On the number of representations of a natural number by certain quaternary quadratic forms

B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh

Abstract In this paper, we find a basis for the space of modular forms of weight 2 on $\Gamma_1(48)$ and then use this basis to find formulas for the number of representations of a positive integer n by certain quaternary quadratic forms which are of the form $\sum_{i=1}^{4} a_i x_i^2$, $\sum_{i=1}^{2} b_i(x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2)$ and $a_1x_1^2 + a_2x_2^2 + b_1(x_3^2 + x_3x_4 + x_4^2)$, where a_i's belong to $\{1, 2, 3, 4, 6, 12\}$ and b_i's belong to $\{1, 2, 4, 8, 16\}$. In [1], A. Alaca et al. considered similar problem for the quaternary forms (which are diagonal) with coefficients 1, 2, 3, 6. Thus, our work extends their results with additional coefficients 4 and 12 and further in our work, we consider two more types of quaternary quadratic forms which are not diagonal. Moreover, our formulas for the diagonal quaternary quadratic forms with coefficients in $\{1, 2, 3, 4, 6, 12\}$ include explicit formulas for the number of representations (of a natural number) by 8 of the Ramanujan's universal quaternary quadratic forms [19]. We also determine some of the universal quadratic forms in the other two types of forms considered in our work.

Key words: representation numbers of quaternary quadratic forms, modular forms of one variable
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1 Introduction

In this paper we consider the problem of finding the number of representations of a natural number by the following three types of quaternary quadratic forms given by

$$\begin{aligned} Q_1 &: a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2,\\ Q_2 &: b_1(x_1^2 + x_1x_2 + x_2^2) + b_2(x_3^2 + x_3x_4 + x_4^2),\\ Q_3 &: a_1x_1^2 + a_2x_2^2 + b_1(x_3^2 + x_3x_4 + x_4^2), \end{aligned}$$

where the coefficients $a_i \in \{1, 2, 3, 4, 6, 12\}$, $1 \leqslant i \leqslant 4$ and $b_i \in \{1, 2, 4, 8, 16\}$, $i = 1, 2$. Without loss of generality we may assume that $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$, $b_1 \leq b_2$ and $\gcd(a_1, a_2, a_3, a_4) = 1$, $\gcd(b_1, b_2) = 1$, $\gcd(a_1, a_2, b_1) = 1$. Finding explicit formula for the number of representations of n by these types of quadratic forms is a classical problem in number theory. The classical formula of Jacobi for the sum of four squares correspond to the quadratic form Q_1 with $(a_1, a_2, a_3, a_4) = (1, 1, 1, 1)$. There are several works in the literature which give formulas for the representation numbers corresponding to quaternary quadratic forms with coefficients. We list some of them here [1, 2, 3, 4, 5, 6, 7, 8, 9, 22]. In [1], A. Alaca et al. considered 35 quadratic forms of type Q_1 with coefficients $a_i \in \{1, 2, 3, 6\}$. They obtained explicit bases for the spaces of modular forms of weight 2 on $\Gamma_0(24)$ with character χ_0 (trivial character modulo 24) or $\chi_d = \left(\frac{d}{\cdot}\right)$ for $d = 8, 12, 24$ and used these bases to determine formulas for the number of representations of a natural number by Q_1, with $a_i \in \{1, 2, 3, 6\}$. However, out of these 35 quadratic forms of type Q_1, formulas for 18 forms appeared in the works [2, 4, 5, 9]. More precisely, denoting the quadratic forms in Q_1 by the quadruple (a_1, a_2, a_3, a_4), the forms $(1, 1, 1, 1), (1, 1, 2, 2), (1, 1, 3, 3), (1, 1, 6, 6), (1, 2, 3, 6), (2, 2, 3, 3)$ were considered in [2], the forms $(1, 1, 1, 3), (1, 1, 2, 6), (1, 2, 2, 3), (1, 3, 3, 3), (1, 3, 6, 6), (2, 3, 3, 6)$ were considered in [4], the forms $(1, 1, 1, 2), (1, 2, 2, 2)$ were considered in [5, 22] and the forms $(1, 1, 2, 3)$, $(1, 2, 2, 6)$, $(1, 3, 3, 6)$, $(2, 3, 6, 6)$ were considered in [9]. There are several works which deal with some of these cases. For details we recommend the reader to look at the references appearing in the works of Williams and his co-authors mentioned here.

The total number of quadratic forms Q_1, with $a_i \in \{1, 2, 3, 4, 6, 12\}$ is 126. Out of this, 35 cases come from the works of [1, 2, 4, 5, 9] (when $a_i \neq 4, 12$) and so we do not consider these cases in our present work. Further, there are 36 cases which have the property that gcd(coefficients) > 1. Therefore, in our work we consider only the remaining 55 cases of quadratic forms Q_1. There are only 4 quadratic forms of type Q_2 and there are 65 quadratic forms of type Q_3 (such that the coefficients have no common factors). In the following table we give the list of quadratic forms Q_i, $i = 1, 2, 3$ considered in our work (55 forms in Q_1, 4 forms in Q_2 and 65 forms in Q_3). These are listed according to the modular forms space (in which the corresponding theta series belong). Note that in place of $M_2(48, \chi)$, we mention only the character χ which is either χ_0 (trivial character modulo 48) or χ_d, $d = 8, 12, 24$.

Table 1. (List of quadratic forms)

Space	(a_1, a_2, a_3, a_4) for Q_1
χ_0	(1, 1, 1, 4), (1, 1, 4, 4), (1, 1, 3, 12), (1, 1, 12, 12), (1, 2, 2, 4), (1, 2, 6, 12), (1, 3, 3, 4), (1, 3, 4, 12), (1, 4, 4, 4), (1, 4, 6, 6), (1, 4, 12, 12), (2, 2, 3, 12), (2, 3, 4, 6), (3, 3, 4, 4), (3, 4, 4, 12)
χ_8	(1, 1, 2, 4), (1, 1, 6, 12), (1, 2, 4, 4), (1, 2, 3, 12), (1, 2, 12, 12), (1, 3, 4, 6), (1, 4, 6, 12), (2, 3, 3, 4), (2, 3, 4, 12), (3, 4, 4, 6)
χ_{12}	(1, 1, 1, 12), (1, 1, 3, 4), (1, 1, 4, 12), (1, 2, 2, 12), (1, 2, 4, 6), (1, 3, 3, 12), (1, 3, 4, 4), (1, 3, 12, 12), (1, 4, 4, 12), (1, 6, 6, 12), (1, 12, 12, 12), (2, 2, 3, 4), (2, 3, 6, 12), (3, 3, 3, 4), (3, 3, 4, 12), (3, 4, 4, 4), (3, 4, 6, 6), (3, 4, 12, 12)
χ_{24}	(1, 1, 2, 12), (1, 1, 4, 6), (1, 2, 3, 4), (1, 2, 4, 12), (1, 3, 6, 12), (1, 4, 4, 6), (1, 6, 12, 12), (2, 3, 3, 12), (2, 3, 4, 4), (2, 3, 12, 12), (3, 3, 4, 6), (3, 4, 6, 12)
	(b_1, b_2) for Q_2
χ_0	(1, 2), (1, 4), (1, 8), (1, 16)
	(a_1, a_2, b_1) for Q_3
χ_0	(1, 3, 1), (1, 3, 2), (1, 3, 4), (1, 3, 8), (1, 3, 16), (1, 12, 1), (1, 12, 2), (1, 12, 4), (1, 12, 8), (1, 12, 16), (2, 6, 1), (3, 4, 1), (3, 4, 2), (3, 4, 4), (3, 4, 8), (3, 4, 16), (4, 12, 1)
χ_8	(1, 6, 1), (1, 6, 2), (1, 6, 4), (1, 6, 8), (1, 6, 16), (2, 3, 1), (2, 3, 2), (2, 3, 4), (2, 3, 8), (2, 3, 16), (2, 12, 1), (4, 6, 1)
χ_{12}	(1, 1, 1), (1, 1, 2), (1, 1, 4), (1, 1, 8), (1, 1, 16), (1, 4, 1), (1, 4, 2), (1, 4, 4), (1, 4, 8), (1, 4, 16), (2, 2, 1), (3, 3, 1), (3, 3, 2), (3, 3, 4), (3, 3, 8), (3, 3, 16), (3, 12, 1), (3, 12, 2), (3, 12, 4), (3, 12, 8), (3, 12, 16), (4, 4, 1), (6, 6, 1), (12, 12, 1)
χ_{24}	(1, 2, 1), (1, 2, 2), (1, 2, 4), (1, 2, 8), (1, 2, 16), (2, 4, 1), (3, 6, 1), (3, 6, 2), (3, 6, 4), (3, 6, 8), (3, 6, 16), (6, 12, 1)

Some of our formulas were also proved in works of K. S. Williams and his co-authors [2, 3, 4, 5, 6, 7, 8] , which we mention in the table below. (These formulas were obtained using different methods.)

Table A. (List of earlier results)

Type	Cases	Ref.
Q_1	(1, 1, 1, 4), (1, 1, 4, 4), (1, 1, 3, 12), (1, 1, 12, 12), (1, 2, 2, 4), (1, 3, 3, 4), (1, 3, 4, 12), (1, 4, 4, 4), (1, 4, 6, 6), (1, 4, 12, 12), (2, 2, 3, 12), (3, 3, 4, 4), (3, 4, 4, 12)	[2]
	(1, 1, 1, 12), (1, 1, 3, 4), (1, 1, 4, 12), (1, 2, 2, 12), (1, 3, 3, 12), (1, 3, 4, 4), (1, 3, 12, 12), (1, 4, 4, 12), (1, 6, 6, 12), (1, 12, 12, 12), (2, 2, 3, 4), (3, 3, 3, 4), (3, 3, 4, 12), (3, 4, 4, 4), (3, 4, 6, 6), (3, 4, 12, 12)	[3]
	(1, 2, 4, 6)	[4]
	(1, 1, 2, 4), (1, 2, 4, 4)	[5]
Q_2	(1, 2), (1, 4)	[6]
Q_3	(1, 1, 1), (1, 1, 2), (1, 1, 4), (3, 3, 1), (3, 3, 2), (3, 3, 4)	[4]
	(1, 1, 8), (1, 4, 2), (3, 12, 2)	[7]
	(1, 3, 1), (1, 3, 2), (1, 3, 4), (1, 4, 4), (2, 2, 1), (6, 6, 1)	[8]

Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{C}$ denote the sets of natural numbers, integers, rational numbers and complex numbers respectively. For $n \in \mathbb{N}$, let the number of representations of n by the quadratic forms Q_1, Q_2 and Q_3 be denoted respectively by

$$\mathcal{N}_1(a_1, a_2, a_3, a_4; n) = \#\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 = n\}, \quad (1)$$

$$\mathcal{N}_2(b_1, b_2; n) = \#\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : b_1(x_1^2+x_1x_2+x_2^2)+b_2(x_3^2+x_3x_4+x_4^2) = n\} \quad (2)$$

and

$$\mathcal{N}_3(a_1, a_2, b_1; n) = \#\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : a_1x_1^2+a_2x_2^2+b_1(x_3^2+x_3x_4+x_4^2) = n\}. \quad (3)$$

We observe that the generating functions corresponding to the quaternary quadratic forms considered in our work are modular forms of weight 2 on $\Gamma_1(48)$. So, we construct explicit bases for the spaces of modular forms of weight 2 on $\Gamma_0(48)$ with character χ (modulo 48) and use them to give formulas for $\mathcal{N}_1(a_1, a_2, a_3, a_4; n)$, $\mathcal{N}_2(b_1, b_2; n)$ and $\mathcal{N}_3(a_1, a_2, b_1; n)$. It is to be noted that in his work [19], S. Ramanujan gave the list of 55 universal quadratic forms of type Q_1. Our work includes 8 out of these 55 forms which are given by $(a_1, a_2, a_3, a_4) = (1, 1, 1, 4), (1, 1, 2, 4), (1, 1, 2, 12), (1, 1, 3, 4), (1, 2, 3, 4), (1, 2, 4, 4), (1, 2, 4, 6)$ and $(1, 2, 4, 12)$. We give explicit formulas for the number of representations of these 8 quadratic forms in Theorem 2.1. In §2.1, we give simplified expressions for some of the formulas obtained in our work and as a consequence deduce that the quadratic form $x_1^2 + x_1x_2 + x_2^2 + \ell(x_3^2 + x_3x_4 + x_4^2)$ is universal when $\ell = 2$ and non-universal when $\ell = 4$. Using the formulas for $\mathcal{N}_2(b_1, b_2; n)$ and $\mathcal{N}_3(a_1, a_2, b_1; n)$, we show the universality and non-universality of some of the forms in these two types.

2 Preliminaries and Statement of Results

We use the theory of modular forms to prove our results and so we first fix our notations and present some of the basic facts on modular forms. For positive integers $k, N \geq 1$ and a Dirichlet character χ modulo N with $\chi(-1) = (-1)^k$, let $M_k(N, \chi)$ denote the $\mathbb{C}$- vector space of holomorphic modular forms of weight k for the congruence subgroup $\Gamma_0(N)$, with character χ. Let us denote by $S_k(N, \chi)$, the subspace of cusp forms in $M_k(N, \chi)$. The modular forms space is decomposed into the space of Eisenstein series (denoted by $\mathcal{E}_k(N, \chi)$) and the space of cusp forms $S_k(N, \chi)$ and one has

$$M_k(N, \chi) = \mathcal{E}_k(N, \chi) \oplus S_k(N, \chi). \quad (4)$$

Explicit basis for the space $\mathcal{E}_k(N, \chi)$ can be obtained using the following construction. For details we refer to [17, 21]. Suppose that χ and ψ are primitive Dirichlet characters with conductors N and M, respectively. For a positive integer $k \geq 2$, let

$$E_{k,\chi,\psi}(z) := c_0 + \sum_{n\geq 1}\left(\sum_{d|n}\psi(d)\chi(n/d)d^{k-1}\right)q^n, \tag{5}$$

where $q = e^{2\pi iz}$ ($z \in \mathbb{C}, \mathrm{Im}(z) > 0$) and

$$c_0 = \begin{cases} 0 & \text{if } N > 1, \\ -\frac{B_{k,\psi}}{2k} & \text{if } N = 1, \end{cases}$$

with $B_{k,\psi}$ denoting the generalized Bernoulli number with respect to the character ψ. Then, the Eisenstein series $E_{k,\chi,\psi}(z)$ belongs to the space $M_k(NM, \chi/\psi)$, provided $\chi(-1)\psi(-1) = (-1)^k$ and $NM \neq 1$. We give a notation to the inner sum in (5):

$$\sigma_{k-1,\chi,\psi}(n) := \sum_{d|n}\psi(d)\chi(n/d)d^{k-1}. \tag{6}$$

In this paper we use the Eisenstein series of the above type with the following 11 pairs of characters given by $(\mathbf{1}, \chi_8)$, $(\chi_8, \mathbf{1})$, $(\mathbf{1}, \chi_{12})$, $(\chi_{12}, \mathbf{1})$, $(\mathbf{1}, \chi_{24})$, $(\chi_{24}, \mathbf{1})$, (χ_{-4}, χ_{-4}), (χ_{-4}, χ_{-3}), (χ_{-3}, χ_{-4}), (χ_{-3}, χ_{-8}), (χ_{-8}, χ_{-3}). For a square-free integer $d \equiv 1 \pmod 4$, the Dirichlet character χ_d (modulo $|d|$) denotes the real quadratic character $\left(\frac{d}{\cdot}\right)$, whereas for a square-free integer $d \equiv 2, 3 \pmod 4$, the Dirichlet character χ_{4d} (modulo $4|d|$) is the real quadratic character $\left(\frac{4d}{\cdot}\right)$. These characters are nothing but the Kronecker symbol. The character $\mathbf{1}$ is the trivial character given by $\mathbf{1}(n) = 1$ for all $n \geq 1$. The constant term c_0 corresponding to each of these 11 pairs is given in the following table.

(χ, ψ)	c_0
$(\chi_8, \mathbf{1})$, $(\chi_{12}, \mathbf{1})$, $(\chi_{24}, \mathbf{1})$, (χ_{-4}, χ_{-4}), (χ_{-4}, χ_{-3}), (χ_{-3}, χ_{-4}), (χ_{-3}, χ_{-8}), (χ_{-8}, χ_{-3})	0
$(\mathbf{1}, \chi_8)$	$-\frac{1}{2}$
$(\mathbf{1}, \chi_{12})$	-1
$(\mathbf{1}, \chi_{24})$	-3

When $\chi = \psi = \mathbf{1}$ (i.e., when $N = M = 1$) and $k \geq 4$, we have $E_{k,\chi,\psi}(z) = -\frac{B_k}{2k}E_k(z)$, where $E_k(z)$ is the normalized Eisenstein series of weight k in the space $M_k(1)$, defined by

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n. \tag{7}$$

In the above $\sigma_r(n)$ is the sum of the r-th powers of the positive divisors of n and B_k is the k-th Bernoulli number defined by $\frac{x}{e^x - 1} = \sum_{m=0}^{\infty}\frac{B_m}{m!}x^m$. We also need the Eisenstein series of weight 2, which is a quasimodular form of weight 2, depth 1 on $SL_2(\mathbb{Z})$ and is given by

$$E_2(z) = 1 - 24\sum_{n\geq 1}\sigma(n)q^n.$$

(Note that $\sigma(n) = \sigma_1(n)$.) Let

$$\eta(z) = q^{1/24}\prod_{n\geq 1}(1-q^n)$$

denote the Dedekind eta function. Then, an eta-quotient is a finite product of integer powers of $\eta(z)$ and we denote it as $\prod_{i=1}^{s}\eta^{r_i}(d_i z)$, where d_i's are positive integers and r_i's are non-zero integers. For more details on holomorphicity/modularity of eta-quotients one may refer to [14].

In the case of the space of cusp forms $S_k(N,\chi)$, we use a basis consisting of newforms of level N and oldforms generated by the newforms of lower level d, $d|N$, χ modulo d, $d \neq N$. However, when $\chi = \chi_{12}$, we construct a basis for the space of newforms, which are not Hecke eigenforms. For a basic theory of newforms we refer to [10, 16] and for details on modular forms, we refer to [15, 17, 21].

We now state the main results of this paper. In the following statements χ denotes a Dirichlet character modulo 48, which is either the principal character modulo 48, denoted as χ_0 or the Kronecker symbol $\chi_d = \left(\frac{d}{\cdot}\right)$, where $d = 8, 12$, or 24. For each such χ, let ℓ_χ denote the dimension of the $\mathbb{C}$- vector space $M_2(48,\chi)$. Then

$$\ell_\chi = \begin{cases} 14 & \text{if } \chi = \chi_0 \text{ or } \chi_{12}, \\ 12 & \text{if } \chi = \chi_8 \text{ or } \chi_{24}. \end{cases}$$

Theorem 2.1 *Let $n \in \mathbb{N}$. For each entry (a_1, a_2, a_3, a_4) corresponding to Q_1 in Table 1, the associated theta series is a modular form of weight 2 on $\Gamma_0(48)$ with character χ. Therefore, using the basis given Table B (in §3.5), we have*

$$\mathcal{N}_1(a_1, a_2, a_3, a_4; n) = \sum_{i=1}^{\ell_\chi} \alpha_{i,\chi} A_{i,\chi}(n), \tag{8}$$

where $A_{i,\chi}(n)$ are the Fourier coefficients of the basis elements $f_{i,\chi}$ and the values of the constants $\alpha_{i,\chi}$'s are given in (§6, Table 2). Explicit formulas for $\mathcal{N}_1(a_1, a_2, a_3, a_4; n)$ are given below for the 8 universal quadratic forms (obtained in Ramanujan's work [19]) corresponding to $(a_1, a_2, a_3, a_4) = (1, 1, 1, 4)$, $(1, 1, 2, 4)$, $(1, 1, 2, 12)$, $(1, 1, 3, 4)$, $(1, 2, 3, 4)$, $(1, 2, 4, 4)$, $(1, 2, 4, 6)$ and $(1, 2, 4, 12)$.

$$
\begin{aligned}
\mathcal{N}_1(1,1,1,4;n) &= 4\sigma(n) - 20\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16) + 2\sigma_{2,\chi_{-4},\chi_{-4}}(n),\\
\mathcal{N}_1(1,1,2,4;n) &= 4\sigma_{2,\chi_8,\mathbf{1}}(n) - 2\sigma_{2,\mathbf{1},\chi_8}(n/2),\\
\mathcal{N}_1(1,1,2,12;n) &= 2\sigma_{2,\chi_{24},\mathbf{1}}(n) - \frac{1}{3}\sigma_{2,\mathbf{1},\chi_{24}}(n/2) + \frac{2}{3}\sigma_{2,\chi_{-3},\chi_{-8}}(n) + \sigma_{2,\chi_{-8},\chi_{-3}}(n/2)\\
&\quad + 16\tau_{2,24,\chi_{24};1}(n/2) + \frac{4}{3}\tau_{2,24,\chi_{24};2}(n) + \frac{16}{3}\tau_{2,24,\chi_{24};2}(n/2),\\
\mathcal{N}_1(1,1,3,4;n) &= 3\sigma_{2,\chi_{12},\mathbf{1}}(n) - 3\sigma_{2,\chi_{12},\mathbf{1}}(n/2) + 12\sigma_{2,\chi_{12},\mathbf{1}}(n/4) - \frac{1}{2}\sigma_{2,\mathbf{1},\chi_{12}}(n)\\
&\quad + \frac{1}{2}\sigma_{2,\mathbf{1},\chi_{12}}(n/2) - \sigma_{2,\mathbf{1},\chi_{12}}(n/4) + \frac{3}{2}\sigma_{2,\chi_{-3},\chi_{-4}}(n) + \frac{3}{2}\sigma_{2,\chi_{-3},\chi_{-4}}(n/2)\\
&\quad + 3\sigma_{2,\chi_{-3},\chi_{-4}}(n/4) - \sigma_{2,\chi_{-4},\chi_{-3}}(n) - \sigma_{2,\chi_{-4},\chi_{-3}}(n/2) - 4\sigma_{2,\chi_{-4},\chi_{-3}}(n/4)\\
&\quad + \tau_{2,48,\chi_{12};1}(n) - \tau_{2,48,\chi_{12};2}(n),\\
\mathcal{N}_1(1,2,3,4;n) &= 2\sigma_{2,\chi_{24},\mathbf{1}}(n) - \frac{1}{3}\sigma_{2,\mathbf{1},\chi_{24}}(n/2) + \frac{2}{3}\sigma_{2,\chi_{-3},\chi_{-8}}(n) + \sigma_{2,\chi_{-8},\chi_{-3}}(n/2)\\
&\quad - 8\tau_{2,24,\chi_{24};1}(n/2) - \frac{2}{3}\tau_{2,24,\chi_{24};2}(n) - \frac{8}{3}\tau_{2,24,\chi_{24};2}(n/2),\\
\mathcal{N}_1(1,2,4,4;n) &= 2\sigma_{2,\chi_8,\mathbf{1}}(n) - 2\sigma_{2,\mathbf{1},\chi_8}(n/2),\\
\mathcal{N}_1(1,2,4,6;n) &= -\sigma_{2,\mathbf{1},\chi_{12}}(n/4) + \frac{3}{2}\sigma_{2,\chi_{12},\mathbf{1}}(n) + \frac{1}{2}\sigma_{2,\chi_{-4},\chi_{-3}}(n) - 3\sigma_{2,\chi_{-3},\chi_{-4}}(n/4),\\
\mathcal{N}_1(1,2,4,12;n) &= \sigma_{2,\chi_{24},\mathbf{1}}(n) - \frac{1}{3}\sigma_{2,\mathbf{1},\chi_{24}}(n/2) + \frac{1}{3}\sigma_{2,\chi_{-8},\chi_{-3}}(n) + \sigma_{2,\chi_{-3},\chi_{-8}}(n/2)\\
&\quad + 4\tau_{2,24,\chi_{24};1}(n/2) + \frac{2}{3}\tau_{2,24,\chi_{24};2}(n) + \frac{4}{3}\tau_{2,24,\chi_{24};2}(n/2).
\end{aligned}
$$

Theorem 2.2 *Let $n \in \mathbb{N}$. Then we have*

$$
\begin{aligned}
\mathcal{N}_2(1,2;n) &= 6\sigma(n) - 12\sigma(n/2) + 18\sigma(n/3) - 36\sigma(n/6),\\
\mathcal{N}_2(1,4;n) &= 6\sigma(n) - 18\sigma(n/2) - 18\sigma(n/3) + 24\sigma(n/4) + 54\sigma(n/6) - 72\sigma(n/12),\\
\mathcal{N}_2(1,8;n) &= \frac{3}{2}\sigma(n) - \frac{9}{2}\sigma(n/2) + \frac{9}{2}\sigma(n/3) + 9\sigma(n/4) - \frac{27}{2}\sigma(n/6) - 12\sigma(n/8)\\
&\quad + 27\sigma(n/12) - 36\sigma(n/24) + \frac{9}{2}\tau_{2,24}(n),\\
\mathcal{N}_2(1,16;n) &= \frac{3}{2}\sigma(n) - \frac{9}{2}\sigma(n/2) - \frac{9}{2}\sigma(n/3) + 9\sigma(n/4) + \frac{27}{2}\sigma(n/6) - 18\sigma(n/8)\\
&\quad - 27\sigma(n/12) + 24\sigma(n/16) + 54\sigma(n/24) + 72\sigma(n/48) + \frac{9}{2}\tau_{2,48}(n),
\end{aligned}
\tag{9}
$$

where $\tau_{2,24}(n)$ and $\tau_{2,48}(n)$ are the n-th Fourier coefficients of $\Delta_{2,24}(z)$ and $\Delta_{2,48}(z)$ respectively, defined in §3.1.

Theorem 2.3 *Let $n \in \mathbb{N}$. For each entry (a_1, a_2, b_1) corresponding to Q_3 in Table 1, the associated theta series is a modular form of weight 2 on $\Gamma_0(48)$ with character χ. Therefore, using the basis given in Table B (in §3.5), we have*

$$\mathcal{N}_3(a_1, a_2, b_1; n) = \sum_{i=1}^{\ell_\chi} \beta_{i,\chi} A_{i,\chi}(n), \tag{10}$$

where $A_{i,\chi}(n)$ are the Fourier coefficients of the basis elements $f_{i,\chi}$ and the values of the constants $\beta_{i,\chi}$'s are given in (§6, Table 3).

Note: Explicit formulas for some of the cases (a_1, a_2, b_1) in the above theorem are given in sections 2.1 and 2.2.

2.1 Simplification of some of the formulas and determining universal property

In this section, we shall simplify some of the formulas given in Theorems 2.1 – 2.3 and discuss about the universal property of the corresponding quadratic forms. In Theorem 2.1, we consider three formulas corresponding to $(1, 1, 1, 4)$, $(1, 1, 2, 4)$ and $(1, 2, 4, 4)$. We first consider the formula for $\mathcal{N}_1(1, 1, 1, 4; n)$, given in Theorem 2.1:

$$\mathcal{N}_1(1, 1, 1, 4; n) = 4\sigma(n) - 20\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16) + 2\sigma_{2,\chi_{-4},\chi_{-4}}(n),$$

where $\sigma_{2,\chi_{-4},\chi_{-4}}(n) = \sum_{d|n} \left(\frac{-4}{d}\right)\left(\frac{-4}{n/d}\right) d$. This twisted divisor sum vanishes when n is even and it is equal to $\left(\frac{-4}{n}\right)\sigma(n)$, when n is odd. Therefore, when n is odd, the representation number becomes $\left(4 + 2\left(\frac{-4}{n}\right)\right)\sigma(n)$. When n is even, using the multiplicative property, it is easy to see that it takes the value $\lambda\sigma(m)$, where $n = 2^\alpha m$, $\alpha \geq 0$ and $\lambda = 12, 8$ or 24 according as $\alpha = 1, 2$ or ≥ 3, respectively. Thus, we have

$$\mathcal{N}_1(1, 1, 1, 4; n) = \begin{cases} \left(4 + 2\left(\frac{-4}{n}\right)\right)\sigma(n) & \text{if } 2 \nmid n, \\ 12\sigma(m) & \text{if } n = 2m, \ m \text{ is odd}, \\ 8\sigma(m) & \text{if } n = 4m, \ m \text{ is odd}, \\ 24\sigma(m) & \text{if } n = 2^\alpha m, \ m \text{ is odd and } \alpha \geq 3. \end{cases} \tag{11}$$

Note that in the case when n is odd, the formula is nothing but $6\sigma(n)$ if $n \equiv 1 \pmod 4$ and $2\sigma(n)$ if $n \equiv 3 \pmod 4$. From this formula, it is clear that $\mathcal{N}_1(1, 1, 1, 4; n) > 0$ for all $n \geq 1$. This shows that the form $x_1^2 + x_2^2 + x_3^2 + 4x_4^2$ is universal.

Next, we consider the quadratic forms corresponding to the cases $(1, 1, 2, 4)$ and $(1, 2, 4, 4)$. For an odd natural number n, we have

$$\mathcal{N}_1(1, 1, 2, 4; n) = 4\sigma_{2,\chi_8,\mathbf{1}}(n) = 4\sum_{d|n}\left(\frac{2}{n/d}\right) d = 4\left(\frac{2}{n}\right)\sum_{d|n}\left(\frac{2}{d}\right) d. \tag{12}$$

When $n = 2^\alpha m$, $\alpha \geq 1$, m odd, then the formula simplifies as follows.

$$\begin{aligned}\mathcal{N}_1(1,1,2,4;n) &= 4\sigma_{2,\chi_8,\mathbf{1}}(n) - 2\sigma_{2,\mathbf{1},\chi_8}(n/2)\\ &= 4\sum_{d|n}\left(\frac{2}{n/d}\right)d - 2\sum_{d|n/2}\left(\frac{2}{d}\right)d \\ &= \left(2^{\alpha+2}\left(\frac{2}{m}\right)-2\right)\sum_{d|m}\left(\frac{2}{d}\right)d.\end{aligned} \tag{13}$$

Combining the above two cases, we get

$$\mathcal{N}_1(1,1,2,4;n) = \begin{cases} 4\left(\frac{2}{n}\right)\sum_{d|n}\left(\frac{2}{d}\right)d & \text{if } n \text{ is odd,}\\ \left(2^{\alpha+2}\left(\frac{2}{m}\right)-2\right)\sum_{d|m}\left(\frac{2}{d}\right)d & \text{if } n=2^{\alpha}m,\ \alpha\geq 1,\ m \text{ is odd.}\end{cases} \tag{14}$$

Now, it is easy to see that for an odd positive integer n, both $\left(\frac{2}{n}\right)$ and $\sum_{d|n}\left(\frac{2}{d}\right)d$ have the same sign (positive or negative). Therefore $\left(\frac{2}{n}\right)\sum_{d|n}\left(\frac{2}{d}\right)d$ is positive when n is odd. Using this fact, it also follows that $\left(2^{\alpha+2}\left(\frac{2}{m}\right)-2\right)\sum_{d|m}\left(\frac{2}{d}\right)d$ is positive for all odd positive integers m, when $\alpha \geq 1$. (Note that one can give explicit values for these twisted divisor sums similar to the one given in Eq.(22).) Thus, for all natural numbers n, we have $\mathcal{N}_1(1,1,2,4;n) > 0$, which implies that the quaternary form $x_1^2 + x_2^2 + 2x_3^2 + 4x_4^2$ is universal.

Using similar arguments as in the case of $\mathcal{N}_1(1,1,2,4;n)$, the formula for $\mathcal{N}_1(1,2,4,4;n)$ given in Theorem 2.1 simplifies as follows (with $n = 2^{\alpha}m$, $\alpha \geq 0$ and m is odd).

$$\begin{aligned}\mathcal{N}_1(1,2,4,4;n) &= 2\sigma_{2,\chi_8,\mathbf{1}}(n) - 2\sigma_{2,\mathbf{1},\chi_8}(n/2)\\ &= \begin{cases} 2\left(\frac{2}{n}\right)\sum_{d|n}\left(\frac{2}{d}\right)d & \text{if } n \text{ is odd,}\\ \left(2^{\alpha+1}\left(\frac{2}{m}\right)-2\right)\sum_{d|m}\left(\frac{2}{d}\right)d & \text{if } n=2^{\alpha}m,\ \alpha\geq 1,\ m \text{ is odd.}\end{cases}\end{aligned} \tag{15}$$

Thus, the form $x_1^2 + 2x_2^2 + 4x_3^2 + 4x_4^2$ is also universal.

Next, we simplify the two formulas for $\mathcal{N}_2(1,2;n)$ and $\mathcal{N}_2(1,4;n)$ given in Theorem 2.2. Using the multiplicative property of the divisor function, it can be seen that

$$\begin{aligned}\mathcal{N}_2(1,2;n) &= 6\sigma(n) - 12\sigma(n/2) + 18\sigma(n/3) - 36\sigma(n/6)\\ &= 6(3^{\beta+1}-2)\sigma(m),\ \text{if } n = 2^{\alpha}3^{\beta}m, \quad \gcd(m,6)=1,\end{aligned} \tag{16}$$

which is always positive and therefore the form $x_1^2 + x_1x_2 + x_2^2 + 2(x_3^2 + x_3x_4 + x_4^2)$ is a universal quadratic form.

Now for the other case $\mathcal{N}_2(1,4;n)$, Theorem 2.2 gives the following formula

$$\mathcal{N}_2(1,4;n) = 6\sigma(n) - 18\sigma(n/2) - 18\sigma(n/3) + 24\sigma(n/4) + 54\sigma(n/6) - 72\sigma(n/12).$$

Now, write $n = 2^\alpha 3^\beta m$, with $\gcd(m,6) = 1$, the above formula reduces to

$$6\sigma(m)\left(\sigma(3^\beta) - 3\sigma(3^{\beta-1})\right)\left(\sigma(2^\alpha) - 3\sigma(2^{\alpha-1}) + 4\sigma(2^{\alpha-2})\right).$$

The value of the factor $(\sigma(2^\alpha) - 3\sigma(2^{\alpha-1}) + 4\sigma(2^{\alpha-2}))$ is 1 when $\alpha = 0$, and 0 if $\alpha = 1$, whereas, it is equal to $2(2^{\alpha-1} - 1)$ for $\alpha \geq 2$. On the other hand, for all $\beta \geq 0$, we have $(\sigma(3^\beta) - 3\sigma(3^{\beta-1})) = 1$. It turns out that the formula does not depend on β and so assuming $n = 2^\alpha 3^\beta m$, m a positive integer with $\gcd(m,6) = 1$, the formula for $\mathcal{N}_2(1,4;n)$ becomes

$$\mathcal{N}_2(1,4;n) = \begin{cases} 6\sigma(m) & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha = 1, \\ 12(2^{\alpha-1} - 1)\sigma(m) & \text{if } \alpha \geq 2. \end{cases} \tag{17}$$

Since $\mathcal{N}_2(1,4;n) = 0$ for all positive integers $n \equiv 2 \pmod 4$, it follows that the corresponding quadratic form $x_1^2 + x_1x_2 + x_2^2 + 4(x_3^2 + x_3x_4 + x_4^2)$ is not a universal form. Further, if $n = 3^\beta$, $\beta \geq 1$, then $\mathcal{N}_2(1,4;n) = 6$ and for $n = 2^\alpha 3^\beta$, $\alpha \geq 2$, we have $\mathcal{N}_2(1,4;n) = 12(2^{\alpha-1} - 1)$.

Finally, we give formulas for the quadratic forms Q_3 corresponding to the cases $(1,3,1)$, $(1,3,2)$ and $(1,3,4)$, which involve only divisor functions $\sigma(n)$. Using these formulas, we show that two of them (corresponding to $(1,3,1)$ and $(1,3,2)$) are universal forms and the third one (corresponding to $(1,3,4)$) is non-universal. Using Table B for the basis elements and Table 3 for the linear combination coefficients, formulas for the cases $(1,3,1)$, $(1,3,2)$ and $(1,3,4)$ are given below.

$$\mathcal{N}_3(1,3,1;n) = 8\sigma(n) - 12\sigma(n/2) - 24\sigma(n/3) + 16\sigma(n/4) + 36\sigma(n/6) - 48\sigma(n/12),$$
$$\mathcal{N}_3(1,3,2;n) = 2\sigma(n) + 6\sigma(n/3) - 8\sigma(n/4) - 24\sigma(n/12),$$
$$\mathcal{N}_3(1,3,4;n) = 2\sigma(n) - 6\sigma(n/2) - 6\sigma(n/3) + 16\sigma(n/4) + 18\sigma(n/6) - 48\sigma(n/12).$$

By writing $n = 2^\alpha 3^\beta m$, $\alpha, \beta \geq 0$ and $\gcd(m,6) = 1$, the above formulas reduce to the following simplified expressions.

$$\mathcal{N}_3(1,3,1;n) = \begin{cases} 8\sigma(m) & \text{if } \alpha = 0, \\ 12(2^\alpha - 1)\sigma(m) & \text{if } \alpha \geq 1. \end{cases} \tag{18}$$

$$\mathcal{N}_3(1,3,2;n) = \begin{cases} 2(3^{\beta+1}-2)\sigma(m) & \text{if } \alpha = 0, \\ 6(3^{\beta+1}-2)\sigma(m) & \text{if } \alpha \geq 1. \end{cases} \tag{19}$$

$$\mathcal{N}_3(1,3,4;n) = \begin{cases} 2\sigma(m) & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha = 1, \\ 12(2^{\alpha-1}-1)\sigma(m) & \text{if } \alpha \geq 2. \end{cases} \tag{20}$$

The above formulas directly imply that the quadratic forms $x_1^2+3x_2^2+\ell(x_3^2+x_3x_4+x_4^2)$ are universal where $\ell = 1, 2$ and the quadratic form $x_1^2+3x_2^2+4(x_3^2+x_3x_4+x_4^2)$ is non-universal.

Among the 65 cases of type Q_3 considered in this work, 16 forms are universal (this is verified using the famous '290' theorem [11]). They are given by the following triplets: $(a_1, a_2, b_1) \in \{(1,3,1), (1,3,2), (1,12,1), (2,6,1), (1,6,1), (1,6,2), (2,3,1), (1,1,1), (1,1,2), (1,4,1), (1,4,2), (2,2,1), (1,2,1), (1,2,2), (1,2,4), (2,4,1)\}$. Among these 16 cases, the formulas for the 6 cases $(1,3,1), (1,3,2), (1,1,1), (1,1,2), (1,4,2), (2,2,1)$ involve only the divisor functions. So, it is easy to verify the universal property from our formulas for these cases. We have just shown (using (18) and (19)) that the cases $(1,3,1)$ and $(1,3,2)$ are universal. In a similar way we show that the remaining 4 cases are also universal by using explicit formulas given by Theorem 2.3. Writing $n = 2^\alpha 3^\beta N$, $\alpha, \beta \geq 0$, $\gcd(N,6) = 1$, the following formulas are obtained for the cases $(1,1,1), (1,1,2), (1,4,2), (2,2,1)$ using Theorem 2.3:

$$\begin{aligned}
\mathcal{N}_3(1,1,1;n) &= (2^{\alpha+2} + (-1)^{\alpha+\beta+\frac{N-1}{2}})(3^{\beta+1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right))F_{12}(N), \\
\mathcal{N}_3(1,1,2;n) &= (2^{\alpha+1} - (-1)^{\alpha+\beta+\frac{N-1}{2}})(3^{\beta+1} + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right))F_{12}(N), \\
\mathcal{N}_3(1,4,2;n) &= \begin{cases} (1 - \frac{1}{2}(-1)^{\beta+\frac{N-1}{2}})(3^{\beta+1} + (-1)^{\beta}\left(\frac{N}{3}\right))F_{12}(N) & \text{if } \alpha = 0, \\ 3(3^{\beta+1} - (-1)^{\beta}\left(\frac{N}{3}\right))F_{12}(N) & \text{if } \alpha = 1, \\ (2^{\alpha-1} - (-1)^{\alpha+\beta+\frac{N-1}{2}})(3^{\beta+1} + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right))F_{12}(N) & \text{if } \alpha \geq 2. \end{cases} \\
\mathcal{N}_3(2,2,1;n) &= \begin{cases} 3(3^{\beta+1} - (-1)^{\beta}\left(\frac{N}{3}\right))F_{12}(N) & \text{if } \alpha = 0, \\ (2^{\alpha} + (-1)^{\alpha+\beta+\frac{N-1}{2}})(3^{\beta+1} - (-1)^{\alpha+\beta}\left(\frac{N}{3}\right))F_{12}(N) & \text{if } \alpha \geq 1. \end{cases}
\end{aligned} \tag{21}$$

In the above, we have used the following notation defined in [6, p. 1542]: For a natural number n,

$$F_{12}(n) = \sum_{d|n} \left(\frac{12}{n/d}\right) d = \prod_{p^\lambda \| n} \frac{p^{\lambda+1} - \left(\frac{12}{p}\right)^{\lambda+1}}{p - \left(\frac{12}{p}\right)}. \tag{22}$$

It is clear from the above definition that for all natural numbers n, $F_{12}(n) > 0$. With the same assumption that $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \geq 0$ and $\gcd(N,6) = 1$, one can check easily the following facts (which are used to prove (21)):

$$
\begin{aligned}
\sum_{d|n} \left(\frac{12}{n/d}\right) d &= 2^{\alpha} 3^{\beta} F_{12}(N), \\
\sum_{d|n} \left(\frac{12}{d}\right) d &= \left(\frac{12}{N}\right) F_{12}(N) \;=\; \left(\frac{-4}{N}\right)\left(\frac{N}{3}\right) F_{12}(N), \\
\sum_{d|n} \left(\frac{-3}{d}\right)\left(\frac{-4}{n/d}\right) d &= (-1)^{\alpha+\beta} 2^{\alpha} \left(\frac{N}{3}\right) F_{12}(N), \\
\sum_{d|n} \left(\frac{-4}{d}\right)\left(\frac{-3}{n/d}\right) d &= (-1)^{\alpha+\beta+(N-1)/2} 3^{\beta} F_{12}(N).
\end{aligned}
\tag{23}
$$

Since $F_{12}(n)$ is positive for all $n \geq 1$, it follows from (21) that all the four quadratic forms corresponding to $(1, 1, 1), (1, 1, 2), (1, 4, 2)$ and $(2, 2, 1)$ are universal.

As mentioned before, there are 10 quadratic forms (of type Q_3) considered in our work, which are universal (by using the '290' theorem) for which our explicit formulas involve Fourier coefficients of cusp forms. So, it will be interesting to get this property using our explicit formulas. Here we would like to mention a very interesting and motivating survey article by J. H. Conway [13] on the 15 and 290 Theorem on the universal property of integral quadratic forms.

2.2 Remarks on equivalence of formulas

As mentioned in the introduction, our results include 36 known formulas (19 corresponding to Q_1, 2 corresponding to Q_2, 15 corresponding to Q_3), which are obtained using different methods. The cases $(1, 2)$ and $(1, 4)$ corresponding to Q_2 were obtained in [6]. Formula for the case $(1, 2)$ given in [6] is same as our formula (Theorem 2.2) and the formula for $(1, 4)$ given in [6, Theorem 15] is equivalent to (17). However, for the remaining 34 cases, some of the earlier formulas have been expressed in a different way. We would like to remark that our formulas are equivalent to these formulas obtained earlier. Here we indicate how these equivalence properties can be realised.

The formulas deduced in the previous section (§2.1) (i.e., simplified versions of actual formulas obtained from our theorems) are exactly the same formulas obtained in the earlier works [2, 4, 5, 7, 8]. Below we mention the formulas along with reference to the earlier result: $\mathcal{N}_1(1, 1, 1, 4; n)$ ([2, Theorem 1.7]), $\mathcal{N}_1(1, 1, 2, 4; n)$, $\mathcal{N}_1(1, 2, 4, 4; n)$ ([5, Theorems 5.3, 5.4]), $\mathcal{N}_3(1, 1, 1; n)$, $\mathcal{N}_3(1, 1, 2; n)$ ([4, Theorems 11.1, 12.1]), $\mathcal{N}_3(1, 3, 1; n)$, $\mathcal{N}_3(1, 3, 2; n)$, $\mathcal{N}_3(1, 3, 4; n)$ ([8, Theorem 1.2 (iii)]), $\mathcal{N}_3(1, 4, 2; n)$ ([7, Theorem 1.3]) and $\mathcal{N}_3(2, 2, 1; n)$ ([8, Theorem 1.4]).

We now show one more formula corresponding to Q_3 for the case $(a_1, a_2, b_1) = (1, 1, 8)$ and deduce the formula obtained in [6, Theorem 1.4] for this case.

By using Table 3 (for the character χ_{12}) and the basis for the space given in Table B, our formula for $\mathcal{N}_3(1, 1, 8; n)$ is obtained by comparing the n-th Fourier coefficients, which is given below.

$$\begin{aligned}\mathcal{N}_3(1,1,8;n) &= \frac{1}{2}\sigma_{2,\mathbf{1},\chi_{12}}(n) - \frac{3}{2}\sigma_{2,\mathbf{1},\chi_{12}}(n/2) + \frac{3}{2}\sigma_{2,\chi_{12},\mathbf{1}}(n) + \frac{1}{2}\sigma_{2,\chi_{-4},\chi_{-3}}(n)\\ &\quad + \frac{3}{2}\sigma_{2,\chi_{-3},\chi_{-4}}(n) + \frac{9}{2}\sigma_{2,\chi_{-3},\chi_{-4}}(n/2)\\ &= \frac{1}{2}\sum_{d|n}\left(\frac{12}{d}\right)d - \frac{3}{2}\sum_{d|n/2}\left(\frac{12}{d}\right)d + \frac{3}{2}\sum_{d|n}\left(\frac{12}{n/d}\right)d + \frac{1}{2}\sum_{d|n}\left(\frac{-3}{d}\right)\left(\frac{-4}{n/d}\right)d\\ &\quad + \frac{3}{2}\sum_{d|n}\left(\frac{-4}{d}\right)\left(\frac{-3}{n/d}\right)d + \frac{9}{2}\sum_{d|n/2}\left(\frac{-4}{d}\right)\left(\frac{-3}{n/d}\right)d.\end{aligned}$$

Now using (22), (23) in the above, we get the following explicit formula for $\mathcal{N}_3(1,1,8;n)$: Let $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \geq 0$ and $\gcd(N,6) = 1$. Then,

$$\begin{aligned}&\mathcal{N}_3(1,1,8;n)\\ &= \begin{cases} 0 & \text{if } n \equiv 3,6,7 \pmod 8,\\ (3^{\beta+1} + (-1)^\beta \left(\frac{N}{3}\right))F_{12}(N) & \text{if } n \equiv 1,5 \pmod 8,\\ (2^{\alpha-1} - (-1)^{\alpha+\beta+\frac{N-1}{2}})(3^{\beta+1} + (-1)^{\alpha+\beta}\left(\frac{N}{3}\right))F_{12}(N) & \text{if } n \equiv 0,2,4 \pmod 8.\end{cases}\end{aligned} \tag{24}$$

The above formula is the same as Theorem 1.4 of [6]. Note that the above formula implies that the corresponding quadratic form is non-universal.

3 Proofs

In this section, we shall take χ to be one of the four characters χ_0, χ_8, χ_{12} or χ_{24} and ℓ_χ is the dimension of the space of modular forms $M_2(48,\chi)$. The main ingredient in proving our theorems is the construction of explicit bases for the spaces $M_2(48,\chi)$. For uniformity, we shall denote these basis elements as $\{f_{i,\chi}(z) : 1 \leq i \leq \ell_\chi\}$ and write their Fourier expansions as

$$f_{i,\chi}(z) = \sum_{n\geq 0} A_{i,\chi}(n) e^{2\pi i n z}. \tag{25}$$

The basis elements $f_{i,\chi}(z)$ are explicitly given in §3.5.

3.1 A basis for $M_2(48,\chi_0)$

The vector space $M_2(48,\chi_0)$ has dimension 14 with
$\dim_{\mathbb{C}} \mathcal{E}_2(48,\chi_0) = 11$ and $\dim_{\mathbb{C}} S_2(48,\chi_0) = 3$. For a, b divisors of N with $a|b$, $(b >$

a), we define $\phi_{a,b}(z)$ to be

$$\phi_{a,b}(z) := \frac{1}{b-a}(bE_2(bz) - aE_2(az)). \tag{26}$$

It is easy to see that $\phi_{a,b} \in M_2(N, \chi_0)$. We need the following two eta-quotients:

$$\Delta_{2,24}(z) = \eta(2z)\eta(4z)\eta(6z)\eta(12z) = \sum_{n=1}^{\infty} \tau_{2,24}(n)q^n \tag{27}$$

$$\Delta_{2,48}(z) = \frac{\eta^4(4z)\eta^4(12z)}{\eta(2z)\eta(6z)\eta(8z)\eta(24z)} = \sum_{n=1}^{\infty} \tau_{2,48}(n)q^n. \tag{28}$$

Using the above functions, we give a basis for the space $M_2(48, \chi_0)$ in the following proposition.

Proposition 3.1 *A basis for the space of Eisenstein series $\mathcal{E}_2(48, \chi_0)$ is given by*

$$\{\phi_{1,b} : b|48 (b > 1), E_{2,\chi_{-4},\chi_{-4}}(z), E_{2,\chi_{-4},\chi_{-4}}(3z)\}$$

and a basis for the space of cusp forms $S_2(48, \chi_0)$ is given by

$$\{\Delta_{2,24}(z), \Delta_{2,24}(2z), \Delta_{2,48}(z)\}.$$

3.2 A basis for $M_2(48, \chi_8)$

The vector space $M_2(48, \chi_8)$ has dimension 12 with $\dim_{\mathbb{C}} \mathcal{E}_2(48, \chi_8) = 8$ and $\dim_{\mathbb{C}} S_2(48, \chi_8) = 4$. For the space of cusp forms, we need the following eta-quotients.

$$\Delta_{2,24,\chi_8;1}(z) = \frac{\eta(z)\eta^4(6z)\eta^2(8z)}{\eta(2z)\eta(3z)\eta(12z)} = \sum_{n=1}^{\infty} \tau_{2,24,\chi_8;1}(n)q^n,$$

$$\Delta_{2,24,\chi_8;2}(z) = \frac{\eta^2(z)\eta(8z)\eta^4(12z)}{\eta(4z)\eta(6z)\eta(24z)} = \sum_{n=1}^{\infty} \tau_{2,24,\chi_8;2}(n)q^n.$$

The following proposition gives a basis of the space $M_2(48, \chi_8)$.

Proposition 3.2 *A basis for the space of Eisenstein series $\mathcal{E}_2(48, \chi_8)$ is given by*

$$\{E_{2,\mathbf{1},\chi_8}(az), a|6, E_{2,\chi_8,\mathbf{1}}(bz), b|6\}$$

and a basis for the space of cusp forms $S_2(48, \chi_8)$ is given by

$$\{\Delta_{2,24,\chi_8;1}(z), \Delta_{2,24,\chi_8;1}(2z), \Delta_{2,24,\chi_8;2}(z), \Delta_{2,24,\chi_8;2}(2z)\}.$$

3.3 A basis for $M_2(48, \chi_{12})$

The vector space $M_2(48, \chi_{12})$ has dimension 14 with $\dim_{\mathbb{C}} S_2(48, \chi_{12}) = 2$. For the space of cusp forms, we use the following eta-quotient.

$$\Delta_{2,48,\chi_{12}}(z) = \frac{\eta^{11}(2z)\eta(6z)\eta(8z)\eta(24z)}{\eta^4(z)\eta^5(4z)\eta(12z)} = \sum_{n=1}^{\infty} a_{2,48,\chi_{12}}(n)q^n.$$

Using the above eta-quotient, we define the following two cusp forms (which are obtained by considering the character twists of the above eta-quotient).

$$\Delta_{2,48,\chi_{12};1}(z) = \sum_{\substack{n\geq 1 \\ n\equiv 1 \pmod 4}} a_{2,48,\chi_{12}}(n)q^n, \qquad \Delta_{2,48,\chi_{12};2}(z) = \sum_{\substack{n\geq 1 \\ n\equiv 3 \pmod 4}} a_{2,48,\chi_{12}}(n)q^n. \tag{29}$$

Using these functions we give a basis for the space $M_2(48, \chi_{12})$ in the following proposition.

Proposition 3.3 *A basis for the space of Eisenstein series $\mathcal{E}_2(48, \chi_{12})$ is given by*

$$\{E_{2,\mathbf{1},\chi_{12}}(az); a|4, E_{2,\chi_{12},\mathbf{1}}(bz); b|4, E_{2,\chi_{-4},\chi_{-3}}(t_1 z); t_1|4, E_{2,\chi_{-3},\chi_{-4}}(t_2 z); t_2|4\}$$

and a basis for the space of cusp forms $S_2(48, \chi_{12})$ is given by

$$\{\Delta_{2,48,\chi_{12};1}(z), \Delta_{2,48,\chi_{12};2}(z)\}.$$

3.4 A basis for $M_2(48, \chi_{24})$

The vector space $M_2(48, \chi_{24})$ has dimension 12 with $\dim_{\mathbb{C}} S_2(48, \chi_{24}) = 4$. We need the following eta-quotients.

$$\Delta_{2,24,\chi_{24};1}(z) = \frac{\eta(z)\eta(4z)\eta^4(6z)\eta^2(24z)}{\eta(2z)\eta(3z)\eta^2(12z)} = \sum_{n=1}^{\infty} \tau_{2,24,\chi_{24};1} q^n,$$
$$\Delta_{2,48,\chi_{24};2}(z) = \frac{\eta^2(z)\eta^4(4z)\eta(6z)\eta(24z)}{\eta^2(2z)\eta(8z)\eta(12z)} = \sum_{n=1}^{\infty} \tau_{2,24,\chi_{24};2} q^n.$$

In the following we give a basis for the space $M_2(48, \chi_{24})$.

Proposition 3.4 *A basis for the space of Eisenstein series $\mathcal{E}_2(48, \chi_{24})$ is given by*

$$\{E_{2,\mathbf{1},\chi_{24}}(az); a|2, E_{2,\chi_{24},\mathbf{1}}(bz); b|2, E_{2,\chi_{-3},\chi_{-8}}(t_1 z); t_1|2, E_{2,\chi_{-8},\chi_{-3}}(t_2 z); t_2|2\}$$

and a basis for the space of cusp forms $S_2(48, \chi_{24})$ is given by

$$\{\Delta_{2,24,\chi_{24};1}(z), \Delta_{2,24,\chi_{24};1}(2z), \Delta_{2,24,\chi_{24};2}(z), \Delta_{2,24,\chi_{24};2}(2z)\}.$$

3.5 Combined table for bases

In this section, we combine all the bases given in Propositions 3.1 to 3.4 in a tabular form along with identifying the elements $f_{i,\chi}(z)$ for each i, $1 \leq i \leq \ell_\chi$.

Table B (List of basis elements)

$f_{1,\chi_0}(z) = \phi_{1,2}(z)$,	$f_{6,\chi_0}(z) = \phi_{1,12}(z)$,	$f_{11,\chi_0}(z) = E_{2,\chi_{-4},\chi_{-4}}(3z)$,
$f_{2,\chi_0}(z) = \phi_{1,3}(z)$,	$f_{7,\chi_0}(z) = \phi_{1,16}(z)$,	$f_{12,\chi_0}(z) = \Delta_{2,24}(z)$,
$f_{3,\chi_0}(z) = \phi_{1,4}(z)$,	$f_{8,\chi_0}(z) = \phi_{1,24}(z)$,	$f_{13,\chi_0}(z) = \Delta_{2,24}(2z)$,
$f_{4,\chi_0}(z) = \phi_{1,6}(z)$,	$f_{9,\chi_0}(z) = \phi_{1,48}(z)$,	$f_{14,\chi_0}(z) = \Delta_{2,48}(z)$.
$f_{5,\chi_0}(z) = \phi_{1,8}(z)$,	$f_{10,\chi_0}(z) = E_{2,\chi_{-4},\chi_{-4}}(z)$,	
$f_{1,\chi_8}(z) = E_{2,\mathbf{1},\chi_8}(z)$,	$f_{5,\chi_8}(z) = E_{2,\chi_8,\mathbf{1}}(z)$,	$f_{9,\chi_8}(z) = \Delta_{2,24,\chi_8;1}(z)$,
$f_{2,\chi_8}(z) = E_{2,\mathbf{1},\chi_8}(2z)$,	$f_{6,\chi_8}(z) = E_{2,\chi_8,\mathbf{1}}(2z)$,	$f_{10,\chi_8}(z) = \Delta_{2,24,\chi_8;1}(2z)$,
$f_{3,\chi_8}(z) = E_{2,\mathbf{1},\chi_8}(3z)$,	$f_{7,\chi_8}(z) = E_{2,\chi_8,\mathbf{1}}(3z)$,	$f_{11,\chi_8}(z) = \Delta_{2,24,\chi_8;2}(z)$,
$f_{4,\chi_8}(z) = E_{2,\mathbf{1},\chi_8}(6z)$,	$f_{8,\chi_8}(z) = E_{2,\chi_8,\mathbf{1}}(6z)$,	$f_{12,\chi_8}(z) = \Delta_{2,24,\chi_8;2}(2z)$.
$f_{1,\chi_{12}}(z) = E_{2,\mathbf{1},\chi_{12}}(z)$,	$f_{6,\chi_{12}}(z) = E_{2,\chi_{12},\mathbf{1}}(4z)$,	$f_{11,\chi_{12}}(z) = E_{2,\chi_{-3},\chi_{-4}}(2z)$,
$f_{2,\chi_{12}}(z)) = E_{2,\mathbf{1},\chi_{12}}(2z)$,	$f_{7,\chi_{12}}(z) = E_{2,\chi_{-4},\chi_{-3}}(z)$,	$f_{12,\chi_{12}}(z) = E_{2,\chi_{-3},\chi_{-4}}(4z)$,
$f_{3,\chi_{12}}(z) = E_{2,\mathbf{1},\chi_{12}}(4z)$,	$f_{8,\chi_{12}}(z) = E_{2,\chi_{-4},\chi_{-3}}(2z)$,	$f_{13,\chi_{12}}(z) = \Delta_{2,48,\chi_{12};1}(z)$,
$f_{4,\chi_{12}}(z) = E_{2,\chi_{12},\mathbf{1}}(z)$,	$f_{9,\chi_{12}}(z) = E_{2,\chi_{-4},\chi_{-3}}(4z)$,	$f_{14,\chi_{12}}(z) = \Delta_{2,48,\chi_{12};2}(z)$.
$f_{5,\chi_{12}}(z) = E_{2,\chi_{12},\mathbf{1}}(2z)$,	$f_{10,\chi_{12}}(z) = E_{2,\chi_{-3},\chi_{-4}}(z)$,	
$f_{1,\chi_{24}}(z) = E_{2,\mathbf{1},\chi_{24}}(z)$,	$f_{5,\chi_{24}}(z) = E_{2,\chi_{-3},\chi_{-8}}(z)$,	$f_{9,\chi_{24}}(z) = \Delta_{2,24,\chi_{24};1}(z)$,
$f_{2,\chi_{24}}(z) = E_{2,\mathbf{1},\chi_{24}}(2z)$,	$f_{6,\chi_{24}}(z) = E_{2,\chi_{-3},\chi_{-8}}(2z)$,	$f_{10,\chi_{24}}(z) = \Delta_{2,24,\chi_{24};1}(2z)$,
$f_{3,\chi_{24}}(z) = E_{2,\chi_{24},\mathbf{1}}(z)$,	$f_{7,\chi_{24}}(z) = E_{2,\chi_{-8},\chi_{-3}}(z)$,	$f_{11,\chi_{24}}(z) = \Delta_{2,24,\chi_{24};2}(z)$,
$f_{4,\chi_{24}}(z) = E_{2,\chi_{24},\mathbf{1}}(2z)$,	$f_{8,\chi_{24}}(z) = E_{2,\chi_{-8},\chi_{-3}}(2z)$,	$f_{12,\chi_{24}}(z) = \Delta_{2,24,\chi_{24};2}(2z)$.

We are now ready to prove the theorems. The generating functions for the two types of quadratic forms considered in this paper, viz., sum of squares and forms of type $x^2 + xy + y^2$ are given respectively by the classical theta function

$$\Theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}, \tag{30}$$

and the function

$$\mathcal{F}(z) = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m^2+mn+n^2) z}. \tag{31}$$

The theta function $\Theta(z)$ is a modular form of weight $1/2$ on $\Gamma_0(4)$ and $\mathcal{F}(z)$ is a modular form of weight 1 on $\Gamma_0(3)$ with character $\left(\frac{\cdot}{3}\right)$ (see [15], [20, Theorem 4], [12] for details). To each quadratic form (a_1, a_2, a_3, a_4) as in the Table 1 (corresponding to the quadratic forms Q_1), the associated theta series is given by

$$\Theta(a_1 z)\Theta(a_2 z)\Theta(a_3 z)\Theta(a_4 z). \tag{32}$$

By using [18, Lemmas 1–3], we see that the above function is a modular form in $M_2(48, \chi)$, where χ is one of the four characters that appear in Table 1. Now using the bases constructed as in Table B, one can express each of the theta products (32) as a linear combination of the respective basis elements. Since $\mathcal{N}_1(a_1, a_2, a_3, a_4; n)$ is the n-th Fourier coefficient of the theta product (32), by comparing the n-th Fourier coefficients, we get the required formulae in Theorem 2.1.

We now briefly demonstrate the case $(1, 1, 1, 4)$. The linear combination coefficients in this case are given by (from Table 2, character χ_0) $0, 0, 5/8, 0, -7/8, 0, 5/4, 0, 0, 2, 0, 0, 0, 0$. Therefore,

$$\begin{aligned}\theta^3(z)\theta(4z) &= \frac{5}{8}\left(\frac{4}{3}E_2(4z) - \frac{1}{3}E_2(z)\right) - \frac{7}{8}\left(\frac{8}{7}E_2(8z) - \frac{1}{7}E_2(z)\right) \\ &\quad + \frac{5}{4}\left(\frac{16}{15}E_2(16z) - \frac{1}{15}E_2(z)\right) + 2E_{2,\chi_{-4},\chi_{-4}}(z) \\ &= -\frac{1}{6}E_2(z) + \frac{5}{6}E_2(4z) - E_2(8z) + \frac{4}{3}E_2(16z) + 2E_{2,\chi_{-4},\chi_{-4}}(z).\end{aligned}$$

Comparing the n-th Fourier coefficients of both the sides, we get

$$\mathcal{N}_1(1, 1, 1, 4; n) = 4\sigma(n) - 20\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16) + 2\sigma_{2,\chi_{-4},\chi_{-4}}(n).$$

Next, for the four quadratic forms given by the pairs $(1, 2), (1, 4), (1, 8), (1, 16)$ in Table 1, the corresponding theta series is the product of the forms $\mathcal{F}(b_1 z)$ and $\mathcal{F}(b_2 z)$. Again by using Lemmas 1 and 3 in [18], these forms belong $M_2(48, \chi_0)$. So, we can express these 4 forms as a linear combination of the basis elements of $M_2(48, \chi_0)$, which we denote as follows. Let $(b_1, b_2) \in \{(1, 2), (1, 4), (1, 8), (1, 16)\}$. Then

$$\mathcal{N}_2(b_1, b_2; n) = \sum_{i=1}^{14} c_i A_{i,\chi_0}(n), \tag{33}$$

where $A_{i,\chi_0}(n)$ are the Fourier coefficients of the basis elements $f_{i,\chi_0}(z)$ (given in Table B). The values of the constants c_i for each pair (b_1, b_2) are given in the following table.

Table C

b_1, b_2	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
1, 2	$\frac{1}{4}$	$\frac{-1}{2}$	0	$\frac{5}{4}$	0	0	0	0	0	0	0	0	0	0
1, 4	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{-3}{4}$	$\frac{-15}{8}$	0	$\frac{11}{4}$	0	0	0	0	0	0	0	0
1, 8	$\frac{3}{32}$	$\frac{-1}{8}$	$\frac{-9}{32}$	$\frac{15}{32}$	$\frac{7}{16}$	$\frac{-33}{32}$	0	$\frac{23}{16}$	0	0	0	$\frac{9}{2}$	0	0
1, 16	$\frac{3}{32}$	$\frac{1}{8}$	$\frac{-9}{32}$	$\frac{-15}{32}$	$\frac{21}{32}$	$\frac{33}{32}$	$\frac{-15}{16}$	$\frac{-69}{32}$	$\frac{47}{16}$	0	0	0	0	$\frac{9}{2}$

The values of c_i are non-zero only in the case of basis elements which are either $\phi_{1,b}(z)$, $b|48$ and $b > 1$ or one of the cusp forms $\Delta_{2,24}(z)$, $\Delta_{2,48}(z)$. The Fourier expansion of the Eisenstein series $\phi_{a,b}(z)$ is given as follows.

$$\phi_{a,b}(z) = 1 + \frac{24a}{b-a}\sum_{n\geq 1}\sigma(n/a)q^n - \frac{24b}{b-a}\sum_{n\geq 1}\sigma(n/b)q^n.$$

By substituting the values of the constants c_i in the expression along with the Fourier expansion of the above basis elements, we get the required formulas in Theorem 2.2.

Finally, the theta series corresponding to each quadratic form Q_3 represented by the triplets (a_1, a_2, b_1) in Table 1 is the product $\Theta(a_1 z)\Theta(a_2 z)\mathcal{F}(b_1 z)$. By using Lemmas 1 to 3 of [18], it can be observed that this theta product is a modular form of weight 2 on $\Gamma_0(48)$ with one of the characters χ_0 or χ_d, $d = 8, 12, 24$ (depending on the triplets (a_1, a_2, b_1)). Formulas in Theorem 2.3 now follow from comparing the Fourier coefficients of these associated modular forms.

This completes the proofs of the theorems.

4 Tables for Theorems 2.1 and 2.3

In this section, we shall give Tables 2 and 3, which give explicit coefficients $\alpha_{i,\chi}$ and $\beta_{i,\chi}$ that appear in Theorem 2.1 and Theorem 2.3.

Table 2 for the character χ_0.

a_1, a_2, a_3, a_4	α_{1,χ_0}	α_{2,χ_0}	α_{3,χ_0}	α_{4,χ_0}	α_{5,χ_0}	α_{6,χ_0}	α_{7,χ_0}	α_{8,χ_0}	α_{9,χ_0}	α_{10,χ_0}	α_{11,χ_0}	α_{12,χ_0}	α_{13,χ_0}	α_{14,χ_0}
1, 1, 1, 4	0	0	$\frac{5}{8}$	0	$-\frac{7}{8}$	0	$\frac{5}{4}$	0	0	2	0	0	0	0
1, 1, 4, 4	$\frac{1}{24}$	0	0	0	$-\frac{7}{24}$	0	$\frac{5}{4}$	0	0	2	0	0	0	0
1, 1, 3, 12	$\frac{1}{12}$	$\frac{1}{6}$	$-\frac{5}{16}$	$-\frac{5}{12}$	$\frac{7}{16}$	$\frac{55}{48}$	$-\frac{5}{8}$	$-\frac{23}{16}$	$\frac{47}{24}$	1	3	0	2	1
1, 1, 12, 12	$\frac{1}{48}$	$\frac{1}{12}$	0	$-\frac{5}{48}$	$\frac{7}{48}$	0	$-\frac{5}{8}$	$-\frac{23}{48}$	$\frac{47}{24}$	1	3	1	2	1
1, 2, 2, 4	$\frac{1}{24}$	0	0	0	$-\frac{7}{24}$	0	$\frac{5}{4}$	0	0	0	0	0	0	0
1, 2, 6, 12	$\frac{1}{96}$	$-\frac{1}{24}$	0	$\frac{5}{96}$	$-\frac{7}{96}$	0	$\frac{5}{16}$	$-\frac{23}{96}$	$\frac{47}{48}$	0	0	$\frac{1}{2}$	1	1
1, 3, 3, 4	$\frac{1}{12}$	$\frac{1}{6}$	$-\frac{5}{16}$	$-\frac{5}{12}$	$\frac{7}{16}$	$\frac{55}{48}$	$-\frac{5}{8}$	$-\frac{23}{16}$	$\frac{47}{24}$	-1	-3	0	-2	1
1, 3, 4, 12	$\frac{1}{16}$	$\frac{1}{12}$	$-\frac{5}{16}$	$-\frac{5}{16}$	$\frac{7}{16}$	$\frac{55}{48}$	$-\frac{5}{8}$	$-\frac{23}{16}$	$\frac{47}{24}$	0	0	0	0	1
1, 4, 4, 4	$\frac{1}{16}$	0	$-\frac{5}{16}$	0	0	0	$\frac{5}{4}$	0	0	1	0	0	0	0
1, 4, 6, 6	$\frac{1}{48}$	$\frac{1}{12}$	0	$-\frac{5}{48}$	$\frac{7}{48}$	0	$-\frac{5}{8}$	$-\frac{23}{48}$	$\frac{47}{24}$	0	0	1	-2	0
1, 4, 12, 12	$\frac{1}{32}$	$\frac{1}{24}$	$-\frac{5}{32}$	$-\frac{5}{32}$	$\frac{7}{24}$	$\frac{55}{96}$	$-\frac{5}{8}$	$-\frac{23}{24}$	$\frac{47}{24}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$
2, 2, 3, 12	$\frac{1}{48}$	$\frac{1}{12}$	0	$-\frac{5}{48}$	$\frac{7}{48}$	0	$-\frac{5}{8}$	$-\frac{23}{48}$	$\frac{47}{24}$	0	0	-1	2	0
2, 3, 4, 6	$\frac{1}{96}$	$-\frac{1}{24}$	0	$\frac{5}{96}$	$-\frac{7}{96}$	0	$\frac{5}{16}$	$-\frac{23}{96}$	$\frac{47}{48}$	0	0	$\frac{1}{2}$	1	-1
3, 3, 4, 4	$\frac{1}{48}$	$\frac{1}{12}$	0	$-\frac{5}{48}$	$\frac{7}{48}$	0	$-\frac{5}{8}$	$-\frac{23}{48}$	$\frac{47}{24}$	-1	-3	-1	-2	1
3, 4, 4, 12	$\frac{1}{32}$	$\frac{1}{24}$	$-\frac{5}{32}$	$-\frac{5}{32}$	$\frac{7}{24}$	$\frac{55}{96}$	$-\frac{5}{8}$	$-\frac{23}{24}$	$\frac{47}{24}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$

Table 2 for the character χ_8.

a_1, a_2, a_3, a_4	α_{1,χ_8}	α_{2,χ_8}	α_{3,χ_8}	α_{4,χ_8}	α_{5,χ_8}	α_{6,χ_8}	α_{7,χ_8}	α_{8,χ_8}	α_{9,χ_8}	α_{10,χ_8}	α_{11,χ_8}	α_{12,χ_8}
1, 1, 2, 4	0	-2	0	0	4	0	0	0	0	0	0	0
1, 1, 6, 12	0	$-\frac{4}{5}$	0	$-\frac{6}{5}$	$\frac{8}{5}$	0	$-\frac{12}{5}$	0	$\frac{8}{5}$	$\frac{32}{5}$	$\frac{4}{5}$	$-\frac{8}{5}$
1, 2, 4, 4	0	-2	0	0	2	0	0	0	0	0	0	0
1, 2, 3, 12	0	$\frac{2}{5}$	0	$-\frac{12}{5}$	$\frac{4}{5}$	0	$\frac{24}{5}$	0	$\frac{4}{5}$	$\frac{24}{5}$	$\frac{2}{5}$	$-\frac{16}{5}$
1, 2, 12, 12	0	$\frac{2}{5}$	0	$-\frac{12}{5}$	$\frac{2}{5}$	0	$\frac{12}{5}$	0	$\frac{12}{5}$	$\frac{24}{5}$	$-\frac{4}{5}$	$-\frac{16}{5}$
1, 3, 4, 6	0	$-\frac{4}{5}$	0	$-\frac{6}{5}$	$\frac{8}{5}$	0	$-\frac{12}{5}$	0	$\frac{8}{5}$	$-\frac{8}{5}$	$-\frac{6}{5}$	$-\frac{8}{5}$
1, 4, 6, 12	0	$-\frac{4}{5}$	0	$-\frac{6}{5}$	$\frac{4}{5}$	0	$-\frac{6}{5}$	0	$\frac{4}{5}$	$\frac{12}{5}$	$\frac{2}{5}$	$-\frac{8}{5}$
2, 3, 3, 4	0	$\frac{2}{5}$	0	$-\frac{12}{5}$	$\frac{4}{5}$	0	$\frac{24}{5}$	0	$-\frac{16}{5}$	$-\frac{16}{5}$	$\frac{12}{5}$	$\frac{24}{5}$
2, 3, 4, 12	0	$\frac{2}{5}$	0	$-\frac{12}{5}$	$\frac{2}{5}$	0	$\frac{12}{5}$	0	$-\frac{8}{5}$	$\frac{4}{5}$	$\frac{6}{5}$	$\frac{4}{5}$
3, 4, 4, 6	0	$-\frac{4}{5}$	0	$-\frac{6}{5}$	$\frac{4}{5}$	0	$-\frac{6}{5}$	0	$\frac{4}{5}$	$-\frac{8}{5}$	$-\frac{8}{5}$	$-\frac{8}{5}$

Table 2 for the character χ_{12}.

a_1, a_2, a_3, a_4	$\alpha_{1,\chi_{12}}$	$\alpha_{2,\chi_{12}}$	$\alpha_{3,\chi_{12}}$	$\alpha_{4,\chi_{12}}$	$\alpha_{5,\chi_{12}}$	$\alpha_{6,\chi_{12}}$	$\alpha_{7,\chi_{12}}$	$\alpha_{8,\chi_{12}}$	$\alpha_{9,\chi_{12}}$	$\alpha_{10,\chi_{12}}$	$\alpha_{11,\chi_{12}}$	$\alpha_{12,\chi_{12}}$	$\alpha_{13,\chi_{12}}$	$\alpha_{14,\chi_{12}}$
1, 1, 1, 12	$-\frac{1}{2}$	$\frac{1}{2}$	-1	3	3	-12	-1	1	4	$\frac{3}{2}$	$\frac{3}{2}$	3	3	1
1, 1, 3, 4	$-\frac{1}{2}$	$\frac{1}{2}$	-1	3	-3	12	-1	-1	-4	$\frac{3}{2}$	$\frac{3}{2}$	3	1	-1
1, 1, 4, 12	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{3}{2}$	0	0	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	$\frac{3}{2}$	3	2	0
1, 2, 2, 12	0	0	-1	$\frac{3}{2}$	0	0	$-\frac{1}{2}$	0	0	0	0	3	1	1
1, 2, 4, 6	0	0	-1	$\frac{3}{2}$	0	0	$\frac{1}{2}$	0	0	0	0	-3	0	0
1, 3, 3, 12	$-\frac{1}{2}$	$\frac{1}{2}$	-1	1	-1	4	1	1	4	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	1	$\frac{1}{3}$
1, 3, 4, 4	0	0	-1	$\frac{3}{2}$	-3	12	$-\frac{1}{2}$	-1	-4	0	0	3	1	-1
1, 3, 12, 12	0	0	-1	$\frac{1}{2}$	-1	4	$\frac{1}{2}$	1	4	0	0	-1	1	$\frac{1}{3}$
1, 4, 4, 12	$-\frac{1}{4}$	$\frac{1}{4}$	-1	$\frac{3}{4}$	$-\frac{3}{2}$	6	$-\frac{1}{4}$	$-\frac{1}{2}$	-2	$\frac{3}{4}$	$\frac{3}{4}$	3	1	0
1, 6, 6, 12	0	0	-1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	0	0	-1	1	$-\frac{1}{3}$
1, 12, 12, 12	$\frac{1}{4}$	$-\frac{1}{4}$	1	$\frac{1}{4}$	$-\frac{1}{2}$	2	$\frac{1}{4}$	$\frac{1}{2}$	2	$\frac{1}{4}$	$\frac{1}{4}$	-1	1	0
2, 2, 3, 4	0	0	-1	$\frac{3}{2}$	0	0	$-\frac{1}{2}$	0	0	0	0	3	-1	-1
2, 3, 6, 12	0	0	-1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	0	0	1	0	0
3, 3, 3, 4	$-\frac{1}{2}$	$\frac{1}{2}$	-1	1	1	-4	1	-1	-4	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	-1	1
3, 3, 4, 12	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	0	$\frac{2}{3}$
3, 4, 4, 4	$\frac{1}{4}$	$-\frac{1}{4}$	-1	$\frac{3}{4}$	$-\frac{3}{2}$	6	$-\frac{1}{4}$	$-\frac{1}{2}$	-2	$-\frac{3}{4}$	$-\frac{3}{4}$	3	0	-1
3, 4, 6, 6	0	0	-1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	0	0	-1	-1	$\frac{1}{3}$
3, 4, 12, 12	$-\frac{1}{4}$	$\frac{1}{4}$	-1	$\frac{1}{4}$	$-\frac{1}{2}$	2	$\frac{1}{4}$	$\frac{1}{2}$	2	$-\frac{1}{4}$	$-\frac{1}{4}$	-1	0	$\frac{1}{3}$

Table 2 for the character χ_{24}.

a_1, a_2, a_3, a_4	$\alpha_{1,\chi_{24}}$	$\alpha_{2,\chi_{24}}$	$\alpha_{3,\chi_{24}}$	$\alpha_{4,\chi_{24}}$	$\alpha_{5,\chi_{24}}$	$\alpha_{6,\chi_{24}}$	$\alpha_{7,\chi_{24}}$	$\alpha_{8,\chi_{24}}$	$\alpha_{9,\chi_{24}}$	$\alpha_{10,\chi_{24}}$	$\alpha_{11,\chi_{24}}$	$\alpha_{12,\chi_{24}}$
1, 1, 2, 12	0	$-\frac{1}{3}$	2	0	$\frac{2}{3}$	0	0	1	0	16	$\frac{4}{3}$	$\frac{16}{3}$
1, 1, 4, 6	0	$-\frac{1}{3}$	2	0	$-\frac{2}{3}$	0	0	-1	8	0	$\frac{8}{3}$	$-\frac{8}{3}$
1, 2, 3, 4	0	$-\frac{1}{3}$	2	0	$\frac{2}{3}$	0	0	1	0	-8	$-\frac{2}{3}$	$-\frac{8}{3}$
1, 2, 4, 12	0	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	0	0	1	0	4	$\frac{2}{3}$	$\frac{4}{3}$
1, 3, 6, 12	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{8}{3}$	$\frac{2}{3}$	0
1, 4, 4, 6	0	$-\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	0	-1	4	0	$\frac{4}{3}$	$-\frac{8}{3}$
1, 6, 12, 12	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{4}{3}$	0
2, 3, 3, 12	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$	$-\frac{8}{3}$	$\frac{16}{3}$	0	$\frac{8}{3}$
2, 3, 4, 4	0	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	0	0	1	0	-8	$-\frac{4}{3}$	$-\frac{8}{3}$
2, 3, 12, 12	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	0	$-\frac{1}{3}$	$-\frac{4}{3}$	$\frac{16}{3}$	0	$\frac{8}{3}$
3, 3, 4, 6	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{8}{3}$	$-\frac{16}{3}$	$-\frac{4}{3}$	0
3, 4, 6, 12	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{2}{3}$	0

Table 3 for the character χ_0.

a_1, a_2, b_1	β_{1,χ_0}	β_{2,χ_0}	β_{3,χ_0}	β_{4,χ_0}	β_{5,χ_0}	β_{6,χ_0}	β_{7,χ_0}	β_{8,χ_0}	β_{9,χ_0}	β_{10,χ_0}	β_{11,χ_0}	β_{12,χ_0}	β_{13,χ_0}	β_{14,χ_0}
1, 3, 1	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{-1}{2}$	$\frac{-5}{4}$	0	$\frac{11}{6}$	0	0	0	0	0	0	0	0
1, 3, 2	0	$\frac{-1}{6}$	$\frac{1}{4}$	0	0	$\frac{11}{12}$	0	0	0	0	0	0	0	0
1, 3, 4	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{-1}{2}$	$\frac{-5}{8}$	0	$\frac{11}{6}$	0	0	0	0	0	0	0	0
1, 3, 8	$\frac{1}{32}$	$\frac{-1}{24}$	$\frac{-7}{32}$	$\frac{5}{32}$	$\frac{7}{16}$	$\frac{-77}{96}$	0	$\frac{23}{16}$	0	0	0	$\frac{3}{2}$	0	0
1, 3, 16	$\frac{1}{32}$	$\frac{1}{24}$	$\frac{-7}{32}$	$\frac{-5}{32}$	$\frac{21}{32}$	$\frac{77}{96}$	$\frac{-15}{16}$	$\frac{-69}{32}$	$\frac{47}{16}$	0	0	0	0	$\frac{3}{2}$
1, 12, 1	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{-5}{16}$	$\frac{-5}{8}$	$\frac{7}{16}$	$\frac{55}{48}$	$\frac{-5}{8}$	$\frac{-23}{16}$	$\frac{47}{24}$	1	3	0	6	3
1, 12, 2	0	$\frac{-1}{12}$	$\frac{5}{32}$	0	$\frac{-7}{32}$	$\frac{55}{96}$	$\frac{5}{16}$	$\frac{-23}{32}$	$\frac{47}{48}$	$\frac{-1}{2}$	$\frac{3}{2}$	0	3	$\frac{3}{2}$
1, 12, 4	$\frac{1}{16}$	$\frac{1}{12}$	$\frac{-5}{16}$	$\frac{-5}{16}$	$\frac{7}{16}$	$\frac{55}{48}$	$\frac{-5}{8}$	$\frac{-23}{16}$	$\frac{47}{24}$	1	3	0	0	0
1, 12, 8	$\frac{1}{64}$	$\frac{-1}{48}$	$\frac{-5}{64}$	$\frac{5}{64}$	0	$\frac{-55}{192}$	$\frac{5}{16}$	0	$\frac{47}{48}$	$\frac{1}{4}$	$\frac{-3}{4}$	$\frac{3}{4}$	0	$\frac{3}{4}$
1, 12, 16	$\frac{1}{64}$	$\frac{1}{48}$	$\frac{-5}{64}$	$\frac{-5}{64}$	$\frac{7}{32}$	$\frac{55}{192}$	$\frac{-5}{8}$	$\frac{-23}{32}$	$\frac{47}{24}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0	$\frac{3}{4}$
2, 6, 1	$\frac{7}{48}$	$\frac{-1}{4}$	$\frac{-3}{16}$	$\frac{35}{48}$	$\frac{7}{24}$	$\frac{-11}{16}$	0	$\frac{23}{24}$	0	0	0	3	0	0
3, 4, 1	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{-5}{16}$	$\frac{-5}{8}$	$\frac{7}{16}$	$\frac{55}{48}$	$\frac{-5}{8}$	$\frac{-23}{16}$	$\frac{47}{24}$	-1	-3	0	-6	3
3, 4, 2	0	$\frac{-1}{12}$	$\frac{5}{32}$	0	$\frac{-7}{32}$	$\frac{55}{96}$	$\frac{5}{16}$	$\frac{-23}{32}$	$\frac{47}{48}$	$\frac{1}{2}$	$\frac{-3}{2}$	0	3	$\frac{-3}{2}$
3, 4, 4	$\frac{1}{16}$	$\frac{1}{12}$	$\frac{-5}{16}$	$\frac{-5}{16}$	$\frac{7}{16}$	$\frac{55}{48}$	$\frac{-5}{8}$	$\frac{-23}{16}$	$\frac{47}{24}$	-1	-3	0	0	0
3, 4, 8	$\frac{1}{64}$	$\frac{-1}{48}$	$\frac{-5}{64}$	$\frac{5}{64}$	0	$\frac{-55}{192}$	$\frac{5}{16}$	0	$\frac{47}{48}$	$\frac{-1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0	$\frac{-3}{4}$
3, 4, 16	$\frac{1}{64}$	$\frac{1}{48}$	$\frac{-5}{64}$	$\frac{-5}{64}$	$\frac{7}{32}$	$\frac{55}{192}$	$\frac{-5}{8}$	$\frac{-23}{32}$	$\frac{47}{24}$	$\frac{-1}{4}$	$\frac{-3}{4}$	$\frac{-3}{4}$	0	$\frac{3}{4}$
4, 12, 1	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{-7}{16}$	$\frac{-15}{16}$	$\frac{7}{16}$	$\frac{77}{48}$	$\frac{-5}{8}$	$\frac{-23}{16}$	$\frac{47}{24}$	0	0	0	0	3

Table 3 for the character χ_8.

a_1, a_2, b_1	β_{1,χ_8}	β_{2,χ_8}	β_{3,χ_8}	β_{4,χ_8}	β_{5,χ_8}	β_{6,χ_8}	β_{7,χ_8}	β_{8,χ_8}	β_{9,χ_8}	β_{10,χ_8}	β_{11,χ_8}	β_{12,χ_8}
1, 6, 1	$\frac{-4}{5}$	0	$\frac{-6}{5}$	0	$\frac{32}{5}$	0	$\frac{-48}{5}$	0	$\frac{24}{5}$	0	$\frac{-12}{5}$	0
1, 6, 2	$\frac{2}{5}$	0	$\frac{-12}{5}$	0	$\frac{8}{5}$	0	$\frac{48}{5}$	0	$\frac{12}{5}$	0	$\frac{-12}{5}$	0
1, 6, 4	$\frac{-4}{5}$	0	$\frac{-6}{5}$	0	$\frac{8}{5}$	0	$\frac{-12}{5}$	0	0	0	$\frac{6}{5}$	0
1, 6, 8	$\frac{2}{5}$	0	$\frac{-12}{5}$	0	$\frac{2}{5}$	0	$\frac{12}{5}$	0	$\frac{6}{5}$	0	0	0
1, 6, 16	$\frac{2}{5}$	$\frac{-6}{5}$	$\frac{3}{5}$	$\frac{-9}{5}$	$\frac{2}{5}$	0	$\frac{-3}{5}$	0	$\frac{6}{5}$	$\frac{18}{5}$	0	$\frac{-12}{5}$
2, 3, 1	$\frac{2}{5}$	0	$\frac{-12}{5}$	0	$\frac{16}{5}$	0	$\frac{96}{5}$	0	0	0	$\frac{12}{5}$	0
2, 3, 2	$\frac{-4}{5}$	0	$\frac{-6}{5}$	0	$\frac{16}{5}$	0	$\frac{-24}{5}$	0	$\frac{-12}{5}$	0	0	0
2, 3, 4	$\frac{2}{5}$	0	$\frac{-12}{5}$	0	$\frac{4}{5}$	0	$\frac{24}{5}$	0	$\frac{-12}{5}$	0	$\frac{6}{5}$	0
2, 3, 8	$\frac{-4}{5}$	0	$\frac{-6}{5}$	0	$\frac{4}{5}$	0	$\frac{-6}{5}$	0	$\frac{6}{5}$	0	$\frac{-6}{5}$	0
2, 3, 16	$\frac{-1}{5}$	$\frac{3}{5}$	$\frac{6}{5}$	$\frac{-18}{5}$	$\frac{1}{5}$	0	$\frac{6}{5}$	0	$\frac{-6}{5}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5}$
4, 6, 1	0	$\frac{-4}{5}$	0	$\frac{-6}{5}$	$\frac{24}{5}$	$\frac{-32}{5}$	$\frac{-36}{5}$	$\frac{48}{5}$	$\frac{24}{5}$	0	$\frac{-18}{5}$	$\frac{-24}{5}$

Table 3 for the character χ_{12}.

a_1, a_2, b_1	$\beta_{1,\chi_{12}}$	$\beta_{2,\chi_{12}}$	$\beta_{3,\chi_{12}}$	$\beta_{4,\chi_{12}}$	$\beta_{5,\chi_{12}}$	$\beta_{6,\chi_{12}}$	$\beta_{7,\chi_{12}}$	$\beta_{8,\chi_{12}}$	$\beta_{9,\chi_{12}}$	$\beta_{10,\chi_{12}}$	$\beta_{11,\chi_{12}}$	$\beta_{12,\chi_{12}}$	$\beta_{13,\chi_{12}}$	$\beta_{14,\chi_{12}}$
1, 1, 1	-1	0	0	12	0	0	-4	0	0	3	0	0	0	0
1, 1, 2	-1	0	0	6	0	0	2	0	0	-3	0	0	0	0
1, 1, 4	-1	0	0	3	0	0	-1	0	0	3	0	0	0	0
1, 1, 8	$\frac{1}{2}$	$\frac{-3}{2}$	0	$\frac{3}{2}$	0	0	$\frac{1}{2}$	0	0	$\frac{3}{2}$	$\frac{9}{2}$	0	0	0
1, 1, 16	$\frac{-1}{4}$	$\frac{3}{4}$	$\frac{-3}{2}$	$\frac{3}{4}$	0	0	$\frac{-1}{4}$	0	0	$\frac{3}{4}$	$\frac{9}{4}$	$\frac{9}{2}$	3	0
1, 4, 1	$\frac{-1}{2}$	$\frac{1}{2}$	-1	6	-3	12	-2	-1	-4	$\frac{3}{2}$	$\frac{3}{2}$	3	3	-3
1, 4, 2	$\frac{-1}{2}$	$\frac{1}{2}$	-1	3	3	-12	1	-1	-4	$\frac{-3}{2}$	$\frac{-3}{2}$	-3	0	0
1, 4, 4	$\frac{-1}{2}$	$\frac{1}{2}$	-1	$\frac{3}{2}$	-3	12	$\frac{-1}{2}$	-1	-4	$\frac{3}{2}$	$\frac{3}{2}$	3	0	0
1, 4, 8	$\frac{1}{4}$	$\frac{-1}{4}$	-1	$\frac{3}{4}$	$\frac{-3}{2}$	6	$\frac{1}{4}$	$\frac{1}{2}$	2	$\frac{3}{4}$	$\frac{3}{4}$	-3	0	0
1, 4, 16	$\frac{-1}{8}$	$\frac{1}{8}$	-1	$\frac{3}{8}$	$\frac{-3}{4}$	3	$\frac{-1}{8}$	$\frac{-1}{4}$	-1	$\frac{3}{8}$	$\frac{3}{8}$	3	$\frac{3}{2}$	0
2, 2, 1	0	-1	0	9	-12	0	-3	-4	0	0	-3	0	0	0
3, 3, 1	-1	0	0	4	0	0	4	0	0	-1	0	0	0	0
3, 3, 2	-1	0	0	2	0	0	-2	0	0	1	0	0	0	0
3, 3, 4	-1	0	0	1	0	0	1	0	0	-1	0	0	0	0
3, 3, 8	$\frac{1}{2}$	$\frac{-3}{2}$	0	$\frac{1}{2}$	0	0	$\frac{-1}{2}$	0	0	$\frac{-1}{2}$	$\frac{-3}{2}$	0	0	0
3, 3, 16	$\frac{-1}{4}$	$\frac{3}{4}$	$\frac{-3}{2}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	$\frac{-1}{4}$	$\frac{-3}{4}$	$\frac{-3}{2}$	0	1
3, 12, 1	$\frac{-1}{2}$	$\frac{1}{2}$	-1	2	-1	4	2	1	4	$\frac{-1}{2}$	$\frac{-1}{2}$	-1	3	1
3, 12, 2	$\frac{-1}{2}$	$\frac{1}{2}$	-1	1	1	-4	-1	1	4	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0
3, 12, 4	$\frac{-1}{2}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	-1	4	$\frac{1}{2}$	1	4	$\frac{-1}{2}$	$\frac{-1}{2}$	-1	0	0
3, 12, 8	$\frac{1}{4}$	$\frac{-1}{4}$	-1	$\frac{1}{4}$	$\frac{-1}{2}$	2	$\frac{-1}{4}$	$\frac{-1}{2}$	-2	$\frac{-1}{4}$	$\frac{-1}{4}$	1	0	0
3, 12, 16	$\frac{-1}{8}$	$\frac{1}{8}$	-1	$\frac{1}{8}$	$\frac{-1}{4}$	1	$\frac{1}{8}$	$\frac{1}{4}$	1	$\frac{-1}{8}$	$\frac{-1}{8}$	-1	0	$\frac{1}{2}$
4, 4, 1	0	0	-1	$\frac{9}{2}$	-9	12	$\frac{-3}{2}$	-3	-4	0	0	3	3	-3
6, 6, 1	0	-1	0	3	-4	0	3	4	0	0	1	0	0	0
12, 12, 1	0	0	-1	$\frac{3}{2}$	-3	4	$\frac{3}{2}$	3	4	0	0	-1	3	1

Table 3 for the character χ_{24}.

a_1, a_2, b_1	$\beta_{1,\chi_{24}}$	$\beta_{2,\chi_{24}}$	$\beta_{3,\chi_{24}}$	$\beta_{4,\chi_{24}}$	$\beta_{5,\chi_{24}}$	$\beta_{6,\chi_{24}}$	$\beta_{7,\chi_{24}}$	$\beta_{8,\chi_{24}}$	$\beta_{9,\chi_{24}}$	$\beta_{10,\chi_{24}}$	$\beta_{11,\chi_{24}}$	$\beta_{12,\chi_{24}}$
1, 2, 1	$\frac{-1}{3}$	0	8	0	$\frac{8}{3}$	0	-1	0	0	0	$\frac{-4}{3}$	0
1, 2, 2	$\frac{-1}{3}$	0	4	0	$\frac{-4}{3}$	0	1	0	-4	0	$\frac{-4}{3}$	0
1, 2, 4	$\frac{-1}{3}$	0	2	0	$\frac{2}{3}$	0	-1	0	0	0	$\frac{2}{3}$	0
1, 2, 8	$\frac{-1}{3}$	0	1	0	$\frac{-1}{3}$	0	1	0	2	0	$\frac{2}{3}$	0
1, 2, 16	$\frac{1}{6}$	$\frac{-1}{2}$	$\frac{1}{2}$	0	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{3}{2}$	0	6	$\frac{2}{3}$	2
2, 4, 1	0	$\frac{-1}{3}$	6	-8	2	$\frac{8}{3}$	0	1	0	-16	-2	$\frac{-16}{3}$
3, 6, 1	$\frac{-1}{3}$	0	$\frac{8}{3}$	0	$\frac{8}{3}$	0	$\frac{-1}{3}$	0	$\frac{8}{3}$	0	$\frac{4}{3}$	0
3, 6, 2	$\frac{-1}{3}$	0	$\frac{4}{3}$	0	$\frac{-4}{3}$	0	$\frac{1}{3}$	0	$\frac{4}{3}$	0	0	0
3, 6, 4	$\frac{-1}{3}$	0	$\frac{2}{3}$	0	$\frac{2}{3}$	0	$\frac{-1}{3}$	0	$\frac{-4}{3}$	0	$\frac{-2}{3}$	0
3, 6, 8	$\frac{-1}{3}$	0	$\frac{1}{3}$	0	$\frac{-1}{3}$	0	$\frac{1}{3}$	0	$\frac{-2}{3}$	0	0	0
3, 6, 16	$\frac{1}{6}$	$\frac{-1}{2}$	$\frac{1}{6}$	0	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{-4}{3}$	-2	$\frac{-2}{3}$	0
6, 12, 1	0	$\frac{-1}{3}$	2	$\frac{-8}{3}$	2	$\frac{8}{3}$	0	$\frac{1}{3}$	4	$\frac{16}{3}$	2	0

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