A REVISIT TO RAMANUJAN-SERRE DERIVATIVE MAP ON QUASIMODULAR FORMS AND SOME APPLICATIONS

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ABSTRACT. We revisit the Ramanujan-Serre derivative map on the space of modular forms, give a re-interpretation as a differential operator on the space of quasimodular forms. We study various algebraic properties of the differential operator, give applications in the direction of the study of the Chazy equation, Niebur's identity, van der Pol's identity and evaluation of convolution sums of divisor functions.

1. INTRODUCTION

The derivative of a modular form is not a modular form. However, there are interesting connections between modular forms and differential operators. Works of Rankin [20], Cohen [4] and Zagier [25] lead to the concept of Rankin-Cohen brackets and works of Ramanujan [19] and Serre lead to the concept of Ramanujan-Serre derivative map which is defined in the following way. Let $f(z)$ be a modular form of weight k for the full modular group $SL_2(\mathbb{Z})$ and $E_2(z)$ be the Eisenstein series of weight 2 which is a quasimodular form. It is well known that the function $\frac{1}{2\pi i}f'(z) - \frac{k}{12}E_2(z)f(z)$ is a modular form of weight $k+2$ for the full modular group, where $f'(z)$ is the derivative of f with respect to z. The result holds when f is a modular form with any level (i.e., with respect to the congruence subgroup $\Gamma_0(N)$) as well. If f is a cusp form, then the resulting function is also a cusp form. Let us denote this map by ϑ_k and write $\vartheta_k(f)(z) = Df(z) - \frac{k}{12}E_2(z)f(z)$, where $D = \frac{1}{2\pi i}\frac{d}{dz}$. This (called the Ramanujan-Serre derivative map) is a linear map from the space of modular forms of weight k to the space of modular forms of weight $k+2$. There are several generalizations and applications of the Ramanujan-Serre derivative map (see [5, 2, 14]) in various contexts.

The Ramanujan-Serre derivative map may also be interpreted in the following way. If $f(z)$ is a modular form of weight k then $Df(z)$ is a quasimodular form of weight $k+2$ and of depth 1. More explicitly, $Df(z)$ has the following transformation

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property with respect to $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$:

$$(cz+d)^{-(k+2)} Df\left(\frac{az+b}{cz+d}\right) = Df(z) + \frac{k}{2\pi i}\frac{c}{(cz+d)}f(z),$$

or equivalently,

$$Df|_{k+2}\gamma(z) = f_0(z) + f_1(z)\left(\frac{c}{cz+d}\right)$$

where $f_0(z) = Df(z)$ and $f_1(z) = \frac{k}{2\pi i}f(z)$. The Eisenstein series $E_2(z)$ has the following transformation property with respect to $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$:

$$E_2|_2\gamma(z) = E_2(z) + \frac{12}{2\pi i}\left(\frac{c}{cz+d}\right).$$

The term $\frac{k}{12}E_2(z)f(z)$ is equal to $\frac{2\pi i}{12}E_2(z)f_1(z)$ (where $f_1(z)$ is the associated function, we call it the first quasi-component of $Df(z)$ with respect to the depth 1) in the definition of the Ramanujan-Serre derivative map which needs to be subtracted to cancel the term f_1 to get back the modularity. One can reinterprete it as $\vartheta_k(f)(z) = Df(z) - \frac{2\pi i}{12}E_2(z)f_1(z)$. Hence, the Ramanujan-Serre derivative map can be reinterpreted as $Df \mapsto Df(z) - \frac{2\pi i}{12}E_2(z)f_1(z)$ (a map from the space of quasi-modular forms of weight $k+2$, depth 1 to the space of modular forms of weight $k+2$). This operator can be generalized in the space of quasimodular forms which is the main result in this article. Also, we give various algebraic properties of the differential operator and give some applications in the direction of study of Chazy equation, Niebur's identity, van der pol's identity and evaluation of convolution sums of divisor functions.

2. Preliminaries and Statement of results

In this section, we shall provide briefly some well-known facts about modular forms, Ramanujan-Serre derivative map and quasimodular forms, which are needed to prove the results. For basic details of the theory of modular forms and quasimodular forms, we refer to [23, 9, 8, 5, 21].

For any positive integer $k \geq 4$, let $M_k(N)$ (resp. $S_k(N)$) denote the vector space of modular forms (resp. cusp forms) of weight k for the congruence subgroup $\Gamma_0(N)$. For even $k \geq 4$, the normalized Eisenstein series of weight k in $M_k(1)$ given by

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n, \tag{1}$$

where B_k is the k-th Bernoulli number defined by

$$\frac{x}{e^x-1} = \sum_{m\geq 0}\frac{B_m}{m!}x^m, \tag{2}$$

and $\sigma_{k-1}(n)=\sum_{d|n} d^{k-1}, q=e^{2\pi iz}$ and z is in the upper half plane $\mathbb{H}$. The first cusp form of weight 12 on the full modular group $\Gamma_0(1)$ is given by the Ramanujan delta function

$$\Delta(z):=\frac{1}{1728}(E_4^3-E_6^2)=\sum_{n\geq 1}\tau(n)q^n. \tag{3}$$

Quasimodular forms We now present some basics of quasimodular forms. Another important Eisenstein series is the weight 2 Eisenstein series E_2 given by

$$E_2(z)=1-24\sum_{n\geq 1}\sigma(n)q^n. \tag{4}$$

This is not a modular form because it doesn't satisfy the required transformation property under the action of $SL_2(\mathbb{Z})$. However, it plays a fundamental role in defining the concept of quasimodular forms, which was formally introduced by M. Kaneko and D. Zagier [8] for the full modular group $\Gamma_0(1)$.

Definition: Let $k\geq 1, s\geq 0$ be positive integers. A holomorphic and polynomially bounded function $f:\mathbb{H}:=\{z\in\mathbb{C}: Im(z)>0\}\to\mathbb{C}$ is said to be a quasimodular form of weight k, depth s on $\Gamma_0(N)$, if there exist holomorphic functions $f_0, f_1, \cdots, f_s$ on $\mathbb{H}$ such that

$$f|_k\gamma(z):=(cz+d)^{-k}f\left(\frac{az+b}{cz+d}\right)=\sum_{i=0}^{s}f_i(z)\left(\frac{c}{cz+d}\right)^i \tag{5}$$

for all $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(N)$ and f_s is not identically vanishing. f_i is called the i-th quasi-component of the quasimodular form f.

Remark 2.1. It is a fact that if f is a quasimodular form of weight k , depth s, not identically zero, then k is even and $s\leq k/2$.

Remark 2.2. The space of quasimodular forms of weight k , depth less than or equal to s is denoted by $\widetilde{M}_k^{\leq s}(N)$. Note that E_2 is a quasimodular form of weight 2 , depth 1 on $SL_2(\mathbb{Z})$. If $f\in M_k(N)$ then $Df\in\widetilde{M}_{k+2}^{\leq 1}(N)$. In general, if $f\in\widetilde{M}_k^{\leq s}(N)$ then $D^rf\in\widetilde{M}_{k+2r}^{\leq s+r}(N)$.

Remark 2.3. It is also a fact that if f is a quasimodular form of weight k , depth s, not identically zero with i-th quasi-component f_i, then $f_i\in\widetilde{M}_{k-2i}^{\leq s-i}(N)$, $f_0=f$ and $f_s\in M_{k-2s}(N)$.

The following transformation property holds [11] for the quasimodular form D^rf.

Lemma 2.1. *Let $f\in\widetilde{M}_k^{\leq s}(N)$ with i-th quasi-component f_i. Then,*

$$D^rf|_{k+2r}\gamma=\sum_{l=0}^{s+r}\left[\sum_{j=0}^{r}\frac{1}{(2\pi i)^j}j!\binom{r}{j}\binom{k+r-l+j-1}{j}D^{(r-j)}f_{l-j}\right]\left(\frac{c}{cz+d}\right)^l \tag{6}$$

for all $r\in\mathbb{Z}_{\geq 0}$ and $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(N)$.

In particular, for $f = E_2$, we get

$$((DE_2)|_4\gamma)(z) = (DE_2)(z) + \frac{2}{2\pi i}\frac{c}{cz+d}E_2(z) + \frac{12}{(2\pi i)^2}\left(\frac{c}{cz+d}\right)^2. \tag{7}$$

We first state the following generalization of the Ramanujan-Serre derivative map on the space of quasimodular forms.

Proposition 2.2. *Let f be a quasimodular form in $\widetilde{M}_k^{\leq s}(N)$ with the s-th quasi-component f_s. Then $f - \left(\frac{2\pi i}{12}\right)^s E_2^s f_s$ is a quasimodular form in $\widetilde{M}_k^{\leq s-1}(N)$. Also, the map $\delta_{k,s} : \widetilde{M}_k^{\leq s}(N) \to \widetilde{M}_k^{\leq s-1}(N)$ given by $\delta_{k,s}(f) = f - \left(\frac{2\pi i}{12}\right)^s E_2^s f_s$ is linear.*

Remark 2.4. Royer [22] used the above fact to prove a structure theorem of quasimodular forms.

Remark 2.5. Let f be a modular form in $M_k(N)$, consider $Df \in \widetilde{M}_{k+2}^{\leq 1}(N)$. In this case $f_1 = \frac{k}{2\pi i}f$ and the map $\delta_{k+2,1}(Df) = Df - \frac{2\pi i}{12}E_2\frac{k}{2\pi i}f = Df - \frac{k}{12}E_2 f$ which is exactly the Ramanujan-Serre derivative of f.

Remark 2.6. Consider $f = E_2 \in \widetilde{M}_2^{\leq 1}(1)$. Here $f_1 = \frac{12}{2\pi i}$ is the first quasi-component, $\delta_{2,1}(E_2) = E_2 - \frac{2\pi i}{12}E_2\frac{12}{2\pi i} = 0$.

Remark 2.7. Note that the restriction of the map $\delta_{k,s}$ to $\widetilde{M}_k^{\leq s-1}(N)$ is the identity map on $\widetilde{M}_k^{\leq s-1}(N)$ and hence it is a surjective map.

Theorem 2.3. *Let f be a quasimodular form in $\widetilde{M}_k^{\leq s}(N)$ with quasi-component f_j for $0 \leq j \leq s$. Consider the map defined by*

$$\delta(f) = f - \left(\frac{2\pi i}{12}\right)E_2 f_1 + \left(\frac{2\pi i}{12}\right)^2 E_2^2 f_2 + ... + (-1)^s\left(\frac{2\pi i}{12}\right)^s E_2^s f_s. \tag{8}$$

Then $\delta(f)$ is a modular form in $M_k(N)$ and δ is a linear map with $\delta = \delta_{k,1} \circ ... \delta_{k,s-1} \circ \delta_{k,s}$ where $\delta_{k,j}$ is defined in Proposition 2.2. The kernel of the map δ is given by $\{E_2 h : h \in \widetilde{M}_{k-2}^{\leq s-1}(N)\}$.

Remark 2.8. One can prove the Ramanujan's identities $DE_2 = \frac{E_2^2 - E_4}{12}, DE_4 = \frac{E_2E_4 - E_6}{3}, DE_6 = \frac{E_2E_6 - E_4^2}{2}$ by applying the map δ to certain quasimodular forms.

Remark 2.9. Note that the restriction of the map δ to the space modular forms $M_k(N)$ is the identity map, so the map δ is surjective. Therefore, by applying the rank-nullity theorem inductively and Theorem 2.3 we get the following dimension formula for the space of quasimodular forms:

$$dim(\widetilde{M}_k^{\leq \frac{k}{2}}(N)) = dim(M_k(N)) + dim(M_{k-2}(N)) + ... + dim(M_4(N)) + dim(M_2(N)) + 1,$$

which can also be deduced from the structure theorem of quasimodular forms given by [8].

3. Proofs

Proof. (Proposition 2.2) Assume that $f \in \widetilde{M}_k^{\leq s}(N)$. By definition of $\delta_{k,s}(f)$, we have

$$(\delta_{k,s}(f))|_k\gamma = f|_k\gamma - \left(\frac{2\pi i}{12}\right)^s (E_2|_2)^s \gamma \; f_s|_{k-2s}\gamma.$$

Now writing the expressions of $f|_k\gamma$ and $E_2|_2$ following (5) and using the fact that $f_s|_{k-2s}\gamma = f_s$, we have

$$\begin{aligned}
(\delta_{k,s}(f))|_k\gamma &= \sum_{j=0}^{s} f_j(z) \left(\frac{c}{cz+d}\right)^j - \left(\frac{2\pi i}{12}\right)^s \left(E_2(z) + \frac{12}{2\pi i}\frac{c}{cz+d}\right)^s f_s \\
&= \sum_{j=0}^{s} f_j(z) \left(\frac{c}{cz+d}\right)^j - \left(\frac{2\pi i}{12}\right)^s \left(\sum_{l=0}^{s} \binom{s}{l} E_2(z)^l \left(\frac{12}{2\pi i}\right)^{(s-l)} \left(\frac{c}{cz+d}\right)^{s-l}\right) f_s \\
&= \big(f_0(z) - \left(\frac{2\pi i}{12}\right)^s f_s(z)E_2^s(z)) + \cdots + \left(f_{s-1}(z) - \frac{2\pi i}{12} f_s(z)E_2(z)\binom{s}{s-1}\right)\Big) \left(\frac{c}{cz+d}\right)^{s-1}.
\end{aligned}$$

Hence, $\delta_{k,s}(f) \in \widetilde{M}_k^{\leq s-1}(N)$. The linearity property of the map $\delta_{k,s}$ follows from the definition of quasimodular forms. This completes the proof of Proposition 2.2. $\square$

Proof. (Theorem 2.3) Suppose $f \in \widetilde{M}_k^{\leq s}(N)$. We shall prove $\delta(f) \in M_k(N)$ using the method of induction on s. The case for $s = 1$ follows from Proposition 2.2. Assume that the statement is true for space of quasimodular forms of depth $\leq s$, we assert that the statement is true for space of quasimodular forms with depth $\leq s+1$. Consider $g \in \widetilde{M}_k^{\leq s+1}(N)$ such that $g|_k\gamma = \sum_{i=0}^{s+1} g_i \left(\frac{c}{cz+d}\right)^i \in \widetilde{M}_k^{\leq s+1}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Then,

$$\begin{aligned}
\delta_{k,s+1}(g)|_k\gamma = \left(g - \left(\frac{2\pi i}{12}\right)^{s+1} E_2^{s+1} g_{s+1}\right)|_k\gamma &= \left[g(z) - \left(\frac{2\pi i}{12}\right)^{s+1} E_2^{s+1}(z) g_{s+1}(z)\right] \\
&+ \left(\frac{c}{cz+d}\right)\left[g_1(z) - \binom{s+1}{1}\left(\frac{2\pi i}{12}\right)^s E_2^s(z) g_{s+1}(z)\right] + \cdots \\
&+ \left(\frac{c}{cz+d}\right)^s \left[g_s(z) - \binom{s+1}{s}\frac{2\pi i}{12} E_2(z) g_{s+1}(z)\right].
\end{aligned}$$

Hence, $\delta_{k,s+1}(g) \in \widetilde{M}_k^{\leq s}(N)$ with j-th quasi-component $g_j(z) - \binom{s+1}{j}\left(\frac{2\pi i}{12}\right)^{s+1-j} E_2(z)^{s+1-j} g_{s+1}(z)$ for $0 \leq j \leq s$. By induction hypothesis, $\delta(\delta_{k,s+1}(g)) \in M_k(N)$. Writing the expression of $\delta(\delta_{k,s+1}(g))$ explicitly we have

$$\begin{aligned}
\delta(\delta_{k,s+1}(g)) &= \sum_{j=0}^{s} (-1)^j \left(\frac{2\pi i}{12}\right)^j E_2^j \left(g_j - \binom{s+1}{j} \left(\frac{2\pi i}{12}\right)^{s+1-j} E_2^{s+1-j} g_{s+1} \right) \\
&= \sum_{j=0}^{s} (-1)^j \left(\frac{2\pi i}{12}\right)^j E_2^j g_j + \left(\frac{2\pi i}{12}\right)^{s+1} E_2^{s+1} g_{s+1} \left(\sum_{j=0}^{s} (-1)^{j+1} \binom{s+1}{j} \right) \\
&= \sum_{j=0}^{s+1} (-1)^j \left(\frac{2\pi i}{12}\right)^j E_2^j g_j,
\end{aligned}$$

which is equal to the expression on right hand side of (8). The linearity of the map is clear (as it is a composition of linear maps). Consider the element $E_2 h$ where $h \in \widetilde{M}_{k-2}^{\leq s-1}(N)$ with

$$h|_{k-2}\gamma = \sum_{j=0}^{s-1} h_j \left(\frac{c}{cz+d}\right)^j$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. The quasi-components of $E_2 h \in \widetilde{M}_k^{\leq s}(N)$ are given by

$$(E_2 h)|_k\gamma = E_2|_2\gamma \, h|_{k-2}\gamma = E_2 h + \sum_{j=1}^{s-1} \left(\frac{c}{cz+d}\right)^j \left(E_2 h_j + \frac{12}{2\pi i} h_{j-1} \right) + \left(\frac{c}{cz+d}\right)^s \frac{12}{2\pi i} h_{s-1}.$$

Applying the map δ to $E_2 h$, we get

$$\delta(E_2 h) = E_2 h + \sum_{j=1}^{s-1} (-E_2)^j \left(\frac{2\pi i}{12}\right)^j (E_2 h_j + \frac{12}{2\pi i} h_{j-1}) + (-E_2)^s \left(\frac{2\pi i}{12}\right)^{s-1} h_{s-1} = 0.$$

Conversely, let $f \in \widetilde{M}_k^{\leq s}(N)$ with the quasi-components f_i for $0 \leq i \leq s$ be such that $\delta(f) = 0$. Writing the definition of $\delta(f)$ we have

$$\delta(f) = \sum_{l=0}^{s} (-E_2)^l \left(\frac{2\pi i}{12}\right)^l f_l = 0$$

Since $f_0 = f$, we get that $f = E_2 h$ where $h = \dfrac{2\pi i}{12}\left(f_1 + \cdots + (-1)^{s-1} \left(\frac{2\pi i}{12}\right)^{s-1} E_2^{s-1} f_s \right)$. Using Remark 2.3, we have $f_i \in \widetilde{M}_{k-2i}^{\leq s-i}(N)$ and $f_s \in M_{k-2s}(N)$, which gives $h \in \widetilde{M}_{k-2}^{\leq s-1}(N)$. This completes the proof of Theorem 2.3. □

4. APPLICATIONS

4.1. **Chazy equation.** The Chazy equation in original form is given by

$$2D^3 E_2 - 2E_2 D^2 E_2 + 3(DE_2)^2 = 0. \tag{9}$$

This identity can be derived as an application of the map δ as follows. Consider $2D^3E_2 + 3(DE_2)^2 \in \widetilde{M}_8^{\leq 4}(1)$. Applying the map δ we get

$$\delta(2D^3E_2 + 3(DE_2)^2) = 2D^3E_2 + 3(DE_2)^2 - 2E_2D^2E_2$$

is a modular form of weight 8 (which is one dimensional space generated by E_8). Now considering the Fourier coefficient one can see that it is 0 multiple of E_8, i.e.,

$$2D^3E_2 + 3(DE_2)^2 - 2E_2D^2E_2 = 0.$$

The above relation may be interpreted as a dependent relation among the quasimodular forms $D^3E_2, (DE_2)^2, E_2D^2E_2$ on the space $\widetilde{M}_8^{\leq 4}(1)$. *One can ask if there are more such relations (or other Chazy type relations) on the set of quasimodular forms of weight* 8 *and of depth* 4. Consider the set of elements of the form $E_2^i(D^jE_2)^k$ of weight 8, depth 4, which is given by $\{D^3E_2, E_2D^2E_2, (DE_2)^2, E_2^4, E_2^2DE_2\}$.

We claim that the set $\{E_2D^2E_2, (DE_2)^2, E_2^4, E_2^2DE_2\}$ is a linearly independent set which can be proved using the δ map. Suppose that

$$c_1E_2D^2E_2 + c_2(DE_2)^2 + c_3E_2^4 + c_4E_2^2DE_2 = 0.$$

Applying the map δ both sides, we get

$$c_2\delta((DE_2)^2) = 0,$$

which is true only when $c_2 = 0$. Hence, the above equation reduces to

$$c_1E_2D^2E_2 + c_3E_2^4 + c_4E_2^2DE_2 = 0,$$

this implies,

$$E_2(c_1D^2E_2 + c_3E_2^3 + c_4E_2DE_2) = 0.$$

As E_2 is not identically zero, we have

$$c_1D^2E_2 + c_3E_2^3 + c_4E_2DE_2 = 0.$$

Using the map δ to the above identity we deduce $c_1 = 0$, repeating similar arguments also give $c_3 = c_4 = 0$. Now we show that Chazy equation $2D^3E_2 - 2E_2D^2E2 + 3(DE_2)^2 = 0$ is the unique possible linear dependence relation possible among the five elements

$$\{D^3E_2, E_2D^2E_2, (DE_2)^2, E_2^4, E_2^2DE_2\}$$

(up to constant multiplication). Let b_i for $i \in \{1, 2, 3, 4, 5\}$ be scalars such that

$$b_1D^3E_2 + b_2E_2D^2E_2 + b_3(DE_2)^2 + b_4E_2^4 + b_5E_2^2DE_2 = 0. \tag{10}$$

Applying the map δ both sides, we get

$$\delta(b_1D^3E_2 + b_2E_2D^2E_2 + b_3(DE_2)^2 + b_4E_2^4 + b_5E_2^2DE_2) = b_1\delta(D^3E_2) + b_3\delta((DE_2)^2) = 0.$$

Substituting the images of D^3E_2 and DE_2^2 under δ gives,

$$(b_1D^3E_2 + b_3(DE_2)^2) - b_1E_2D^2E_2 + E_2^2DE_2(\frac{b_1}{4} - \frac{b_3}{6}) + E_2^4(\frac{b_3}{144} - \frac{18b_1}{12^3}) = 0.$$

Comparing first few Fourier coefficients and solving for b_1 and b_3 we get $\{(b_1, b_3) = (\frac{2r}{3}, r) : r \in \mathbb{C}\}$. Substituting this in (10), we get

$$r(\frac{2}{3}D^3E_2 + (DE_2)^2) + b_2E_2D^2E_2 + b_4E_2^4 + b_5E_2^2DE_2 = 0.$$

Substituting $\frac{2}{3}D^3E_2 + (DE_2)^2 = \frac{2}{3}E_2D^2E_2$ from the Chazy equation (9), we get

$$(\frac{2r}{3} + b_2)E_2D^2E_2 + b_4E_2^4 + b_5E_2^2DE_2 = 0$$

As shown earlier the set $\{E_2D^2E_2, E_2^4, E_2^2DE_2\}$ is linearly independent, which implies $b_2 = -\frac{2r}{3}$ and $b_4 = b_5 = 0$. Substituting these b_i's into (10), we have exactly one chazy-like relation up to constant.

4.2. **Niebur's Identity.** Niebur [12] proved the following relation between the Ramanujan delta function Δ and Eisenstein series E_2

$$\Delta = \frac{2}{3}D^3E_2DE_2 - \frac{3}{4}(D^2E_2)^2 - \frac{1}{24}E_2D^4E_2 \tag{11}$$

using differential equation satisfied by E_2. This gives the following interesting identity of Ramanujan tau function $\tau(n)$ in terms of divisor function $\sigma(n)$:

$$\tau(n) = n^4\sigma(n) - 24\sum_{m=1}^{n-1}(35m^4 - 52m^3n + 18m^2n^2)\sigma(m)\sigma(n-m). \tag{12}$$

One can prove (11) by applying the map δ on $8DE_2D^3E_2 - 9(D^2E_2)^2$ and simplifying using Chazy equation (9).

Next, we discuss the more interesting question: *are there any other such relations possible among* $E_2^i(D^jE_2)^{j_1}(D^kE_2)^{k_1} \in \widetilde{M}_{12}^{\leq 6}(1)$ *of weight 12 , depth 6 and Ramanujan delta function* Δ. The set of such forms is $\{(E_2DE_2)^2, (DE_2)^3, E_2^4DE_2, E_2^3D^2E_2, E_2DE_2D^2E_2, E_2^6, E_2^2D^3E_2, (D^2E_2)^2, DE_2D^3E_2, E_2D^4E_2, D^5E_2\}$. We note the following identites due to the Chazy equation:

$$\begin{aligned}
(DE_2)^3 &= \frac{2}{3}E_2DE_2D^2E_2 - \frac{2}{3}DE_2D^3E_2, \\
E_2^2D^3E_2 &= E_2D^4E_2 + 2E_2DE_2D^2E_2, \\
E_2^3D^2E_2 &= E_2D^4E_2 + 2E_2DE_2D^2E_2 + \frac{3}{2}(E_2DE_2)^2, \\
D^5E_2 &= E_2D^4E_2 - 2(D^2E_2)^2 - 2DE_2D^3E_2.
\end{aligned}$$

Hence, it is enough to consider the set $\{(E_2DE_2)^2, E_2^4DE_2, E_2DE_2D^2E_2, E_2^6, (D^2E_2)^2, DE_2D^3E_2, E_2D^4E_2\}$. *We claim that Niebur's Identity* (11) *is the unique possible relation between these seven elements and Ramanujan delta function* Δ. Let $c_i \in \mathbb{C}$ for $i \in \{1, \cdots, 7\}$ be such that

$$c_1(D^2E_2)^2 + c_2DE_2D^3E_2 + c_3(E_2DE_2)^2 + c_4(E_2^4DE_2) + c_5(E_2DE_2D^2E_2) + c_6E_2^6 + c_7E_2D^4E_2 = \Delta. \tag{13}$$

As the kernel of δ is $\{E_2h : h \in \widetilde{M}_{k-2}^{\leq s-1}(1)\}$, applying the map δ on both sides of (13), we get

$$\delta(c_1(D^2E_2)^2 + c_2DE_2D^3E_2) = \Delta.$$

Solving for c_1 and c_2 we get, $c_1 = \frac{-3}{4}$ and $c_2 = \frac{2}{3}$. Substituting these values in (13), we get

$$\frac{-3}{4}(D^2E_2)^2+\frac{2}{3}DE_2D^3E_2+c_3(E_2DE_2)^2+c_4(E_2^4DE_2)+c_5(E_2DE_2D^2E_2)+c_6E_2^6+c_7E_2D^4E_2 = \Delta. \tag{14}$$

Using the Niebur's identity (11) we have

$$\frac{-3}{4}(D^2E_2)^2 + \frac{2}{3}DE_2D^3E_2 = \Delta + \frac{1}{12}E_2D^4E_2.$$

On substitution the above identity, (14) reduces to

$$c_3(E_2DE_2)^2 + c_4(E_2^4DE_2) + c_5(E_2DE_2D^2E_2) + c_6E_2^6 + (c_7 + \frac{1}{12})E_2D^4E_2 = 0. \tag{15}$$

One can show that that the set $\{(E_2DE_2)^2, (E_2^4DE_2), (E_2DE_2D^2E_2), E_2^6, E_2D^4E_2\}$ is linearly independent using the map δ again, as a consequence we have $c_1 = \frac{-3}{4}, c_2 = \frac{2}{3}, c_7 = \frac{-1}{12}$ and $c_3 = c_4 = c_5 = c_6 = 0$ which asserts the claim.

4.3. **van der Pol's Identity.** B. van der Pol [13] derived the following identity relating $\tau(n)$ to sum-of-divisors functions.

$$\tau(n) = n^2\sigma_3(n) + 60\sum_{m=1}^{n-1}(2n-3m)(n-3m)\sigma_3(m)\sigma_3(n-m). \tag{16}$$

Using the relation between $\sigma_7(n)$ and $\sigma_3(n)$ the above identity is equivalent to

$$\tau(n) = n^2\sigma_7(n) - 540\sum_{m=1}^{n-1}m(n-m)\sigma_3(m)\sigma_3(n-m). \tag{17}$$

The above identities correspond to the modular identities

$$4E_4D^2E_4 - 10(DE_4)^2 = 960\Delta. \tag{18}$$

and

$$2D^2E_8 - 9(DE_4)^2 = 960\Delta, \tag{19}$$

respectively. These kind of identities are reproved using different methods like Masss operator [10], Rankin-Cohen bracket [14], and an elementary method [15]. One can prove the above identity using the map δ. Applying δ to the quasimodular form $\frac{1}{240}E_4D^2E_4 - \frac{1}{192}(DE_4)^2 \in \widetilde{M}_{12}^{\leq 6}(1)$ we have

$$\delta\left(\frac{1}{240}E_4D^2E_4 - \frac{1}{192}(DE_4)^2\right) = \Delta$$

which gives (18). One can also prove other van der Pol's type identities given in [10] using the map δ.

5. CONVOLUTION SUMS:

Let $\mathbb{N}$ be the set of positive integers. For $r, n \in \mathbb{N}$, let $\sigma_r(n) = \sum_{d|n, d\in\mathbb{N}} d^r$ be the divisor function. If n is not a positive integer, set $\sigma_r(n) = 0$ and we write $\sigma(n)$ for $\sigma_1(n)$. For $a, b, r, s, n \in \mathbb{N}$, we define $W_{a,b}^{r,s}(n)$ by

$$W_{a,b}^{r,s}(n) := \sum_{\substack{l,m\in\mathbb{N}\\ al+bm=n}} \sigma_r(l)\sigma_s(m). \tag{20}$$

These are referred to as the convolution sums of the divisor functions. When $r = s = 1$, it is denoted simply by $W_{a,b}(n)$. Further, we write $W_{1,a}(n) = W_{a,1}(n) = W_a(n)$. Evaluation of these sums has a long history, going back to the works of Besge, Glaisher and Ramanujan [3, 6, 19]. In the literature there are various methods used to obtain these convolution sums (namely, elementary evaluation, using the theory of modular forms and quasimodular forms and also using (p, k) parametrization etc.,). We refer to [7] and the book by K. S. Williams [24] for more details about the history of this problem.

In this section we use the map δ on the space of quasimodular forms and demonstrate that one can evaluate convolution sum explicitly. Note that the convolution sums $W_N(n)$ appears as a Fourier coefficient of the quasimodular form of weight 4, depth 2 on $\Gamma_0(N)$,

$$E_2(z)E_2(Nz) = 1 - 24\sum_{n=1}^{\infty}\left(\sigma(n) + \sigma\left(\frac{n}{N}\right)\right)q^n + 576\sum_{n=1}^{\infty} W_N(n)q^n.$$

We state the following known results on evaluation of convolution sum [16], give another proof using the map δ.

Theorem 5.1 ([16], Corollary 2.3)**.** *Let N be a natural number with $dim(M_4(N)) = m$ and $\{h_1, h_2, ..., h_m\}$ be a basis of the vector space $M_4(N)$ with Fourier expansion $h_i = \sum_{n=0}^{\infty} a_{h_i}(n)q^n$, then there exist constants c_i such that*

$$W_N(n) = \left(\frac{1}{24} - \frac{n}{4N}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{N}\right) + \sum_{i=1}^{m} c_i a_{h_i}(n). \tag{21}$$

Let $a > 1, b > 1$ be natural numbers with $lcm(a, b) = N$ and dim $(M_4(N)) = l$ with $\{g_1, g_2, ..., g_l\}$ be a basis of the vector space $M_4(N)$ with Fourier expansions $g_i = \sum_{n=0}^{\infty} a_{g_i}(n)q^n$, then there exist constants c_i such that

$$W_{a,b}(n) = \left(\frac{1}{24} - \frac{n}{4a}\right)\sigma\left(\frac{n}{b}\right) + \left(\frac{1}{24} - \frac{n}{4b}\right)\sigma\left(\frac{n}{a}\right) + \sum_{i=1}^{l} c_i a_{g_i}(n). \tag{22}$$

Proof. We apply δ map on $DE_2(Nz)$ to conclude

$$\delta(DE_2(Nz)) = DE_2(Nz) - \frac{1}{6N}E_2(z)E_2(Nz) + \frac{1}{12N^2}E_2^2(z)$$

is a modular form on the space $M_4(N)$. Now substituting $E_2^2(z) = 12DE_2(z) + E_4(z)$ (Ramanujan's identity) we conclude that

$$DE_2(Nz) - \frac{1}{6N}E_2(z)E_2(Nz) + \frac{1}{12N^2}12DE_2(z)$$

is a modular form on the space $M_4(N)$. Writing in terms of the given basis we have

$$DE_2(Nz) - \frac{1}{6N}E_2(z)E_2(Nz) + \frac{1}{N^2}DE_2(z) = \sum_{i=1}^{m} \lambda_i h_i.$$

for some constants λ_i. Now comparing the n-th Fourier coefficients and some algebraic manipulations, we get the required identity of $W_N(n)$. To get the formula for $W_{a,b}(n)$ we consider $E_2(az)E_2(bz) \in \widetilde{M}_4^{\leq 2}(N)$ where $N = lcm(a, b)$. Applying the map δ we have

$$\delta(E_2(az)E_2(bz)) = E_2(az)E_2(bz) - \frac{1}{b}E_2(z)E_2(az) - \frac{1}{a}E_2(z)E_2(bz) + \frac{1}{ab}E_2^2(z) \in M_4(N).$$

Writing as a linear combination of the basis elements,

$$E_2(az)E_2(bz) - \frac{1}{b}E_2(z)E_2(az) - \frac{1}{a}E_2(z)E_2(bz) + \frac{1}{ab}E_2^2(z) = \sum_{i=1}^{l} \lambda_i g_i.$$

for some constants λ_i. Writing in terms of Fourier coefficients, we get

$$\begin{aligned}\sum_{i=1}^{l} \lambda_i a_{g_i}(n) = &- 24\sigma\left(\frac{n}{b}\right) - 24\sigma\left(\frac{n}{a}\right) + 576W_{a,b}(n) - \frac{1}{b}\left[-24\sigma(n) - 24\sigma\left(\frac{n}{a}\right) + 576W_a(n)\right] \\ &- \frac{1}{a}\left[-24\sigma(n) - 24\sigma\left(\frac{n}{b}\right) + 576W_b(n)\right] + \frac{1}{ab}[-48\sigma(n) + 576W_1(n)].\end{aligned}$$

Substituting the expressions for $W_a(n), W_b(n)$ and $W_1(n)$ from (21) we get the required expression for $W_{a,b}(n)$. □

Theorem 5.2 ([16], Corollary 2.2). *Let N be a natural number and $dim(M_{k+2}(N)) = l$. Let $\{h_1, \cdots, h_l\}$ be a basis of the vector space $M_{k+2}(N)$ with Fourier expansion $h_i(z) = \sum_{n=0}^{\infty} a_{h_i}(n)q^n$, then there exist constants λ_i such that*

$$W_{1,N}^{1,k-1}(n) = \left(\frac{1}{24} - \frac{n}{2k}\right)\sigma_{k-1}\left(\frac{n}{N}\right) + \frac{B_k}{2k}\sigma(n) - \frac{NB_k}{4k^2}\sum_{i=1}^{l} \lambda_i a_{h_i}(n)$$

where B_k is the k-th Bernoulli Number.

Proof. Let us consider the quasimodular form $DE_k(Nz)$ for an even integer $k \geq 4$, then for any $\gamma \in \Gamma_0(N)$

$$DE_k(Nz)|_{k+2}\gamma = DE_k(Nz) + \frac{1}{2\pi i}\frac{c}{(cz+d)}\frac{k}{N}E_2(z)E_2(Nz)$$

By the definition and property of δ,

$$\delta(DE_k(Nz)) = DE_k(Nz) - \frac{k}{12N}E_2(z)E_k(Nz)$$

is a modular form of weight $k+2$. There exist constants λ_i such that

$$\delta(DE_k(Nz)) = DE_k(Nz) - \frac{k}{12N}E_2(z)E_k(Nz) = \sum_{i=1}^{l} \lambda_i f_i. \tag{23}$$

Comparing the Fourier coefficients both sides of (23), we get the required expression for $W_{1,N}^{1,k-1}(n)$. □

5.1. **Example.** For $N = 12$, $\dim(M_4(12)) = 9$. Consider the basis given by $\{E_4(z), E_4(2z), E_4(3z), E_4(4z), E_4(6z), E_4(12z), \Delta_{4,6}(z), \Delta_{4,6}(2z), \Delta_{4,12}(z)\}$ with the Fourier series expansions:

$$E_4(z) = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n, \Delta_{4,6}(z) = \sum_{n\geq 1}\tau_{4,6}(n)q^n, \Delta_{4,12}(z) = \sum_{n\geq 1}\tau_{4,12}(n)q^n.$$

Applying Theorem 5.1 we have

$$DE_2(12z) - \frac{1}{72}E_2(z)E_2(12z) + \frac{1}{1728}E_2^2(z) = \frac{11}{21600}E_4(z) - \frac{1}{4800}E_4(2z) - \frac{1}{1600}E_4(3z)$$
$$- \frac{1}{900}E_4(4z) - \frac{3}{1600}E_4(6z) - \frac{1}{100}E_4(12z) + \frac{1}{10}\Delta_{4,6}(z) + \frac{2}{5}\Delta_{4,6}(2z) + \frac{1}{12}\Delta_{4,12}(z).$$

Comparing the n-th Fourier coefficient we have

$$W_{12}(n) = \left(\frac{1}{24} - \frac{n}{48}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{12}\right) + \frac{1}{480}\sigma_3(n) + \frac{1}{160}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{160}\sigma_3\left(\frac{n}{3}\right)$$
$$+ \frac{1}{30}\sigma_3\left(\frac{n}{4}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{6}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{12}\right) - \frac{1}{80}\tau_{4,6}(n) - \frac{1}{20}\tau_{4,6}(\frac{n}{2}) - \frac{1}{96}\tau_{4,12}(n).$$

This gives the formula for $W_{12}(n)$ established in [1, 18].

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