$
c'_{λ_2}	2	16	$2\left(\dfrac{13 \times 2^{3\lambda_2 - 1} + 60}{2^3 - 1} \right)$
c''_{λ_2}	1	8	$6\left(\dfrac{2^{3\lambda_2} + 20}{2^3 - 1} \right)$

For the remaining 33 cases, we use the basis $M_4(12)$ from Table 2, and in Table 5 below we give the corresponding linear combination coefficients $(\lambda_{4,12;j}, 1 \leq j \leq 9)$ in these cases.

Table 5

$\mathbf{a}$	$\lambda_{4,12;1}$	$\lambda_{4,12;2}$	$\lambda_{4,12;3}$	$\lambda_{4,12;4}$	$\lambda_{4,12;5}$	$\lambda_{4,12;6}$	$\lambda_{4,12;7}$	$\lambda_{4,12;8}$	$\lambda_{4,12;9}$
$(1^3,3^2)$	$\frac{11}{600}$	$\frac{-2}{75}$	$\frac{33}{200}$	0	$\frac{-6}{25}$	0	$\frac{8}{5}$	0	0
$(1,3^2,6^2)$	$\frac{1}{200}$	$\frac{1}{300}$	$\frac{-11}{200}$	0	$\frac{-11}{300}$	0	$\frac{4}{5}$	0	0
$(1^3,6^2)$	$\frac{1}{120}$	$\frac{-1}{60}$	$\frac{3}{40}$	0	$\frac{-3}{20}$	0	4	0	0
$(1,6^4)$	$\frac{1}{600}$	$\frac{1}{150}$	$\frac{-11}{600}$	0	$\frac{-11}{150}$	0	$\frac{8}{5}$	0	0
$(1,2^2,6^2)$	$\frac{1}{200}$	$\frac{-1}{75}$	$\frac{9}{200}$	0	$\frac{-3}{25}$	0	$\frac{4}{5}$	0	0
$(2^3,3,6)$	$\frac{1}{300}$	$\frac{1}{75}$	$\frac{-1}{50}$	0	$\frac{-2}{25}$	0	$\frac{-4}{5}$	0	0
$(2,3^3,6)$	0	0	$\frac{1}{12}$	0	$\frac{-1}{6}$	0	0	0	0
$(2,3,6^3)$	0	0	$\frac{1}{20}$	0	$\frac{-2}{15}$	0	0	0	0
$(1^2,2,3,6)$	$\frac{1}{100}$	$\frac{1}{150}$	$\frac{-3}{50}$	0	$\frac{-1}{25}$	0	$\frac{8}{5}$	0	0
$(2^3,3^2)$	$\frac{1}{120}$	$\frac{-1}{40}$	$\frac{-3}{40}$	$\frac{4}{15}$	$\frac{9}{40}$	$\frac{-12}{5}$	0	0	-2
$(2,3^4)$	0	0	$\frac{2}{15}$	0	0	$\frac{-32}{15}$	0	0	0
$(1^3,3,6)$	$\frac{1}{120}$	0	$\frac{1}{8}$	$\frac{-2}{15}$	0	-2	0	0	4
$(1,3,6^3)$	$\frac{1}{240}$	$\frac{-1}{80}$	$\frac{-17}{240}$	$\frac{2}{15}$	$\frac{17}{80}$	$\frac{-34}{15}$	0	0	1
$(1^2,2,6^2)$	$\frac{1}{120}$	$\frac{-1}{40}$	$\frac{-3}{40}$	$\frac{4}{15}$	$\frac{9}{40}$	$\frac{-12}{5}$	0	0	2
$(2,3^2,6^2)$	0	0	$\frac{1}{15}$	0	$\frac{1}{15}$	$\frac{-32}{15}$	0	0	0
$(1,2^2,3,6)$	$\frac{1}{240}$	$\frac{1}{240}$	$\frac{1}{16}$	$\frac{-2}{15}$	$\frac{1}{16}$	-2	0	0	1
$(1^2,3^3)$	$\frac{1}{100}$	$\frac{1}{25}$	$\frac{-21}{100}$	0	$\frac{-21}{25}$	0	$\frac{8}{5}$	0	0
$(2^4,3)$	$\frac{1}{150}$	0	$\frac{3}{50}$	$\frac{-8}{75}$	0	$\frac{-24}{25}$	$\frac{-8}{5}$	$\frac{-32}{5}$	0
$(2^2,3^3)$	$\frac{1}{150}$	$\frac{-1}{100}$	$\frac{-7}{50}$	$\frac{4}{75}$	$\frac{21}{100}$	$\frac{-28}{25}$	$\frac{-4}{15}$	$\frac{-128}{15}$	$\frac{-4}{3}$
$(1^2,2^2,3)$	$\frac{1}{75}$	$\frac{-1}{150}$	$\frac{3}{25}$	$\frac{-8}{75}$	$\frac{-3}{50}$	$\frac{-24}{25}$	$\frac{4}{5}$	$\frac{32}{5}$	0
$(1^2,3,6^2)$	$\frac{1}{150}$	$\frac{-1}{100}$	$\frac{-7}{50}$	$\frac{4}{75}$	$\frac{21}{100}$	$\frac{-28}{25}$	$\frac{16}{15}$	$\frac{32}{15}$	$\frac{4}{3}$
$(1^3,2,6)$	$\frac{1}{60}$	$\frac{-1}{40}$	$\frac{-3}{20}$	$\frac{2}{15}$	$\frac{9}{40}$	$\frac{-6}{5}$	0	0	2
$(1,2,6^3)$	$\frac{1}{600}$	0	$\frac{13}{200}$	$\frac{-2}{75}$	0	$\frac{-26}{25}$	$\frac{4}{15}$	$\frac{-16}{15}$	$\frac{4}{3}$
$(2^2,3,6^2)$	$\frac{1}{300}$	$\frac{-1}{150}$	$\frac{-7}{100}$	$\frac{4}{75}$	$\frac{7}{50}$	$\frac{-28}{25}$	$\frac{-4}{5}$	$\frac{-16}{5}$	0
$(1,2,3^2,6)$	$\frac{1}{300}$	$\frac{-1}{600}$	$\frac{13}{100}$	$\frac{-2}{75}$	$\frac{-13}{200}$	$\frac{-26}{25}$	$\frac{8}{15}$	$\frac{64}{15}$	0
$(1,2^3,3)$	$\frac{1}{80}$	$\frac{-3}{80}$	$\frac{-9}{80}$	$\frac{2}{5}$	$\frac{27}{80}$	$\frac{-18}{5}$	0	0	-1
$(1,2,3^3)$	$\frac{1}{200}$	0	$\frac{39}{200}$	$\frac{-2}{25}$	0	$\frac{-78}{25}$	$\frac{-8}{15}$	$\frac{32}{15}$	$\frac{4}{3}$
$(1^4,6)$	$\frac{1}{50}$	0	$\frac{9}{50}$	$\frac{-8}{25}$	0	$\frac{-72}{25}$	$\frac{16}{5}$	$\frac{-64}{5}$	0
$(1^2,6^3)$	$\frac{1}{200}$	$\frac{-3}{200}$	$\frac{-21}{200}$	$\frac{4}{25}$	$\frac{63}{200}$	$\frac{-84}{25}$	$\frac{32}{15}$	$\frac{16}{15}$	$\frac{2}{3}$
$(1^2,3^2,6)$	$\frac{1}{100}$	$\frac{-1}{50}$	$\frac{-21}{100}$	$\frac{4}{25}$	$\frac{21}{50}$	$\frac{-84}{25}$	$\frac{8}{5}$	$\frac{32}{5}$	0
$(1^2,2^2,6)$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{9}{100}$	$\frac{-8}{25}$	$\frac{9}{100}$	$\frac{-72}{25}$	$\frac{8}{5}$	$\frac{16}{5}$	0
$(2^2,3^2,6)$	$\frac{1}{200}$	$\frac{-3}{200}$	$\frac{-21}{200}$	$\frac{4}{25}$	$\frac{63}{200}$	$\frac{-84}{25}$	$\frac{-8}{15}$	$\frac{-64}{15}$	$\frac{-2}{3}$
$(1,2,3,6^2)$	$\frac{1}{400}$	$\frac{1}{400}$	$\frac{39}{400}$	$\frac{-2}{25}$	$\frac{39}{400}$	$\frac{-78}{25}$	$\frac{16}{15}$	$\frac{32}{15}$	$\frac{1}{3}$

Out of these 33 cases, 11 can be given in simpler form. These 11 cases are: $(1^3,3^2)$, $(1,3^2,6^2)$, $(1^3,6^2)$, $(1,6^4)$, $(1,2^2,6^2)$, $(2^3,3,6)$, $(2,3^3,6)$, $(2,3,6^3)$, $(1^2,2,3,6)$, $(2,3^4)$, $(2,3^2,6^2)$. We list the formulas corresponding to these cases in the following corollary.

Corollary 4.3. *Let n be a natural number and write it as $n = 2^{\lambda_2}3^{\lambda_3}m$, where $\gcd(m,6) = 1$. Then we have the following formulas (using data from Table 5):*

$$r_5(\mathbf{a}; n^2) = c_{\mathbf{a},\lambda_2} d_{\lambda_3} s_1(m) + e_{\mathbf{a}} \tau_{4,6}(2^{\lambda_2} 3^{\lambda_3}) C_2(m), \tag{21}$$

$$r_5(2, 3^4; n^2) = 96 \left(\frac{2^{3\lambda_2+1}+5}{2^3-1} \right) \sigma_3(3^{\lambda_3-1}) s_2(m), \tag{22}$$

$$r_5(2, 3^2, 6^2; n^2) = 80 \left(\frac{2^{3\lambda_2}+6}{2^3-1} \right) \sigma_3(3^{\lambda_3-1}) s_2(m), \tag{23}$$

where $s_1(m)$, $s_2(m)$ are defined by (9), (10) and for $m \geq 1$, $\gcd(m, 6) = 1$, $C_2(m)$ is defined by

$$C_2(m) = \tau_{4,6}(m) \prod_{p \geq 5} \left(1 - p \frac{\tau_{4,6}(m/p)}{\tau_{4,6}(m)} \right). \tag{24}$$

The constants $c_{\mathbf{a},\lambda_2}$, d_{λ_3} and $e_{\mathbf{a}}$ for the 9 vectors $\mathbf{a}$ are given in the table below.

$\mathbf{a}$	$c_{\mathbf{a},\lambda_2}$	d_{λ_3}	$e_{\mathbf{a}}$
$(1^3, 3^2)$	$\frac{4}{5} \left(\frac{9 \times 2^{3\lambda_2+3}+5}{2^3-1} \right)$	$\left(\frac{2 \times 3^{3\lambda_3+2}-5}{3^3-1} \right)$	$\frac{8}{5}$
$(1, 3^2, 6^2)$	$\frac{2}{5} \left(\frac{13 \times 2^{3\lambda_2+1}-5}{2^3-1} \right)$	$\left(\frac{16 \times 3^{3\lambda_3}+10}{3^3-1} \right)$	$\frac{4}{5}$
$(1^3, 6^2)$	$2 \left(\frac{3 \times 2^{3\lambda_2+1}+1}{2^3-1} \right)$	$\left(\frac{4 \times 3^{3\lambda_3+2}-10}{3^3-1} \right)$	4
$(1, 6^4)$	$\frac{2}{5} \left(\frac{3 \times 2^{3\lambda_2+2}-5}{2^3-1} \right)$	$\left(\frac{16 \times 3^{3\lambda_3}+10}{3^3-1} \right)$	$\frac{8}{5}$
$(1, 2^2, 6^2)$	$\frac{2}{5} \left(\frac{2^{3\lambda_2+4}+5}{2^3-1} \right)$	$\left(\frac{4 \times 3^{3\lambda_3+2}-10}{3^3-1} \right)$	$\frac{4}{5}$
$(2^3, 3, 6)$	$\frac{4}{5} \left(\frac{3 \times 2^{3\lambda_2+2}-5}{2^3-1} \right)$	$\left(\frac{7 \times 3^{3\lambda_3+1}+5}{3^3-1} \right)$	$\frac{-4}{5}$
$(2, 3^3, 6)$	$20 \left(\frac{3 \times 2^{3\lambda_2+1}+1}{2^3-1} \right)$	$\sigma_3(3^{\lambda_3-1})$	0
$(2, 3, 6^3)$	$4 \left(\frac{2^{3\lambda_2+4}+5}{2^3-1} \right)$	$\sigma_3(3^{\lambda_3-1})$	0
$(1^2, 2, 3, 6)$	$\frac{4}{5} \left(\frac{13 \times 2^{3\lambda_2+1}-5}{2^3-1} \right)$	$\left(\frac{11 \times 3^{3\lambda_3+1}-7}{3^3-1} \right)$	$\frac{8}{5}$

Note: In the above corollary, $\lambda_3 \geq 1$ when $\mathbf{a} = (2, 3^3, 6)$ and $(2, 3, 6^3)$.

4.2. **Septenary forms.** In this section we discuss the septenary case $\ell = 7$. We find formulas when the coefficients a_i belong to the set $\{1, 2, 4\}$. There are 27 forms corresponding to the choice of a_i's in this set. Let $i_1 + i_2 + i_3 = 7$. i.e., the coefficient 1 appears i_1 times, 2 appears i_2 times and 4 appears i_3 times. The theta series associated to these 27 forms are given by $\Theta^{i_1}(z) \Theta^{i_2}(2z) \Theta^{i_3}(4z)$. They belong to the one of the spaces $M_{7/2}(8)$, $M_{7/2}(8, \chi_2)$, $M_{7/2}(16)$, $M_{7/2}(16, \chi_2)$ according as $i_3 = 0, i_2$ even; $i_3 = 0, i_2$ odd; $i_3 \neq 0, i_2$

even; and $i_3 \neq 0, i_2$ odd respectively. Since $(\ell - 1)/2$ is odd in this case, we take $D = -4$ and apply the Shimura map $\mathcal{S}_D$ on these theta products and by using Theorem 2.1 the images of these functions are in $M_6(4)$ or $M_6(8)$ according as $i_3 = 0$ or not. We now use the following basis for these spaces:

<p align="center">List of basis for $M_6(4)$ and $M_6(8)$</p>

Space	Basis elements	Dimension
$M_6(4)$	$E_6(z), E_6(2z), E_6(4z), \Delta_{6,4}(z)$	4
$M_6(8)$	$E_6(z), E_6(2z), E_6(4z), E_6(8z), \Delta_{6,4}(z), \Delta_{6,4}(2z), \Delta_{6,8}(z)$	7

The newforms $\Delta_{k,N}(z)$ that appear in the above table are defined below.

$$\Delta_{6,4}(z) = \eta^{12}(2z) = \sum_{n \geq 1} \tau_{6,4}(n)q^n,$$

$$\Delta_{6,8}(z) = \frac{4}{45}\left[\frac{1}{5376}E_6(z) + \frac{5}{1792}E_6(2z) + \frac{5}{112}E_6(4z) + \frac{20}{21}E_6(8z) - \frac{405}{32}\Delta_{6,4}(z) \right.$$
$$\left. - \frac{135}{2}\Delta_{6,4}(2z) - E_2(z)E_4(8z) + 3DE_4(8z) \right] = \sum_{n \geq 1} \tau_{6,8}(n)q^n.$$

We have another expression for $\Delta_{6,8}(z)$, which we give below.

$$\Delta_{6,8}(z) = \frac{-1}{90}\left[\frac{5}{336}E_6(z) + \frac{5}{112}E_6(2z) + \frac{5}{28}E_6(4z) + \frac{16}{21}E_6(8z) + \frac{135}{2}\Delta_{6,4}(z) \right.$$
$$\left. + 810\Delta_{6,4}(2z) - E_2(8z)E_4(z) + \frac{3}{8}DE_4(z) \right].$$

In the above, Df denotes the derivative of the modular form f with respect to z, which is given by $\frac{1}{2\pi i}f'(z)$. It is known that $Df(z) - \frac{k}{12}E_2(z)f(z)$ is a modular form of weight $k + 2$, where f is a modular form of weight k and $E_2(z) = 1 - 24\sum_{n \geq 1}\sigma(n)q^n$ is the normalised Eisenstein series of weight 2 (which is a quasimodular form), where $\sigma(n)$ is the sum of the positive divisors of n.

We use the basis elements given in the above table to obtain the following explicit expressions for $\mathcal{S}_{-4}(\Theta_{\mathbf{a}}(z))$, where $\mathbf{a} = (1^{i_1}, 2^{i_2}, 4^{i_3})$, $i_1 + i_2 + i_3 = 7$, with either $i_3 = 0$ or $i_3 \neq 0$, i_2 is even. There are 18 vectors $\mathbf{a}$ with these conditions.

<p align="center">**Table 6**</p>

$\mathbf{a}$	$\mathcal{S}_{-4}(\Theta_{\mathbf{a}})(z)$	$\mathbf{a}$	$\mathcal{S}_{-4}(\Theta_{\mathbf{a}})(z)$
$(1^6, 2)$	$\frac{-31}{42}E_6(z) - \frac{16}{21}E_6(2z)$	$(1^5, 4^2)$	$\frac{-47}{252}E_6(z) - \frac{4}{63}E_6(2z)$
$(1^5, 2^2)$	$\frac{-127}{252}E_6(z) + \frac{16}{63}E_6(2z)$	$(1^3, 4^4), (1^2, 4^5),$	
$(1^4, 2^3)$	$\frac{-5}{14}E_6(z) - \frac{8}{7}E_6(2z)$	$(1, 4^6), (1, 2^2, 4^4)$	$\frac{-1}{36}E_6(z) - \frac{2}{9}E_6(2z)$
$(1^3, 2^4),$		$(1^3, 2^2, 4^2),$	
$(1^4, 2^2, 4)$	$\frac{-1}{4}E_6(z)$	$(1^2, 2^4, 4), (1, 2^6)$	$\frac{-31}{252}E_6(z) - \frac{8}{63}E_6(2z)$
$(1^2, 2^5)$	$\frac{-1}{6}E_6(z) - \frac{4}{3}E_6(2z)$	$(1^2, 2^2, 4^3),$	
$(1^6, 4)$	$\frac{-127}{252}E_6(z) + \frac{16}{63}E_6(2z)$	$(1, 2^4, 4^2), (1^4, 4^3)$	$\frac{-5}{84}E_6(z) - \frac{4}{21}E_6(2z)$

Using the above data, we deduce some of the results and identities proved in [7]. Put $n = 2^k m$, $k \geq 0, m \geq 1$ are integers, m odd. Comparing the Fourier coefficients and simplifying (using the multiplicative property of the divisor function $\sigma_5(n)$) as done earlier, we get the following formulas as corollary to our main theorem. When n is even, the formulas given below for $r_7(\mathbf{a}; 4n^2)$ (derived from our method) are the same 18 formulas obtained in Theorems 2.1 to 2.3 in [7].

Corollary 4.4. *For an integer $k \geq 1$, we have the following identities:*

$$r_7(1^6, 2; 2^{2k}m^2) = 12\left(\frac{2^{5k+5} - 63}{2^5 - 1}\right) s_5(m), \tag{25}$$

$$r_7(1^5, 2^2; 2^{2k}m^2) = r_7(1^6, 4; 2^{2k}m^2) = \left(\frac{250 \times 2^{5k} - 126}{2^5 - 1}\right) s_6(m), \tag{26}$$

$$r_7(1^4, 2^3; 2^{2k}m^2) = \left(\frac{198 \times 2^{5k} - 756}{2^5 - 1}\right) s_5(m), \tag{27}$$

$$r_7(1^3, 2^4; 2^{2k}m^2) = r_7(1^4, 2^2, 4; 2^{2k}m^2) = 126\left(\frac{2^{5k} - 1}{2^5 - 1}\right) s_6(m), \tag{28}$$

$$r_7(1^2, 2^5; 2^{2k}m^2) = \left(\frac{105 \times 2^{5k} - 756}{2^5 - 1}\right) s_5(m), \tag{29}$$

$$r_7(1^5, 4^2; 2^{2k}m^2) = \left(\frac{95 \times 2^{5k} - 126}{2^5 - 1}\right) s_6(m), \tag{30}$$

$$r_7(1^3, 4^4; 2^{2k}m^2) = r_7(1^2, 4^5; 2^{2k}m^2) = r_7(1, 4^6; 2^{2k}m^2)$$
$$= r_7(1, 2^2, 4^4; 2^{2k}m^2) = \left(\frac{35 \times 2^{5k-1} - 126}{2^5 - 1}\right) s_6(m), \tag{31}$$

$$r_7(1^3, 2^2, 4^2; 2^{2k}m^2) = r_7(1^2, 2^4, 4; 2^{2k}m^2) = r_7(1, 2^6; 2^{2k}m^2)$$
$$= \left(\frac{2^{5k+6} - 126}{2^5 - 1}\right) s_6(m), \tag{32}$$

$$r_7(1^2, 2^2, 4^3; 2^{2k}m^2) = r_7(1, 2^4, 4^2; 2^{2k}m^2) = r_7(1^4, 4^3; 2^{2k}m^2)$$
$$= \left(\frac{33 \times 2^{5k} - 126}{2^5 - 1}\right) s_6(m), \tag{33}$$

where $s_5(m)$ and $s_6(m)$ are given by (13) and (14).

Remark 4.3. Formulas (25), (27), (29) given above are exactly the same as in Lemma 3.1 of [7], except for the fact that $r_7(1^6, 2; m^2) = 12s_5(m)$ (see Remark 4.7). Formulas (32) ($\mathbf{a} = (1, 2^6)$), (26) ($\mathbf{a} = (1^6, 4)$), (31) ($\mathbf{a} = (1, 4^6)$) are the same as in Lemmas 4.1, 4.2 and Lemma 5.1 of [7], except for the fact that $r_7(m^2) = 14s_6(m)$. We also note that the same formula for $r_7(n^2)$ can be derived using the Shimura correspondence and this fact was observed in [9, §3, p.372]. We also note that the above corollary proves all the results stated in Theorems 2.1, 2.2, 2.3 of [7], when n is an even integer.

4.2.1. **Remaining 9 cases.** We now consider the remaining 9 formulas, where $i_3 \neq 0$ and i_2 is odd and obtain new formulas. We again take $D = -4$ and use Theorem 2.1 to obtain the following explicit images under the Shimura lifting:

$$\mathcal{S}_{-4}\left(\Theta(z)\Theta(2z)\Theta^5(4z)\right) = \frac{-1}{42}E_6(z) + \frac{1}{21}E_6(2z) - \frac{32}{21}E_6(4z),$$

$$\mathcal{S}_{-4}\left(\Theta(z)\Theta^3(2z)\Theta^3(4z)\right) = \mathcal{S}_{-4}\left(\Theta^2(z)\Theta(2z)\Theta^4(4z)\right)$$

$$= \frac{-1}{21}E_6(z) + \frac{1}{14}E_6(2z) - \frac{32}{21}E_6(4z) - 4\Delta_{6,4}(z),$$

$$\mathcal{S}_{-4}\left(\Theta(z)\Theta^5(2z)\Theta(4z)\right) = \mathcal{S}_{-4}\left(\Theta^2(z)\Theta^3(2z)\Theta^2(4z)\right)$$

$$= \frac{-2}{21}E_6(z) + \frac{5}{42}E_6(2z) - \frac{32}{21}E_6(4z) - 4\Delta_{6,4}(z),$$

$$\mathcal{S}_{-4}\left(\Theta^3(z)\Theta(2z)\Theta^3(4z)\right) = \frac{-2}{21}E_6(z) + \frac{5}{42}E_6(2z) - \frac{32}{21}E_6(4z) - 12\Delta_{6,4}(z),$$

$$\mathcal{S}_{-4}\left(\Theta^3(z)\Theta^3(2z)\Theta(4z)\right) = \frac{-4}{21}E_6(z) + \frac{3}{14}E_6(2z) - \frac{32}{21}E_6(4z) - 4\Delta_{6,4}(z),$$

$$\mathcal{S}_{-4}\left(\Theta^4(z)\Theta(2z)\Theta^2(4z)\right) = \frac{-4}{21}E_6(z) + \frac{3}{14}E_6(2z) - \frac{32}{21}E_6(4z) - 20\Delta_{6,4}(z),$$

$$\mathcal{S}_{-4}\left(\Theta^5(z)\Theta(2z)\Theta(4z)\right) = \frac{-8}{21}E_6(z) + \frac{17}{42}E_6(2z) - \frac{32}{21}E_6(4z) - 20\Delta_{6,4}(z).$$

From the above, we deduce the following 9 formulas which are new (and not obtained in the work of Cooper-Lam-Ye [7]). Write $n = 2^{\lambda_2}m$, $\lambda_2 \geq 1$, $2 \nmid m$. Let $m = \prod_{p \text{ odd}} p^{\lambda_p}$, $\lambda_p \geq 0$ and let $s_5(m)$ be as defined in (13). It is a fact that $\tau_{6,4}(n) = 0$, when n is an even positive integer and so, we define the function $C_3(m)$ as follows:

$$C_3(m) = \tau_{6,4}(m)\prod_{p \geq 3}\left(1 - p^2\left(\frac{-2}{p}\right)\frac{\tau_{6,4}(m/p)}{\tau_{6,4}(m)}\right). \tag{34}$$

Using the multiplicative property of $\sigma_5(n)$ and $\tau_{6,4}(n)$, we get the following new formulas (as a consequence of the explicit Shimura images stated above):

Corollary 4.5. *For a natural number $n = 2^{\lambda_2}m$, $\lambda_2 \geq 1$, $m \geq 1$ odd, we have*

$$r_7(1, 2, 4^5; n^2) = 12\left|\frac{2^{5\lambda_2} - 63}{2^5 - 1}\right| s_5(m), \tag{35}$$

$$r_7(1, 2^3, 4^3; n^2) = r_7(1^2, 2, 4^4; n^2) = \begin{cases} 24s_5(m) - 4C_3(m) & \text{if } \lambda_2 = 1, \\ 756\left(\frac{2^{5\lambda_2-5} - 1}{2^5 - 1}\right)s_5(m) & \text{if } \lambda_2 \geq 2, \end{cases} \tag{36}$$

$$r_7(1, 2^5, 4; n^2) = r_7(1^2, 2^3, 4^2; n^2) = \begin{cases} 48s_5(m) - 4C_3(m) & \text{if } \lambda_2 = 1, \\ \dfrac{375 \times 2^{5\lambda_2-3} - 756}{2^5 - 1}s_5(m) & \text{if } \lambda_2 \geq 2, \end{cases} \tag{37}$$

$$r_7(1^3, 2, 4^3; n^2) = \begin{cases} 48s_5(m) - 12C_3(m) & \text{if } \lambda_2 = 1, \\ \dfrac{375 \times 2^{5\lambda_2 - 3} - 756}{2^5 - 1} s_5(m) & \text{if } \lambda_2 \geq 2, \end{cases} \tag{38}$$

$$r_7(1^3, 2^3, 4; n^2) = \begin{cases} 96s_5(m) - 4C_3(m) & \text{if } \lambda_2 = 1, \\ \dfrac{747 \times 2^{5\lambda_2 - 3} - 756}{2^5 - 1} s_5(m) & \text{if } \lambda_2 \geq 2, \end{cases} \tag{39}$$

$$r_7(1^4, 2, 4^2; n^2) = \begin{cases} 96s_5(m) - 20C_3(m) & \text{if } \lambda_2 = 1, \\ \dfrac{747 \times 2^{5\lambda_2 - 3} - 756}{2^5 - 1} s_5(m) & \text{if } \lambda_2 \geq 2, \end{cases} \tag{40}$$

$$r_7(1^5, 2, 4; n^2) = \begin{cases} 192s_5(m) - 20C_3(m) & \text{if } \lambda_2 = 1, \\ \dfrac{2982 \times 2^{5\lambda_2 - 4} - 756}{2^5 - 1} s_5(m) & \text{if } \lambda_2 \geq 2. \end{cases} \tag{41}$$

Remark 4.4. As mentioned earlier, we could use only the Shimura map $\mathcal{S}_{-4}$ in the case when $\ell = 7$ (more generally $\ell \equiv 3 \pmod{4}$) in order to apply Theorem 2.1. So, we could get formulas only for $4n^2$ instead of n^2. In this connection, we refer to the concluding remarks in [7], where they also mention that these formulas (for $4n^2$) can be obtained using methods of [5]. However, explicit formulas for these cases have not been given in their work. The above corollary gives explicit formulas in this case and they are new. In the next section, we determine these formulas for n^2 for any natural number $n \geq 1$ (under an assumption).

Remark 4.5. Using the formulas given in the above corollary, we obtain several relations connecting $r_7(\mathbf{a}; n^2)$ and $C_3(m)$. Some of them are listed in the following Proposition. It is possible that these relations can be obtained using different methods.

Proposition 4.6. *Let $n \geq 1$ be an integer and m be the odd part of n. We have the following relations, which arise using the identities listed in Corollary 4.5. When $n \equiv 0 \pmod{4}$, we have*

$$r_7(1, 2^5, 4; n^2) = r_7(1^2, 2^3, 4^2; n^2) = r_7(1^3, 2, 4^3; n^2), \tag{42}$$
$$r_7(1^3, 2^3, 4; n^2) = r_7(1^4, 2, 4^2; n^2). \tag{43}$$

When $n \equiv 2 \pmod{4}$, we have

$$4C_3(m) = r_7(1, 2^5, 4; n^2) - 2\, r_7(1, 2^3, 4^3; n^2) = r_7(1^2, 2^3, 4^2; n^2) - 2\, r_7(1, 2^3, 4^3; n^2)$$
$$= r_7(1, 2^5, 4; n^2) - 2\, r_7(1^2, 2, 4^4; n^2) = r_7(1^2, 2^3, 4^2; n^2) - 2\, r_7(1^2, 2, 4^4; n^2), \tag{44}$$

$$8C_3(m) = r_7(1, 2^5, 4; n^2) - r_7(1^3, 2, 4^3; n^2) = r_7(1^2, 2^3, 4^2; n^2) - 2r_7(1^3, 2, 4^3; n^2), \tag{45}$$

$$r_7(1, 2^5, 4; n^2) = 4\big(r_7(1^2, 2, 4^4; n^2) - r_7(1^3, 2, 4^3; n^2)\big), \tag{46}$$

where $C_3(m)$ is given by (34).

4.2.2. **Conjectural formulas.** In this section, we assume that the Shimura map $\mathcal{S}_{-1}$ maps the space $M_{k+1/2}(4N, \chi)$ into the space $M_{2k}(2N)$, where χ is a quadratic Dirichlet character modulo $4N$. All the results presented in this section are based on this assumption. Therefore, it implies that $\mathcal{S}_{-1}(\Theta^{i_1}(z)\Theta^{i_2}(2z)\Theta^{i_3}(4z)) \in M_6(8)$, where $(1^{i_1}, 2^{i_2}, 4^{i_3}) \in \{(1, 2, 4^5), (1, 2^3, 4^3), (1^2, 2, 4^4), (1, 2^5, 4), (1^2, 2^3, 4^2), (1^3, 2, 4^3), (1^3, 2^3, 4), (1^4, 2, 4^2), (1^5, 2, 4)\}$ and the explicit images of these functions are given below.

$$\mathcal{S}_{-1}\left(\Theta(z)\Theta(2z)\Theta^5(4z)\right) = \frac{-1}{1344}E_6(z) + \frac{1}{1344}E_6(2z) + \frac{1}{42}E_6(4z) - \frac{32}{21}E_6(8z)$$
$$+ \frac{5}{8}\Delta_{6,4}(z) + \Delta_{6,8}(z),$$

$$\mathcal{S}_{-1}\left(\Theta(z)\Theta^3(2z)\Theta^3(4z)\right) = \frac{-1}{672}E_6(z) + \frac{1}{672}E_6(2z) + \frac{1}{42}E_6(4z) - \frac{32}{21}E_6(8z)$$
$$+ \frac{1}{4}\Delta_{6,4}(z) - 4\Delta_{6,4}(2z) + \Delta_{6,8}(z),$$

$$\mathcal{S}_{-1}\left(\Theta^2(z)\Theta(2z)\Theta^4(4z)\right) = \frac{-1}{672}E_6(z) + \frac{1}{672}E_6(2z) + \frac{1}{42}E_6(4z) - \frac{32}{21}E_6(8z)$$
$$+ \frac{5}{4}\Delta_{6,4}(z) - 4\Delta_{6,4}(2z) + 2\Delta_{6,8}(z),$$

$$\mathcal{S}_{-1}\left(\Theta(z)\Theta^5(2z)\Theta(4z)\right) = \frac{-1}{336}E_6(z) + \frac{1}{336}E_6(2z) + \frac{1}{42}E_6(4z) - \frac{32}{21}E_6(8z)$$
$$+ \frac{1}{2}\Delta_{6,4}(z) - 4\Delta_{6,4}(2z),$$

$$\mathcal{S}_{-1}\left(\Theta^2(z)\Theta^3(2z)\Theta^2(4z)\right) = \frac{-1}{336}E_6(z) + \frac{1}{336}E_6(2z) + \frac{1}{42}E_6(4z) - \frac{32}{21}E_6(8z)$$
$$+ \frac{1}{2}\Delta_{6,4}(z) - 4\Delta_{6,4}(2z) + 2\Delta_{6,8}(z),$$

$$\mathcal{S}_{-1}\left(\Theta^3(z)\Theta(2z)\Theta^3(4z)\right) = \frac{-1}{336}E_6(z) + \frac{1}{336}E_6(2z) + \frac{1}{42}E_6(4z) - \frac{32}{21}E_6(8z)$$
$$- \frac{3}{2}\Delta_{6,4}(z) - 12\Delta_{6,4}(2z) + 3\Delta_{6,8}(z),$$

$$\mathcal{S}_{-1}\left(\Theta^3(z)\Theta^3(2z)\Theta(4z)\right) = \frac{-1}{168}E_6(z) + \frac{1}{168}E_6(2z) + \frac{1}{42}E_6(4z) - \frac{32}{21}E_6(8z)$$
$$+ \Delta_{6,4}(z) - 4\Delta_{6,4}(2z) + 2\Delta_{6,8}(z),$$

$$\mathcal{S}_{-1}\left(\Theta^4(z)\Theta(2z)\Theta^2(4z)\right) = \frac{-1}{168}E_6(z) + \frac{1}{168}E_6(2z) + \frac{1}{42}E_6(4z) - \frac{32}{21}E_6(8z)$$
$$+ \Delta_{6,4}(z) - 20\Delta_{6,4}(2z) + 4\Delta_{6,8}(z),$$

$$\mathcal{S}_{-1}\left(\Theta^5(z)\Theta(2z)\Theta(4z)\right) = \frac{-1}{84}E_6(z) + \frac{1}{84}E_6(2z) + \frac{1}{42}E_6(4z) - \frac{32}{21}E_6(8z)$$
$$- 20\Delta_{6,4}(2z) + 4\Delta_{6,8}(z).$$

As done before, we can simplify the above formulas. We write $n = 2^{\lambda_2}m$, with $\lambda_2 \geq 0$, $2 \nmid m$ and m is expressed as before. Let $s_5(m)$ be as defined in (13). In this case $\tau_{6,8}(n)$

appear and they are zero when n is even. For $m \geq 1$ odd, let $C_3(m)$ be as defined by (34) and we let

$$C_4(m) = \tau_{6,8}(m) \prod_{p \geq 3} \left(1 - p^2 \left(\frac{-2}{p} \right) \frac{\tau_{6,8}(m/p)}{\tau_{6,8}(m)} \right). \tag{47}$$

Using the above explicit expressions given in terms of the basis elements of $M_6(8)$ and using the multiplicative properties of $\sigma_5(n)$, $\tau_{6,4}(n)$, $\tau_{6,8}(n)$, the following (conjectural) formulas are obtained.

Conjectural formulas. Let $m \geq 1$ be an odd integer. Then, we have

$$r_7(1, 2, 4^5; m^2) = \frac{3}{8} s_5(m) + \frac{5}{8} C_3(m) + C_4(m), \tag{48}$$

$$r_7(1, 2^3, 4^3; m^2) = \frac{3}{4} s_5(m) + \frac{1}{4} C_3(m) + C_4(m), \tag{49}$$

$$r_7(1^2, 2, 4^4; m^2) = \frac{3}{4} s_5(m) + \frac{5}{4} C_3(m) + 2C_4(m), \tag{50}$$

$$r_7(1, 2^5, 4; m^2) = \frac{3}{2} s_5(m) + \frac{1}{2} C_3(m), \tag{51}$$

$$r_7(1^2, 2^3, 4^2; m^2) = \frac{3}{2} s_5(m) + \frac{1}{2} C_3(m) + 2C_4(m), \tag{52}$$

$$r_7(1^3, 2, 4^3; m^2) = \frac{3}{2} s_5(m) + \frac{3}{2} C_3(m) + 3C_4(m), \tag{53}$$

$$r_7(1^3, 2^3, 4; m^2) = 3s_5(m) + C_3(m) + 2C_4(m), \tag{54}$$

$$r_7(1^4, 2, 4^2; m^2) = 3s_5(m) + C_3(m) + 4C_4(m), \tag{55}$$

$$r_7(1^5, 2, 4; m^2) = 6s_5(m) + 4C_4(m). \tag{56}$$

Remark 4.6. When n is an even positive integer, the formulas for $r_7(\mathbf{a}; n^2)$ obtained using our assumption (i.e., using the Shimura map $\mathcal{S}_{-1}$) coincide with the formulas proved in Corollary 4.5 (which was obtained using the Shimura map $\mathcal{S}_{-4}$).

Remark 4.7. Under the assumption made at the beginning of this section, we also have $\mathcal{S}_{-1}(\Theta^6(z)\Theta(2z)) \in M_6(4)$ and writing it in terms of the basis elements mentioned in §4.2, we get

$$\mathcal{S}_{-1}(\Theta^6(z)\Theta(2z)) = \frac{-1}{42} E_6(z) + \frac{1}{21} E_6(2z) - \frac{32}{21} E_6(4z).$$

Like we did earlier, writing $n = 2^\lambda m$, $\lambda \geq 0$ and $m \geq 1$ odd, the above identity can be simplified (after comparing the n-th Fourier coefficients and taking Möbius inversion) and we get the following formula (which is exactly the first formula in Theorem 2.1 of [7]):

$$r_7(1^6, 2; n^2) = 12 \left| \frac{2^{5\lambda+5} - 63}{2^5 - 1} \right| s_5(m), \tag{57}$$

where $s_5(m)$ is given by (13).

4.3. **Examples for $\ell = 9$.** In this section, we shall see some examples in the case $\ell = 9$. Here we consider three cases where the coefficients a_i belong to the sets $\{1, 2\}$, $\{1, 3\}$ and $\{1, 2, 4\}$. In the first two cases there are 8 forms each corresponding to the vectors $\mathbf{a} = (1^{i_1}, b^{i_2})$, where $b = 2, 3$ and $1 \leq i_1 \leq 8$, with $i_1 + i_2 = 9$. In the third case, there are 36 cases: $\mathbf{a} = (1^{i_1}, 2^{i_2}, 4^{i_3})$, $i_1 + i_2 + i_3 = 9$, (8 cases if $i_2 = 0$, 12 cases if $i_2, i_3 \neq 0$, i_2 is even and 16 cases if $i_2, i_3 \neq 0$, i_2 is odd). For the first two sets, the theta series corresponding to these vectors belong to the space $M_{9/2}(4b, \chi)$, where χ is the character $\chi_b^{i_2}$, $b = 2, 3$. For the third set, the theta series belongs to $M_{9/2}(16, \chi_2^{i_2})$. For the application of Theorem 2.1, we can take $t = D = 1$ and we see that the Shimura map $\mathcal{S}_1$ maps $\Theta_{\mathbf{a}}(z)$ to the space $M_8(2M)$, $M = 2, 3, 4$, respectively corresponding to the three sets. Therefore, by constructing bases for the spaces $M_8(N)$, $N = 4, 6, 8$, we derive the formulas for the number of representations of $r_9(\mathbf{a}; n^2)$ corresponding to these 52 vectors $\mathbf{a}$. Below we give the necessary details and state the formulas as corollaries.

<div align="center">List of basis for $M_8(N)$, $N = 4, 6, 8$</div>

Space	Basis elements
$M_8(4)$	$E_8(z), E_8(2z), E_8(4z), \Delta_{8,2}(z), \Delta_{8,2}(2z)$
$M_8(6)$	$E_8(z), E_8(2z), E_8(3z), E_8(6z), \Delta_{8,2}(z), \Delta_{8,2}(3z), \Delta_{8,3}(z), \Delta_{8,3}(2z), \Delta_{8,6}(z)$
$M_8(8)$	$E_8(z), E_8(2z), E_8(4z), E_8(8z), \Delta_{8,2}(z), \Delta_{8,2}(2z), \Delta_{8,2}(4z), \Delta_{8,8;1}(z), \Delta_{8,8;2}(z)$

The newforms appearing in the above table are given below.

$$\Delta_{8,2}(z) = \eta^8(z)\eta^8(2z) = \sum_{n \geq 1} \tau_{8,2}(n)q^n, \tag{58}$$

$$\Delta_{8,3}(z) = \eta^{12}(z)\eta^4(3z) + 18\eta^9(z)\eta^4(3z)\eta^3(9z) + 81\eta^6(z)\eta^4(3z)\eta^6(9z) = \sum_{n=1}^{\infty} \tau_{8,3}(n)q^n, \tag{59}$$

$$\Delta_{8,6}(z) = \frac{1}{240}\Big(E_4(z)E_4(6z) - E_4(2z)E_4(3z)\Big) = \sum_{n=1}^{\infty} \tau_{8,6}(n)q^n. \tag{60}$$

In the case of level 8, there are two newforms denoted as $\Delta_{8,8;j}(z)$, $j = 1, 2$, which are given below.

$$\Delta_{8,8;1}(z) = \frac{7}{3231360}E_8(z) - \frac{1687}{3231360}E_8(2z) - \frac{1687}{201960}E_8(4z) + \frac{224}{25245}E_8(8z) + \frac{-1385}{2244}\Delta_{8,2}(z)$$
$$+ \frac{7450}{561}\Delta_{8,2}(2z) + \frac{15680}{51}\Delta_{8,2}(4z) + \frac{128}{33}\Delta_{8,2}(z)E_4(8z) - \frac{224}{99}G_{4,8}(z), \tag{61}$$

$$\Delta_{8,8;2}(z) = -\frac{1}{489600}E_8(z) + \frac{241}{489600}E_8(2z) + \frac{241}{30600}E_8(4z) - \frac{32}{3825}E_8(8z) - \frac{77}{68}\Delta_{8,2}(z)$$
$$- \frac{446}{17}\Delta_{8,2}(2z) - \frac{4928}{17}\Delta_{8,2}(4z) + \frac{32}{15}G_{4,8}(z), \tag{62}$$

where $G_{4,8}(z)$ is defined as

$$G_{4,8}(z) = \frac{1}{240}\Big(E_4(z)E_4(8z) - E_4(2z)E_4(4z)\Big). \tag{63}$$

For $\mathbf{a} = (1^{i_1}, b^{i_2})$, $i_1 + i_2 = 9$, $b = 2, 3$, we use the above bases and express the image of $\Theta_{\mathbf{a}}(z)$ under the Shimura map $\mathcal{S}_1$. Explicit Shimura images are given in the following table (Table 7 for $b = 2$, and Table 8 for $b = 3$). When $\mathbf{a} = (1^{i_1}, 2^{i_2}, 4^{i_3})$, $i_1 + i_2 + i_3 = 9$, $i_j \geq 1$ we give the Shimura images for these vectors in Table 9 ($i_3 \neq 0$, i_2 is even, including the case $i_2 = 0$) and in Table 10 ($i_3 \neq 0$, i_2 is odd). In Table 10, we give only the linear combination coefficients $\lambda_{8,8;j}$, $1 \leq j \leq 9$. These are the constants when the Shimura image is written as a linear combination of basis elements listed above for the space $M_8(8)$.

Table 7

$\mathbf{a}$	$\mathcal{S}_1(\Theta_{\mathbf{a}})(z)$
$(1^7, 2^2)$	$\frac{13}{816}E_8(z) - \frac{23}{510}E_8(2z) + \frac{108}{17}\Delta_{8,2}(z)$
$(1^5, 2^4)$	$\frac{11}{1360}E_8(z) - \frac{19}{510}E_8(2z) + \frac{104}{17}\Delta_{8,2}(z)$
$(1^3, 2^6)$	$\frac{1}{240}E_8(z) - \frac{1}{30}E_8(2z) + 4\Delta_{8,2}(z)$
$(1, 2^8)$	$\frac{3}{1360}E_8(z) - \frac{8}{255}E_8(2z) + \frac{16}{17}\Delta_{8,2}(z)$
$(1^2, 2^7)$	$\frac{11}{4080}E_8(z) - \frac{33}{1360}E_8(2z) + \frac{1408}{255}E_8(4z) + \frac{46}{17}\Delta_{8,2}(z) + \frac{176}{17}\Delta_{8,2}(2z)$
$(1^4, 2^5)$	$\frac{11}{2040}E_8(z) - \frac{11}{408}E_8(2z) + \frac{1408}{255}E_8(4z) + \frac{92}{17}\Delta_{8,2}(z) + \frac{448}{17}\Delta_{8,2}(2z)$
$(1^6, 2^3)$	$\frac{11}{1020}E_8(z) - \frac{11}{340}E_8(2z) + \frac{1408}{255}E_8(4z) + \frac{116}{17}\Delta_{8,2}(z) + \frac{448}{17}\Delta_{8,2}(2z)$
$(1^8, 2)$	$\frac{11}{510}E_8(z) - \frac{11}{255}E_8(2z) + \frac{1408}{255}E_8(4z) + \frac{96}{17}\Delta_{8,2}(z) + \frac{1536}{17}\Delta_{8,2}(2z)$

Table 8

$\mathbf{a}$	$\mathcal{S}_1(\Theta_{\mathbf{a}})(z)$
$(1^7, 3^2)$	$\frac{91}{9840}E_8(z) + \frac{2457}{3280}E_8(3z) + \frac{392}{41}\Delta_{8,3}(z)$
$(1^5, 3^4)$	$\frac{137}{33456}E_8(z) - \frac{16}{2091}E_8(2z) - \frac{2877}{3280}E_8(3z) + \frac{336}{205}E_8(6z) + \frac{224}{51}\Delta_{8,2}(z)$ $+288\Delta_{8,2}(3z) + \frac{448}{123}\Delta_{8,3}(z) - \frac{8192}{123}\Delta_{8,3}(2z)$
$(1^3, 3^6)$	$\frac{7}{9840}E_8(z) + \frac{497}{656}E_8(3z) + \frac{232}{41}\Delta_{8,2}(z)$
$(1, 3^8)$	$\frac{137}{167280}E_8(z) - \frac{16}{10455}E_8(2z) - \frac{146179}{167280}E_8(3z) + \frac{17072}{10455}E_8(6z) + \frac{224}{255}\Delta_{8,2}(z)$ $+\frac{1568}{85}\Delta_{8,2}(3z) + \frac{448}{615}\Delta_{8,3}(z) - \frac{8192}{615}\Delta_{8,3}(2z)$
$(1^2, 3^7)$	$\frac{587}{418200}E_8(z) - \frac{33569}{313650}E_8(2z) - \frac{1106767}{418200}E_8(3z) + \frac{11106157}{156825}E_8(6z) - \frac{784}{17}\Delta_{8,2}(z)$ $-\frac{51072}{17}\Delta_{8,2}(3z) + \frac{15736}{369}\Delta_{8,3}(z) - \frac{316352}{369}\Delta_{8,3}(2z) + \frac{4144}{45}\Delta_{8,6}(z)$
$(1^4, 3^5)$	$\frac{16567}{5645700}E_8(z) - \frac{315263}{1411425}E_8(2z) - \frac{563591}{332100}E_8(3z) + \frac{9684934}{83025}E_8(6z) - \frac{55424}{459}\Delta_{8,2}(z)$ $-\frac{73952}{9}\Delta_{8,2}(3z) - \frac{130880}{1107}\Delta_{8,3}(z) - \frac{2497024}{1107}\Delta_{8,3}(2z) + \frac{33152}{135}\Delta_{8,6}(z)$
$(1^6, 3^3)$	$\frac{10423}{1254600}E_8(z) - \frac{147467}{313650}E_8(2z) - \frac{7181323}{1254600}E_8(3z) + \frac{24943411}{156825}E_8(6z) - \frac{8848}{51}\Delta_{8,2}(z)$ $-\frac{192128}{17}\Delta_{8,2}(3z) - \frac{19528}{123}\Delta_{8,3}(z) - \frac{395840}{123}\Delta_{8,3}(2z) + \frac{5104}{15}\Delta_{8,6}(z)$
$(1^8, 3)$	$\frac{49459}{2822850}E_8(z) - \frac{89542}{1411425}E_8(2z) - \frac{3071569}{2822850}E_8(3z) + \frac{146224612}{1411425}E_8(6z) - \frac{218624}{2295}\Delta_{8,2}(z)$ $-\frac{4683328}{765}\Delta_{8,2}(3z) - \frac{511808}{5535}\Delta_{8,3}(z) - \frac{9875968}{5535}\Delta_{8,3}(2z) + \frac{26368}{135}\Delta_{8,6}(z)$

Table 9

$\mathbf{a}$	$\mathcal{S}_1(\mathbf{\Theta_a})(z)$
$(1,4^8)$	$\frac{1}{4080}E_8(z) + \frac{1}{510}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{32}{17}\Delta_{8,2}(z) + \frac{288}{17}\Delta_{8,2}(2z)$
$(1^2,4^7)$	$\frac{1}{2040}E_8(z) - \frac{121}{4080}E_8(2z) + \frac{64}{17}\Delta_{8,2}(z) + 32\Delta_{8,2}(2z)$
$(1^3,4^6)$	$\frac{1}{1360}E_8(z) - \frac{25}{408}E_8(2z) + \frac{8}{255}E_8(4z) + \frac{96}{17}\Delta_{8,2}(z) + \frac{800}{17}\Delta_{8,2}(2z)$
$(1^4,4^5)$	$\frac{1}{1020}E_8(z) - \frac{251}{4080}E_8(2z) + \frac{8}{255}E_8(4z) + \frac{128}{17}\Delta_{8,2}(z) + \frac{1072}{17}\Delta_{8,2}(2z)$
$(1^5,4^4)$	$\frac{1}{680}E_8(z) + \frac{1}{1360}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{158}{17}\Delta_{8,2}(z) + \frac{1376}{17}\Delta_{8,2}(2z)$
$(1^6,4^3)$	$\frac{1}{340}E_8(z) + \frac{509}{4080}E_8(2z) - \frac{8}{51}E_8(4z) + \frac{180}{17}\Delta_{8,2}(z) + \frac{1712}{17}\Delta_{8,2}(2z)$
$(1^7,4^2)$	$\frac{7}{1020}E_8(z) + \frac{67}{272}E_8(2z) - \frac{24}{85}E_8(4z) + \frac{182}{17}\Delta_{8,2}(z) + \frac{2048}{17}\Delta_{8,2}(2z)$
$(1^8,4)$	$\frac{4}{255}E_8(z) + \frac{19}{80}E_8(2z) - \frac{24}{85}E_8(4z) + \frac{144}{17}\Delta_{8,2}(z) + \frac{2048}{17}\Delta_{8,2}(2z)$
$(1,2^2,4^6)$	$\frac{1}{4080}E_8(z) + \frac{1}{510}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{32}{17}\Delta_{8,2}(z) + \frac{288}{17}\Delta_{8,2}(2z)$
$(1,2^4,4^4)$	$\frac{1}{2040}E_8(z) + \frac{7}{4080}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{30}{17}\Delta_{8,2}(z) + \frac{288}{17}\Delta_{8,2}(2z)$
$(1,2^6,4^2)$	$\frac{1}{1020}E_8(z) + \frac{1}{816}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{26}{17}\Delta_{8,2}(z) + \frac{288}{17}\Delta_{8,2}(2z)$
$(1^2,2^2,4^5)$	$\frac{1}{2040}E_8(z) + \frac{1}{4080}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{64}{17}\Delta_{8,2}(z) + \frac{560}{17}\Delta_{8,2}(2z)$
$(1^2,2^4,4^3)$	$\frac{1}{1020}E_8(z) + \frac{1}{816}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{60}{17}\Delta_{8,2}(z) + \frac{560}{17}\Delta_{8,2}(2z)$
$(1^2,2^6,4^1)$	$\frac{1}{510}E_8(z) + \frac{1}{4080}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{52}{17}\Delta_{8,2}(z) + \frac{288}{17}\Delta_{8,2}(2z)$
$(1^3,2^2,4^4)$	$\frac{1}{1020}E_8(z) + \frac{1}{816}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{94}{17}\Delta_{8,2}(z) + \frac{832}{17}\Delta_{8,2}(2z)$
$(1^3,2^4,4^2)$	$\frac{1}{510}E_8(z) + \frac{1}{4080}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{86}{17}\Delta_{8,2}(z) + \frac{832}{17}\Delta_{8,2}(2z)$
$(1^4,2^2,4^3)$	$\frac{1}{510}E_8(z) + \frac{1}{4080}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{120}{17}\Delta_{8,2}(z) + \frac{1104}{17}\Delta_{8,2}(2z)$
$(1^4,2^4,4)$	$\frac{1}{255}E_8(z) - \frac{7}{4080}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{104}{17}\Delta_{8,2}(z) + \frac{832}{17}\Delta_{8,2}(2z)$
$(1^5,2^2,4^2)$	$\frac{1}{255}E_8(z) - \frac{7}{4080}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{138}{17}\Delta_{8,2}(z) + \frac{1376}{17}\Delta_{8,2}(2z)$
$(1^6,2^2,4)$	$\frac{2}{255}E_8(z) - \frac{23}{4080}E_8(2z) - \frac{8}{255}E_8(4z) + \frac{140}{17}\Delta_{8,2}(z) + \frac{1376}{17}\Delta_{8,2}(2z)$

Table 10

$\mathbf{a}$	$\lambda_{8,8;1}$	$\lambda_{8,8;2}$	$\lambda_{8,8;3}$	$\lambda_{8,8;4}$	$\lambda_{8,8;5}$	$\lambda_{8,8;6}$	$\lambda_{8,8;7}$	$\lambda_{8,8;8}$	$\lambda_{8,8;9}$
$(1,2,4^7)$	$\frac{11}{65280}$	$\frac{-11}{65280}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{125}{136}$	13	$\frac{1536}{17}$	$\frac{7}{8}$	$\frac{1}{8}$
$(1,2^3,4^5)$	$\frac{11}{32640}$	$\frac{-11}{32640}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{57}{68}$	10	$\frac{3712}{17}$	1	0
$(1,2^5,4^3)$	$\frac{11}{16320}$	$\frac{-11}{16320}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{23}{34}$	12	$\frac{5888}{17}$	1	0
$(1,2^7,4)$	$\frac{11}{8160}$	$\frac{-11}{8160}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{23}{17}$	16	$\frac{5888}{17}$	0	0
$(1^2,2,4^6)$	$\frac{11}{32640}$	$\frac{-11}{32640}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{125}{68}$	18	$\frac{3712}{17}$	$\frac{7}{4}$	$\frac{1}{4}$
$(1^2,2^3,4^4)$	$\frac{11}{16320}$	$\frac{-11}{16320}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{57}{34}$	20	$\frac{5888}{17}$	2	0
$(1^2,2^5,4^2)$	$\frac{11}{8160}$	$\frac{-11}{8160}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{23}{17}$	16	$\frac{5888}{17}$	2	0

Table 10 contd.

$(1^3, 2, 4^5)$	$\frac{11}{16320}$	$\frac{-11}{16320}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{91}{34}$	20	$\frac{8064}{17}$	$\frac{11}{4}$	$\frac{1}{4}$
$(1^3, 2^3, 4^3)$	$\frac{11}{8160}$	$\frac{-11}{8160}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{40}{17}$	32	$\frac{10240}{17}$	3	0
$(1^3, 2^5, 4)$	$\frac{11}{4080}$	$\frac{-11}{4080}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{46}{17}$	24	$\frac{5888}{17}$	2	0
$(1^4, 2, 4^4)$	$\frac{11}{8160}$	$\frac{-11}{8160}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{57}{17}$	24	$\frac{14592}{17}$	4	0
$(1^4, 2^3, 4^2)$	$\frac{11}{4080}$	$\frac{-11}{4080}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{46}{17}$	40	$\frac{14592}{17}$	4	0
$(1^5, 2, 4^3)$	$\frac{11}{4080}$	$\frac{-11}{4080}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{63}{17}$	40	$\frac{23296}{17}$	$\frac{11}{2}$	$\frac{-1}{2}$
$(1^5, 2^3, 4)$	$\frac{11}{2040}$	$\frac{-11}{2040}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{58}{17}$	40	$\frac{14592}{17}$	4	0
$(1^6, 2, 4^2)$	$\frac{11}{2040}$	$\frac{-11}{2040}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{58}{17}$	72	$\frac{32000}{17}$	7	-1
$(1^7, 2, 4)$	$\frac{11}{1020}$	$\frac{-11}{1020}$	$\frac{-11}{510}$	$\frac{1408}{255}$	$\frac{48}{17}$	104	$\frac{32000}{17}$	7	-1

Simplifications and some congruence relations: We now present simplified versions of some of the formulas (4 from Table 7, 2 each from Tables 8 and 9). These are corresponding to the vectors $(1^7, 2^2), (1^5, 2^4), (1^3, 2^6), (1, 2^8)$ from Table 7, $(1^7, 3), (1^3, 3^6)$ from Table 8 and $(1, 4^8), (1^2, 4^7)$ from Table 9.

For an odd positive integer m, let

$$C_5(m) = \tau_{8,2}(m) \prod_{p \geq 3} \left(1 - p^3 \frac{\tau_{8,2}(m/p)}{\tau_{8,2}(m)} \right), \tag{64}$$

$$C_6(m) = \tau_{8,3}(m) \prod_{p \geq 5} \left(1 - p^3 \frac{\tau_{8,3}(m/p)}{\tau_{8,3}(m)} \right). \tag{65}$$

We use the same notation for the prime factorisation of a natural number n, with λ_p denoting the highest power of p dividing n and m is the odd part of n. The 8 formulas are listed in the following corollary.

Corollary 4.7. *For $n = 2^{\lambda_2} m$, $m \geq 1$, we have*

$$r_9(1^7, 2^2; n^2) = \frac{2}{17} \left(\frac{1017 \times 2^{7\lambda_2+3} + 119}{2^7 - 1} \right) s_7(m) + \frac{108}{17} \tau_{8,2}(2^{\lambda_2}) \, C_5(m), \tag{66}$$

$$r_9(1^5, 2^4; n^2) = \frac{2}{17} \left(\frac{509 \times 2^{7\lambda_2+3} + 119}{2^7 - 1} \right) s_7(m) + \frac{104}{17} \tau_{8,2}(2^{\lambda_2}) C_5(m), \tag{67}$$

$$r_9(1^3, 2^6; n^2) = 2 \left(\frac{15 \times 2^{7\lambda_2+3} + 7}{2^7 - 1} \right) s_7(m) + 2\tau_{8,2}(2^{\lambda_2}) C_5(m), \tag{68}$$

$$r_9(1, 2^8; n^2) = \frac{2}{17} \left(\frac{8 \times 2^{7\lambda_2+7} + 119}{2^7 - 1} \right) s_7(m) + \frac{8}{17} \tau_{8,2}(2^{\lambda_2}) C_5(m), \tag{69}$$

$$r_9(1, 4^8; n^2) = \frac{2}{17} \left(\frac{67 \times 2^{7\lambda_2+1} + 247}{2^7 - 1} \right) s_7(m) + \frac{32}{17} \left(\tau_{8,2}(2^{\lambda_2}) + 9\tau_{8,2}(2^{\lambda_2-1}) \right) C_5(m), \tag{70}$$

$$r_9(1^2, 4^7; n^2) = \frac{2}{17} \left(\frac{135 \times 2^{7\lambda_2} + 119}{2^7 - 1} \right) s_7(m) + \left(\frac{64}{17} \tau_{8,2}(2^{\lambda_2}) + 32\tau_{8,2}(2^{\lambda_2-1}) \right) C_5(m), \tag{71}$$

where $s_7(m)$ is defined by (15) and $C_5(m)$ is defined by (64). When $n = 2^{\lambda_2} 3^{\lambda_3} m$, with $\gcd(m, 6) = 1$, we have

$$r_9(1^7, 3^2; n^2) = \frac{364}{41} \sigma_7(2^{\lambda_2}) \left(\frac{14 \times 3^{7\lambda_3+4} - 41}{3^7 - 1} \right) s_7(m) + \frac{392}{41} \tau_{8,3}(2^{\lambda_2} 3^{\lambda_3}) C_6(m), \tag{72}$$

$$r_9(1^3, 3^6; n^2) = \frac{14}{41} \sigma_7(2^{\lambda_2}) \left(\frac{1084 \times 3^{7\lambda_3+1} - 1066}{3^7 - 1} \right) s_7(m) + \frac{232}{41} \tau_{8,3}(2^{\lambda_2} 3^{\lambda_3}) C_6(m), \tag{73}$$

where $C_6(m)$ is defined by (65).

From the above formulas, one can derive certain congruences for the Fourier coefficients of the two newforms of weight 8 and levels 2 and 3.

Corollary 4.8. *Let $\lambda \geq 1$ be an integer. Then for an odd prime p, we have*

$$\tau_{8,2}(p^\lambda) \equiv \sigma_7(p^\lambda) \pmod{17}. \tag{74}$$

For odd primes $p \geq 5$, we have

$$\tau_{8,3}(p^\lambda) \equiv \sigma_7(p^\lambda) \pmod{41}. \tag{75}$$

In particular, for any integer $\lambda \geq 1$, we have

$$\tau_{8,2}(17^\lambda) \equiv 1 \pmod{17}, \tag{76}$$

$$\tau_{8,3}(41^\lambda) \equiv 1 \pmod{41}. \tag{77}$$

Remark 4.8. The cases in Table 9 correspond to forms of weight 9/2 on $\Gamma_0(16)$ with trivial character and so by the Shimura map these forms are mapped to the space $M_8(8)$ and we used a basis of this space to express the images. However, by looking at the image functions, all of them are in the subspace $M_8(4)$. This may be due to some properties, which can be explored.

5. Concluding remarks

We have mentioned that in some cases we could not get the required formulas for the number of representations of n^2 and instead we could get only the formulas for $4n^2$. For example the case when $\ell \equiv 3 \pmod 4$ and $N_{\mathbf{a}}$ is even, we only have the mapping property of the Shimura-Kohnen map. However, when we consider a square-free integer $t \equiv 3 \pmod 4$, then by taking $D = -t$, which is a negative fundamental discriminant, the application of the Shimura-Kohnen map gives rise to formulas for $r_\ell(\mathbf{a}; |t|n^2)$. So, instead of formulas for n^2 we get formulas for $|t|n^2$, where t is a square-free integer and $-t$ is a fundamental discriminant. In the case of quadratic forms of odd number of squares, only representation numbers of squares are known so far and formulas for non-square integers are not yet proved. Though there are some restrictions, the method of Shimura correspondence gives formulas for a wider class. One also gets formulas in the case when $\ell \equiv 1 \pmod 4$ (in this case we can take t to be positive square-free which is congruent to 1 modulo 4). Towards this direction, for the case $\ell = 7$, we present some formulas for the number of representations of $3n^2$ for certain coefficient vectors $\mathbf{a}$.

We consider the coefficients in the sets $\{1, 2\}$, $\{1, 3\}$ and $\{1, 2, 4\}$. By using the basis respectively for the spaces $M_6(N)$, $N = 4, 6, 8$, we obtain the following formulas. The basis elements of $M_6(4)$ and $M_6(8)$ were given in §4.2. Here we give a basis for $M_6(6)$, consisting the following 7 modular forms: $E_6(z)$, $E_6(2z)$, $E_6(3z)$, $E_6(6z)$, $\Delta_{6,3}(z)$, $\Delta_{6,3}(2z)$, $\Delta_{6,6}(z)$. To derive our formulas, we denote these bases by $f_{N,j}(z)$, $1 \le j \le \dim_{\mathbb{C}}(M_6(N))$, where $N = 4, 6, 8$ and the dimensions of the spaces are $4, 7, 7$ respectively. So, by the application of Theorem 2.1, we get the following expression (taking $D = -3$, $\ell = 7$).

$$\mathcal{S}_{-3}(\Theta_{\mathbf{a}}(z)) = \sum_j \lambda_{\mathbf{a},N,j} f_{N,j}(z), \tag{78}$$

where the constants $\lambda_{\mathbf{a},N,j}$ depend on the vector $\mathbf{a}$. By comparing the n-th Fourier coefficients and taking the Möbius inversion, we get the following formula:

$$r_7(\mathbf{a}; 3n^2) = \sum_{\substack{d|n \\ \gcd(d,2N_{\mathbf{a}})=1}} \mu(d)\psi_{\mathbf{a}}(d)d^2 \sum_j \lambda_{\mathbf{a},N,j} a_{f_{N,j}}(n/d), \tag{79}$$

where j runs from 1 to the dimension of the respective space, $a_{f_{N,j}}(n)$ denotes the n-th Fourier coefficients of the basis elements.

In the following tables, we give the explicit values of the constants $\lambda_{\mathbf{a},N,j}$, where $\mathbf{a}$ belongs to the three sets $\{1, 2\}$, $\{1, 3\}$, $\{1, 2, 4\}$.

Table 11.1 for the coefficients $\lambda_{\mathbf{a},4,j}$, $1 \leq j \leq 4$, $\mathbf{a} \in \{1,2\}$

$\rightarrow j$ $\downarrow \mathbf{a}$	1	2	3	4
$1^5, 2^2$	$\frac{-5}{21}$	$\frac{-20}{63}$	0	0
$1^3, 2^4$	$\frac{-1}{9}$	$\frac{-4}{9}$	0	0
$1, 2^6$	$\frac{-1}{21}$	$\frac{-32}{63}$	0	0

$\rightarrow j$ $\downarrow \mathbf{a}$	1	2	3	4
$1^6, 2$	$\frac{-23}{63}$	$\frac{46}{63}$	$\frac{-1472}{63}$	0
$1^4, 2^3$	$\frac{-23}{126}$	$\frac{23}{42}$	$\frac{-1472}{63}$	-12
$1^2, 2^5$	$\frac{-23}{252}$	$\frac{115}{252}$	$\frac{-1472}{63}$	-6

Table 11.2 for the coefficients $\lambda_{\mathbf{a},6,j}$, $1 \leq j \leq 7$, $\mathbf{a} \in \{1,3\}$

$\rightarrow j$ $\downarrow \mathbf{a}$	1	2	3	4	5	6	7
$1^5, 3^2$	$\frac{-1769}{11466}$	$\frac{1952}{5733}$	$\frac{783}{1274}$	$\frac{-864}{637}$	$\frac{200}{13}$	$\frac{1280}{13}$	$\frac{-64}{7}$
$1^3, 3^4$	$\frac{-50}{819}$	0	$\frac{-45}{91}$	0	$\frac{-192}{13}$	0	0
$1, 3^6$	$\frac{-29}{1638}$	$\frac{32}{819}$	$\frac{87}{182}$	$\frac{-96}{91}$	$\frac{40}{13}$	$\frac{256}{13}$	0

$\rightarrow j$ $\downarrow \mathbf{a}$	1	2	3	4	5	6	7
$1^6, 3$	$\frac{-113}{364}$	$\frac{226}{91}$	$\frac{243}{364}$	$\frac{-486}{91}$	$\frac{72}{13}$	$\frac{-576}{13}$	0
$1^4, 3^3$	$\frac{-49}{468}$	$\frac{-98}{117}$	$\frac{-9}{52}$	$\frac{-18}{13}$	$\frac{-192}{13}$	$\frac{-1536}{13}$	0
$1^2, 3^5$	$\frac{-5}{364}$	$\frac{10}{91}$	$\frac{135}{364}$	$\frac{-270}{91}$	$\frac{40}{13}$	$\frac{-320}{13}$	0

Table 11.3 for the coefficients $\lambda_{\mathbf{a},8,j}$, $1 \leq j \leq 7$, $\mathbf{a} \in \{1,2,4\}$

$\rightarrow j$ $\downarrow \mathbf{a}$	1	2	3	4	5	6	7
$1^6, 4$	$\frac{-20}{63}$	$\frac{145}{63}$	$\frac{-160}{63}$	0	0	0	0
$1^5, 4^2$	$\frac{-10}{63}$	$\frac{15}{7}$	$\frac{-160}{63}$	0	0	0	0
$1^4, 4^3$	$\frac{-4}{63}$	$\frac{65}{63}$	$\frac{-32}{21}$	0	0	0	0
$1^3, 4^4$	$\frac{-1}{63}$	$\frac{-2}{63}$	$\frac{-32}{63}$	0	0	0	0
$1^2, 4^5$	0	$\frac{-5}{9}$	0	0	0	0	0
$1, 4^6$	0	$\frac{-5}{9}$	0	0	0	0	0

$\rightarrow j$ $\downarrow \mathbf{a}$	1	2	3	4	5	6	7
$1^4, 2^2, 4$	$\frac{-8}{63}$	$\frac{5}{63}$	$\frac{-32}{63}$	0	0	0	0
$1^3, 2^2, 4^2$	$\frac{-4}{63}$	$\frac{1}{63}$	$\frac{-32}{63}$	0	0	0	0
$1^2, 2^4, 4$	$\frac{-4}{63}$	$\frac{1}{63}$	$\frac{-32}{63}$	0	0	0	0
$1^2, 2^2, 4^3$	$\frac{-2}{63}$	$\frac{-1}{63}$	$\frac{-32}{63}$	0	0	0	0
$1, 2^4, 4^2$	$\frac{-2}{63}$	$\frac{-1}{63}$	$\frac{-32}{63}$	0	0	0	0
$1, 2^2, 4^4$	$\frac{-1}{63}$	$\frac{-2}{63}$	$\frac{-32}{63}$	0	0	0	0

$\rightarrow j$ $\downarrow \mathbf{a}$	1	2	3	4	5	6	7
$1^5, 2, 4$	$\frac{-23}{126}$	$\frac{23}{126}$	$\frac{23}{63}$	$\frac{-1472}{63}$	0	120	8
$1^4, 2, 4^2$	$\frac{-23}{252}$	$\frac{23}{252}$	$\frac{23}{63}$	$\frac{-1472}{63}$	-6	120	8
$1^3, 2^3, 4$	$\frac{-23}{252}$	$\frac{23}{252}$	$\frac{23}{63}$	$\frac{-1472}{63}$	-6	24	4
$1^3, 2, 4^3$	$\frac{-23}{504}$	$\frac{23}{504}$	$\frac{23}{63}$	$\frac{-1472}{63}$	-9	72	6
$1^2, 2^3, 4^2$	$\frac{-23}{504}$	$\frac{23}{504}$	$\frac{23}{63}$	$\frac{-1472}{63}$	-3	24	4
$1^2, 2, 4^4$	$\frac{-23}{1008}$	$\frac{23}{1008}$	$\frac{23}{63}$	$\frac{-1472}{63}$	$\frac{-15}{2}$	24	4
$1, 2^5, 4$	$\frac{-23}{504}$	$\frac{23}{504}$	$\frac{23}{63}$	$\frac{-1472}{63}$	-3	24	0
$1, 2^3, 4^3$	$\frac{-23}{1008}$	$\frac{23}{1008}$	$\frac{23}{63}$	$\frac{-1472}{63}$	$\frac{-3}{2}$	24	2
$1, 2, 4^5$	$\frac{-23}{2016}$	$\frac{23}{2016}$	$\frac{23}{63}$	$\frac{-1472}{63}$	$\frac{-15}{4}$	0	2

We now give some of the formulas from the above data in explicit form.

Corollary 5.1. *For a natural number $n = 2^{\lambda_2}m$, with m odd, we have the following formulas for the number of representations.*

$$r_7(1^6, 2; 3n^2) = 184 \left| \frac{2^{5\lambda_2+5} - 63}{2^5 - 1} \right| s_5(m), \tag{80}$$

$$r_7(1^5, 2^2; 3n^2) = 40 \left(\frac{25 \times 2^{5\lambda_2+2} - 7}{2^5 - 1} \right) s_6(m), \tag{81}$$

$$r_7(1^4, 2^3; 3n^2) = \begin{cases} 92\, s_5(m) - 12 C_3(m) & \text{if } \lambda_2 = 0, \\ 92 \left(\frac{33 \times 2^{5\lambda_2} - 126}{2^5 - 1} \right) s_5(m) & \text{if } \lambda_2 \geq 1, \end{cases} \tag{82}$$

$$r_7(1^3, 2^4; 3n^2) = 56 \left(\frac{9 \times 2^{5\lambda_2+2} - 5}{2^5 - 1} \right) s_6(m), \tag{83}$$

$$r_7(1^2, 2^5; 3n^2) = \begin{cases} 46\, s_5(m) - 6 C_3(m) & \text{if } \lambda_2 = 0, \\ 46 \left(\frac{35 \times 2^{5\lambda_2} - 252}{2^5 - 1} \right) s_5(m) & \text{if } \lambda_2 \geq 1, \end{cases} \tag{84}$$

$$r_7(1, 2^6; 3n^2) = 8 \left(\frac{2^{5\lambda_2+7} - 35}{2^5 - 1} \right) s_6(m), \tag{85}$$

$$r_7(1^2, 2^2, 4^3; 3n^2) = r_7(1, 2^4, 4^2; 3n^2) = \begin{cases} 16\, s_6(m) & \text{if } \lambda_2 = 0, \\ 8 \left(\frac{66 \times 2^{5\lambda_2} - 35}{2^5 - 1} \right) s_6(m) & \text{if } \lambda_2 \geq 1, \end{cases} \tag{86}$$

$$r_7(1^3, 2^2, 4^2; 3n^2) = r_7(1^2, 2^4, 4; 3n^2) = \begin{cases} 32\, s_6(m) & \text{if } \lambda_2 = 0, \\ 8 \left(\frac{2^{5\lambda_2+7} - 35}{2^5 - 1} \right) s_6(m) & \text{if } \lambda_2 \geq 1, \end{cases} \tag{87}$$

$$r_7(1^4, 2^2, 4; 3n^2) = \begin{cases} 64\, s_6(m) & \text{if } \lambda_2 = 0, \\ 2072\, \sigma_5(2^{5\lambda_2-1}) s_6(m) & \text{if } \lambda_2 \geq 1, \end{cases} \tag{88}$$

$$r_7(1^6, 4; 3n^2) = \begin{cases} 160\, s_6(m) & \text{if } \lambda_2 = 0, \\ 40 \left(\frac{25 \times 2^{5\lambda_2+2} - 7}{2^5 - 1} \right) s_6(m) & \text{if } \lambda_2 \geq 1, \end{cases} \tag{89}$$

$$r_7(1^5, 4^2; 3n^2) = \begin{cases} 80\, s_6(m) & \text{if } \lambda_2 = 0, \\ 40 \left(\frac{19 \times 2^{5\lambda_2+1} - 7}{2^5 - 1} \right) s_6(m) & \text{if } \lambda_2 \geq 1, \end{cases} \tag{90}$$

$$r_7(1^4, 4^3; 3n^2) = \begin{cases} 32\, s_6(m) & \text{if } \lambda_2 = 0, \\ 8 \left(\frac{33 \times 2^{5\lambda_2+1} - 35}{2^5 - 1} \right) s_6(m) & \text{if } \lambda_2 \geq 1, \end{cases} \tag{91}$$

$$r_7(1^3, 4^4; 3n^2) = r_7(1^2, 4^5; 3n^2) = r_7(1, 4^6; 3n^2) = r_7(1, 2^2, 4^4; 3n^2) \tag{92}$$

$$= 280\, \sigma_5(2^{\lambda_2-1}) s_6(m), \text{ if } \lambda_2 \geq 1, \tag{93}$$

$$r_7(1^3, 4^4; 3m^2) = 35\, r_7(1, 2^2, 4^4; 3m^2) = 280\, s_6(m), \tag{94}$$

$$r_7(1^3, 3^4; 3n^2) = \frac{40}{13} \sigma_5(2^{\lambda_2}) \left(\frac{31 \times 3^{5\lambda_3+4} - 91}{3^5 - 1} \right) s_6'(m) - \frac{192}{13} \tau_{6,3}(2^{\lambda_2} 3^{\lambda_3}) C_7(m). \tag{95}$$

In the last formula, we take $n = 2^{\lambda_2} 3^{\lambda_3} m$, $\gcd(m, 6) = 1$. *The functions* $s_5(m)$ *and* $s_6(m)$ *are defined as in* (13), (14) *respectively and the functions* $s_6'(m)$ *and* $C_7(m)$ *are defined below.*

$$s_6'(m) = \prod_{p \geq 5} \left(\frac{p^{5\lambda_p + 5} - 1}{p^5 - 1} - p^2 \left(\frac{-4}{p} \right) \frac{p^{5\lambda_p} - 1}{p^5 - 1} \right), \tag{96}$$

$$C_7(m) = \tau_{6,3}(m) \prod_{p \geq 5} \left(1 - p^2 \left(\frac{-4}{p} \right) \frac{\tau_{6,3}(m/p)}{\tau_{6,3}(m)} \right). \tag{97}$$

Remark 5.1. We remark that numbers of the form $3m^2$, where m is odd are not represented by the forms corresponding to $(1^2, 4^5)$ and $(1, 4^6)$. That is $r_7(1^2, 4^5; 3m^2) = r_7(1, 4^6; 3m^2) = 0$, if m is odd. This fact can also be checked by elementary arguments. When n is even these numbers of representations (of $3n^2$) are equal to the number of representations of $3n^2$ by the form corresponding to $(1^3, 4^4)$ (see Eq.(93)). There are many relations among these representation numbers. We will mention some of them here.

$$r_7(1^6, 4; 3n^2) = r_7(1^5, 2^2; 3n^2) \text{ (if } 2|n),$$
$$r_7(1^4; 2^3; 3m^2) = 2 \; r_7(1^2, 2^5; 3m^2),$$
$$r_7(1^4, 4^3; 3m^2) = r_7(1^3, 2^2, 4^2; 3m^2) \; = r_7(1^2, 2^4, 4; 3m^2)$$
$$= 2 \; r_7(1^2, 2^2, 4^3; 3m^2) \; = \; 2r_7(1^2, 2^4, 4; 3m^2),$$
$$r_7(1^6, 4; 3m^2) = r_7(1^4, 2^2, 4; 3m^2)$$
$$= 2 \; r_7(1^4, 4^3; 3m^2) \; = \; 2 \; r_7(1^5, 4^2; 3m^2).$$

Remark 5.2. There is one more congruence from the last formula in the above corollary. It is easy to derive the following congruence using (95):

$$\tau_{6,3}(p^\lambda) \equiv \sigma_5(p^\lambda) \pmod{13}, \tag{98}$$

where $p \neq 3$ is a prime and $\lambda \geq 1$ is an integer.

Further results. In our earlier works we determined explicit bases for the spaces of modular forms $M_4(N)$, $N = 4, 6, 8, 12, 14, 24, 28, 32, 48$ and $M_6(N')$, $N' = 4, 6, 8, 12$. Therefore, it is possible to use the method described in our work to include coefficients a_i in larger sets, which we have done and computed explicit description of the images under the Shimura maps in the case when $\ell = 5$ (quinary) and $\ell = 7$ (septenary). Specifically, in the quinary case we can take a_i in the following sets: $\{1, 2, 3, 4, 6, 12, 24\}$, $\{1, 2, 4, 8, 16\}$, $\{1, 2, 7, 14\}$, $\{1, 11\}$. In the septenary case we can take a_i in the set: $\{1, 2, 3, 4, 6\}$. Detailed list of tables for the linear combination coefficients are available. Since they occupy large space (more than 10 pages), we have not presented in this paper. From these data, one can write down more than 500 explicit formulas for the number of representations of quadratic forms corresponding to these a_i's. We will be uploading these details in preprint form for reference.

As a final remark, we mention that it is difficult to make use of Hurwitz method in the general set up. Also by using this method one gets formulas for only squares. So,

at the moment using Shimura map seems to give new and general results. However, our approach is also not giving a complete answer to the original question of obtaining explicit formulas for the number of representations of any natural number.

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