NONVANISHING OF L-FUNCTIONS AND POINCARÉ SERIES FOR JACOBI FORMS OF MATRIX INDEX

SHIVANSH PANDEY AND BRUNDABAN SAHU

Abstract. W. Kohnen introduced kernel functions to study the nonvanishing of L-functions attached to Hecke eigenforms. Y. Martin defined L-functions for Jacobi forms of arbitrary index and studied the analytic properties of these L-functions. In this paper, we study the nonvanishing of L-functions and Poincaré series for Jacobi forms defined on $\mathcal{H} \times \mathbb{C}^{g,1}$ using kernel functions.

1. Introduction

Study of L-functions associated to automorphic forms plays an important role in analytic number theory. For example nonvanishing of Riemann zeta function is the key point in the proof of prime number theorem. However, finding zero-free region for L-functions for automorphic forms is not an easy problem. Let $f(\tau) = \sum_{n\geqslant 1} a(n)e^{2\pi i n\tau}$ be a normalized Hecke eigenforms of weight k for the full modular group $\Gamma = SL_2(\mathbb{Z})$. Let $L^*(f,s) = (2\pi)^{-s}\Gamma(s)\sum_{n\geqslant 1} a(n)n^{-s}$ be completed L-function associated to f. $L^*(f,s)$ has an Euler product for $Re(s) \geq \frac{k+1}{2}$ and all the zeroes of $L^*(f,s)$ can occur only inside critical strip $\frac{k-1}{2} \leq Re(s) \leq \frac{k+1}{2}$. Generalized Riemann hypothesis states that all zeroes of $L^*(f,s)$ can occur only on line $Re(s) = \frac{k}{2}$. In this direction, Kohnen [8] proved that given $t_0 \in \mathbb{R}$ and $\epsilon > 0$ there exists $k_0(t_0,\epsilon)$ such that for sufficiently large weight $k > k_0$ there exists a Hecke eigenforms for which $L^*(f,s)$ does not vanish at any point on the line segments $Im(s) = t_0$, $\frac{k-1}{2} \leq Re(s) \leq \frac{k}{2} - \epsilon$, $\frac{k}{2} + \epsilon \leq Re(s) \leq \frac{k+1}{2}$. For this purpose Kohnen constructed the following kernel functions defined by

$$R_{k,s}(\tau) = \gamma_{k,s} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} (c\tau + d)^{-k} \left(\frac{a\tau + b}{c\tau + d}\right)^{-s},$$

where $\tau \in \mathcal{H}$ and $1 \leq Re(s) \leq k-1$ and $\gamma_{k,s} = \frac{1}{2}e^{\pi i s/2}\Gamma(s)\Gamma(k-s)$. Kohnen also computed the Fourier coefficients of $R_{k,s}$ explicitly, proved that first Fourier coefficient does not vanish for large k with $s = \sigma + it_0$, $\frac{k}{2} + \epsilon \leq Re(s) \leq \frac{k+1}{2}$ and obtained a nonvanishing result for L-functions on average.

Kohnen's approach is adopted for obtaining nonvanishing result for L-functions associated to various automorphic forms like half-integral weight modular forms [10], Siegel modular

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forms [6], Hilbert modular forms [14], derivatives of L-functions [7, 9] and product of L-functions [4].

Jacobi forms appears as Fourier-Jacobi coefficients of Siegel modular forms. Berndt [2] associated $2m$ many L-functions to any Jacobi forms of weight k and index m. In the case of Jacobi forms analytic behaviours of L-functions are different to the case of modular forms as L-functions associated to Jacobi forms do not have Euler product. Following the work of Kohnen on nonvanishing of L-functions for modular forms and work of Martin [11] on kernel functions, we obtained a nonvanishing result on average for Jacobi forms defined on $\mathcal{H}\times\mathbb{C}$ [13].

Jacobi forms for arbitrary degree defined on $\mathcal{H}^n\times\mathbb{C}^{j,n}$ were studied by Ziegler [16]. Martin [12] associated a set of L-functions to a Jacobi forms of arbitrary degree and studied their analytic continuation. Aim of this paper is to study kernel functions for Jacobi forms on $\mathcal{H}\times\mathbb{C}^{g,1}$ analogous to the kernel functions considered by Martin. Using the method of Kohnen we obtain results for nonvanishing of L-functions and Poincaré series for Jacobi forms of matrix index.

2. **Preliminaries**

Let $\mathbb{C}$ be complex plane and $\mathcal{H}$ be complex upper half-plane. Let R^n denote the set of row vectors of order $1\times g$ and $R^{g,1}$ be set of column vectores of order $g\times 1$ in the ring R. We denote by $e(x)=e^{2\pi ix}$. Consider the Jacobi group Γ_g^J of degree g for full modular group $\Gamma=SL_2(\mathbb{Z})$ defined as $\Gamma_g^J=\{(M,X): M\in\Gamma,\ X=(\lambda,\mu)\in\mathbb{Z}^{g,1}\times\mathbb{Z}^{g,1}\}$ with usual group law. The Jacobi group acts on the space $\mathcal{H}\times\mathbb{C}^{g,1}$ via

$$\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda,\mu)\right)\cdot(\tau,z)=\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right).$$

Let k be a positive integer and $\mathcal{M}$ be a positive definite symmetric half-integral matrix of order $g\times g$. For $h=\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda,\mu)\right)\in\Gamma_g^J$, we define the automorphic factor w.r.t. h by

$$j_{h,k,\mathcal{M}}(h,\tau,z):=(c\tau+d)^{-k}e\left(\frac{-c}{c\tau+d}\mathcal{M}[z+\lambda\tau+\mu]+\mathcal{M}[\lambda]\tau+2\lambda^t\mathcal{M}z\right),$$

where $A[X]=X^tAX$ for matrices A and X of suitable orders. We define the action of Jacobi group Γ_g^J on the set of all holomorphic functions $f:\mathcal{H}\times\mathbb{C}^{g,1}\to\mathbb{C}$ as

$$(f|_{k,\mathcal{M}}h)(\tau,z)=j_{h,k,\mathcal{M}}(h,\tau,z)f(h\cdot(\tau,z)).$$

Definition 2.1. *Let k be a positive integer and $\mathcal{M}$ be a symmetric positive definite half-integral $g\times g$ matrix. A holomorphic function $f:\mathcal{H}\times\mathbb{C}^{g,1}\to\mathbb{C}$ is said to be a Jacobi form of weight k and index $\mathcal{M}$ if it satisfies*

$$(f|_{k,\mathcal{M}}h)(\tau,z)=f(\tau,z)\ \textit{for every}\ h\in\Gamma_g^J$$

and f has a Fourier expansion of the form

$$f(\tau,z)=\sum_{\substack{n\in\mathbb{Z},R\in\mathbb{Z}^g\\ n\geqslant\frac{1}{4}\mathcal{M}^{-1}[R^t]}}c(n,R)e(n)e(Rz).$$

Furthermore if the inequality is strict in the above sum we say that f is a Jacobi cusp form.

We denote the space of all Jacobi forms of weight k and index $\mathcal{M}$ over $\mathbb{C}$ by $J_{k,\mathcal{M}}$ and Jacobi cusp forms by $J^{cusp}_{k,\mathcal{M}}$. The space of all Jacobi cusp forms is a finite dimensional Hilbert space w.r.t. Petersson inner product defined as

$$\langle f,g\rangle = \int\limits_{\Gamma_g^J\backslash\mathcal{H}\times\mathbb{C}^{g,1}} f(\tau,z)\overline{g(\tau,z)}y^k e(-4\pi\mathcal{M}[v]y^{-1})dV_g^J,$$

where $\tau = x+iy,\ z = u+iv$ and $dV_g^J = dxdydudv$. We also define $z = p\tau+q,\ p,q\in\mathbb{R}^{g,1}$ and $\mu_{k,\mathcal{M}}(\tau,z) := y^{\frac{k}{2}}e(iy\mathcal{M}[p])$. Then one can rewrite above inner product as

$$\langle f,g\rangle = \int\limits_{\Gamma_g^J\backslash\mathcal{H}\times\mathbb{C}^{g,1}} f(\tau,z)\overline{g(\tau,z)}y^k e(2iy\mathcal{M}[p])y^{-2}dxdydpdq.$$

Fourier coefficients of a Jacobi form of weight k and index $\mathcal{M}$ satisfy $c(n,R) = c(n',R')$ whenever $R\equiv R' \pmod{2\mathcal{M}}$ and $n-\frac{1}{4}\mathcal{M}^{-1}[R^t] = n'-\frac{1}{4}\mathcal{M}^{-1}[R'^t]$. Denote $c_\mu(N) = c(n,R)$ whenever $N = 4n-\mathcal{M}^{-1}[R^t]$. Hence any Jacobi form of weight k and index $\mathcal{M}$ can be written as

$$f(\tau,z) = \sum_{\mu\in\mathbb{Z}^g \pmod{2\mathcal{M}}} f_\mu(\tau)\Theta_{\mathcal{M},\mu}(\tau,z)$$

where $f_\mu(\tau) = \sum\limits_{N=0}^{\infty} c_\mu(N)e(\frac{N}{4}\tau)$ and $\Theta_{\mathcal{M},\mu}(\tau,z) = \sum\limits_{\substack{R\in\mathbb{Z}^g\\ R\equiv\mu \pmod{2\mathcal{M}}}} e(\frac{1}{4}\mathcal{M}^{-1}[R^t])e(Rz)$. The above representation is called the theta decomposition of f. To any Jacobi cusp form f we associate $\overline{f}(\tau,z) = \overline{f(-\overline{\tau},-\overline{z})}$. The $\overline{f}$ has Fourier coefficients $\overline{c_\mu(N)}$ in the corresponding theta decomposition. For more details on Jacobi forms of arbitrary degree we refer to [16].

For any Jacobi cusp form of weight k and index $\mathcal{M}$ with theta decomposition $f(\tau,z) = \sum\limits_{\mu \pmod{2\mathcal{M}}} f_\mu(\tau)\Theta_{\mathcal{M},\mu}(\tau,z)$, we define a set of Dirichlet series

$$L_\mu(f,s) = \sum_{D=1}^{\infty} c_\mu(D)\left(\frac{D}{4|\mathcal{M}|}\right)^{-s} \tag{1}$$

for every $\mu\in\mathbb{Z}^g\setminus\mathbb{Z}^g(2\mathcal{M})$. We also define the completed Dirichlet series by

$$\Lambda_\mu(f,s) = (2\pi)^{-s}\Gamma(s)L_\mu(f,s). \tag{2}$$

In [12] Martin studied analytic properties of these Dirichlets series and established a set functional equations.

Theorem 2.2. [12] *Let k be a positive even integer and $\mathcal{M}$ be symmetric, positive definite, half-integral matrix of order $g\times g$. Let $f:\mathcal{H}\times\mathbb{C}^{g,1}\to\mathbb{C}$ be a Jacobi form of weight k and index $\mathcal{M}$. Then for any μ the completed Dirichlet series $\Lambda_\mu(f,s)$ have analytic continuation to whole complex plane and they satisfy*

$$\frac{1}{\sqrt{2^g|\mathcal{M}|}}\sum_{\beta \pmod{2\mathcal{M}}}(e(-\beta(2\mathcal{M})^{-1}\mu^t)+e(\beta(2\mathcal{M})^{-1}\mu^t))\Lambda_\beta(f,s) = i^k\Lambda_\mu(f,k-s-\frac{g}{2}).$$

Now we define Jacobi Poincaré series for Γ_g^J.

Definition 2.3. *Let k be a positive integer, $\mathcal{M}$ be a symmetric, positive definite, half-integral $g\times g$ matrix. Let $n \in \mathbb{Z}$ and $R \in \mathbb{Z}^g$ such that $n-\frac{1}{4}\mathcal{M}^{-1}[R^t] > 0$. We define $(n, R)^{th}$ Poincaré series by*

$$P_{k,\mathcal{M};n,R}(\tau, z) = \sum_{\gamma\in\Gamma^J_{g,\infty}\backslash\Gamma^J_g} e(n\tau)e(Rz)|_{k,\mathcal{M}}\gamma(\tau, z)$$

where $(\tau, z) \in \mathcal{H} \times \mathbb{C}^{g,1}$ and $\Gamma^J_{g,\infty} = \{\left(\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}, (0, \mu)\right) : n \in \mathbb{Z},\ \mu \in \mathbb{Z}^{g,1}\}$.

The set of all Poincaré series of weight k and index $\mathcal{M}$ generates the space of Jacobi cusp forms of weight k and index $\mathcal{M}$. The Poincaré series have interesting property mentioned below:

Theorem 2.4. *The Poincaré series $P_{k,\mathcal{M};n,R} \in J^{cusp}_{k,\mathcal{M}}$. For $f \in J^{cusp}_{k,\mathcal{M}}$ we have*

$$\langle f, P_{k,\mathcal{M};n,R}\rangle = 2^{(g-1)(k-\frac{g}{2}-1)}\pi^{-k+\frac{g}{2}+1}|\mathcal{M}|^{k-(\frac{g+3}{2})}D^{-k+\frac{g}{2}+1}\Gamma(k-\frac{g}{2}-1)c(n, R)$$

and

$$P_{k,\mathcal{M};n,R}(\tau, z) = \sum_{\substack{n'\in\mathbb{Z},\ R'\in\mathbb{Z}^g\\ n'\geqslant\frac{1}{4}\mathcal{M}^{-1}[R'^t]}} p_{k,\mathcal{M};n,R}(n', R')e(n'\tau + R'z)$$

where

$$\begin{aligned} p_{k,\mathcal{M};n,R}(n', R') &= \delta_{\mathcal{M}}(n, R, n', R') + (-1)^k\delta_{\mathcal{M}}(n, R, n', -R')\\ &+ i^k\pi 2^{1-\frac{g}{2}}|\mathcal{M}|^{-\frac{1}{2}}\left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{g}{2}-\frac{1}{2}}\sum_{c\geqslant 1}\Bigg(H_{\mathcal{M}}(n, R, n', R')\\ &+ (-1)^kH_{\mathcal{M}}(n, R, n', -R')\Bigg)J_{k-\frac{g}{2}-1}\left(\frac{\pi\sqrt{DD'}}{2^{g-1}|\mathcal{M}|c}\right), \end{aligned}$$

where $D = det\left(2\begin{pmatrix}n & \frac{1}{2}R\\ \frac{1}{2}R^t & M\end{pmatrix}\right)$, $D' = det\left(2\begin{pmatrix}n' & \frac{1}{2}R'\\ \frac{1}{2}R'^t & M\end{pmatrix}\right)$ and

$$\delta_{\mathcal{M}}(n, R, n', R') = \begin{cases}1 \text{ if } D = D', R' \equiv R(\mathbb{Z}^g 2\mathcal{M})\\ 0 \text{ otherwise}\end{cases}$$

and

$$H_{\mathcal{M}}(n, R, n', R') = c^{-\frac{g}{2}-1}\sum_{x(c)y(c^*)} e_c((\mathcal{M}[x] + Rx + n)\overline{y} + n'y + R'x)e_{2c}(R'\mathcal{M}^{-1}R^t).$$

Here y runs over $(\mathbb{Z}/c\mathbb{Z})^*$ with $\overline{y}y \equiv 1(c)$ and x runs over $(\mathbb{Z}^{g,1}/c\mathbb{Z}^{g,1})$.

For more properties of Poincaré series see [3].

3. Kernel Functions

Let $k > 2g+4$ be a positive even integer and $\mathcal{M}$ be a symmetric positive definite half-integral $g \times g$ matrix. For $t_0 \in (2\mathcal{M})^{-1}\mathbb{Z}^{g,1}$ and $s \in \mathbb{C}$ with $1 < Re(s) < k-2g-1$ we define the kernel functions

$$\Omega^{k,\mathcal{M}}_{t_0,s}(\tau,z) = \sum_{h \in H^J_g \backslash \Gamma^J_g} \phi_{t_0,s}(\tau,z)|_{k,\mathcal{M}}h(\tau,z), \tag{3}$$

where $\phi_{t_0,s}(\tau,z) = \frac{1}{\tau^s}e(-\dfrac{1}{\tau}\mathcal{M}[z-t_0])$ and $H^J_g = \{(Id,(\lambda,0)) : \lambda \in \mathbb{Z}^{g,1}\}$. A set of all coset representatives for $H^J_g \setminus \Gamma^J_g$ is given by $\{(Id,(0,\nu))(M,(0,0)) : M \in \Gamma,\ \nu \in \mathbb{Z}^{g,1}\}$. We now study the analytic properties of these kernel functions.

Theorem 3.1. *Let k be a positive integer, $\mathcal{M}$ be a positive definite symmetric matrix of order g with $k > 2g+4$ and $t_0 \in (2\mathcal{M})^{-1}\mathbb{Z}^{g,1}$. If $1 < Re(s) < k-2g-1$ then*

$$\Omega^{k,\mathcal{M}}_{t_0,s} \in J^{cusp}_{k,\mathcal{M}}.$$

Before proving the Theorem 3.1 we state a lemma (without proof) which we will use.

Lemma 3.2. *For every $(\tau,z) \in \mathcal{H} \times \mathbb{C}^{g,1}$, there exists $r = r(\tau,z) \geq 0$ such that the image of $B(\tau,\frac{1}{2}) \times D(z,\frac{1}{2})$ under any $M \in \Gamma$ is contained in $B(M(\tau),\frac{1}{2}) \times D(0,r)$.*

Proof. We only need to prove that the kernel function is absolutely and uniformly convergent on compact subsets of $\mathcal{H} \times \mathbb{C}^{g,1}$ as transformation property is satisfied by the definition and cusp condition will be proved later when we compute the Fourier expansion of $\Omega^{k,\mathcal{M}}_{t_0,s}$. Both the functions $\phi_{t_0,s}$ and $\phi_{t_0,s}|_{k,\mathcal{M}}h$ are holomorphic functions on $\mathcal{H} \times \mathbb{C}^{g,1}$ for any $h \in H^J_g \setminus \Gamma^J_g$. We have

$$|\phi_{t_0,s}|_{k,\mathcal{M}}h(\tau,z)| \leqslant \frac{2^g\Gamma(1+\frac{g}{2})}{\pi^{1+\frac{g}{2}}r_0^2} \int\limits_{D(\tau_0,r_0)\times D(z,\frac{1}{2})} |\phi_{t_0,s}|_{k,\mathcal{M}}h((\tau',z'))|dx'dy'du'dv'.$$

$\square$

The map $(\tau',z') \mapsto \mu_{k,\mathcal{M}}(\tau',z')y'^{-g-2}$ is continuous and hence there exists a positive no. $m_{(\tau,z)}$ such that

$$1 \leqslant \frac{\mu_{k,\mathcal{M}}(\tau',z')y'^{-g-2}}{m_{(\tau,z)}}$$

for all $(\tau',z') \in D(\tau_0,r_0) \times D(z,\frac{1}{2})$. Hence rewriting the above equation we get

$$|\phi_{t_0,s}|_{k,\mathcal{M}}h(\tau,z)| \leqslant \frac{2^g\Gamma(1+\frac{g}{2})}{\pi^{1+\frac{g}{2}}r_0^2 m_{\tau,z}} \int\limits_{B(\tau,\frac{1}{2})\times D(z,\frac{1}{2})} |\phi_{t_0,s}|_{k,\mathcal{M}}h(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')dV(\tau',z').$$

Summing over all the elements $h \in H^J_g \setminus \Gamma^J_g$ we obtain

$$2^{-g}\pi^{1+\frac{g}{2}}\frac{r_0^2}{\Gamma(1+\frac{g}{2})}m_{\tau,z} \sum_{h \in \mathcal{H}^J_g \backslash \Gamma^J_g} |\phi_{t_0,s}|_{k,\mathcal{M}}h(\tau,z)|$$

$$\begin{aligned}
&\leqslant \sum_{h\in\mathcal{H}_g^J\backslash\Gamma_g^J}\int_{B(\tau,\frac{1}{2})\times D(z,\frac{1}{2})} |\phi_{t_0,s}(h(\tau',z'))|\mu_{k,\mathcal{M}}(h(\tau',z'))dV(\tau',z')\\
&= \sum_{h\in\mathcal{H}_g^J\backslash\Gamma_g^J}\int_{h(B(\tau,\frac{1}{2})\times D(z,\frac{1}{2}))} |\phi_{t_0,s}(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')dV(\tau'z')\\
&= \sum_{M\in\Gamma}\sum_{\nu\in\mathbb{Z}^{g,1}}\int_{[Id,0,\nu].M(B(\tau,\frac{1}{2})\times D(z,\frac{1}{2}))} |\phi_{t_0,s}(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')dV(\tau',z')\\
&\leqslant \sum_{M\in\Gamma}\sum_{\nu\in\mathbb{Z}^{g,1}}\int_{[Id,0,\nu].(B(M(\tau),\frac{1}{2})\times D(0,r))} |\phi_{t_0,s}(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')dV(\tau',z')\\
&= \sum_{M\in\Gamma}\sum_{\nu\in\mathbb{Z}^{g,1}}\int_{(B(M(\tau),\frac{1}{2})}\int_{D(\nu,r))} |\phi_{t_0,s}(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')dV(\tau',z').
\end{aligned}$$

Now we estimate the inner integral

$$\begin{aligned}
&\sum_{\nu\in\mathbb{Z}^{g,1}}\int_{D(\nu,r))} |\phi_{t_0,s}(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')dp'dq'\\
&\leqslant 2rg\int_{\cup_{\nu\in\mathbb{Z}^{g,1}}D(\nu,r))} |\phi_{t_0,s}(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')y'^{-g}du'dv'dp'dq'.
\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
&\sum_{\nu\in\mathbb{Z}^{g,1}}\int_{D(\nu,r))} |\phi_{t_0,s}(\tau',z')|\mu_{k,\mathcal{M}}(\tau',z')dp'dq'\\
&\leqslant \sqrt{\frac{2^{3g}}{|\mathcal{M}|}}R^{2g}|\frac{1}{\tau'^{s-g}}|y'^{\frac{k-3g}{2}}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
&2^{-g}\pi^{1+\frac{g}{2}}\frac{r_0^2}{\Gamma(1+\frac{g}{2})}m_{\tau,z}\sum_{h\in\mathcal{H}_g^J\backslash\Gamma_g^J}|\phi_{t_0,s}|_{k,\mathcal{M}}h((\tau,z))|\\
&\leqslant \sqrt{\frac{2^{3g}}{|\mathcal{M}|}}R^{2g}\sum_{M\in\Gamma}\int_{(B(M(\tau),\frac{1}{2})}|\frac{1}{\tau'^{s-g}}|y'^{\frac{k-3g-4}{2}}dx'dy'.
\end{aligned}$$

We estimate $m_{\tau,z}$ whenever $k>2g+4$ to get

$$\sum_{h\in\mathcal{H}_g^J\backslash\Gamma_g^J}|\phi_{t_0,s}|_{k,\mathcal{M}}h((\tau,z))| \ll \frac{(1+y^2)^g}{y^{(\frac{k}{2}+g)}}\sum_{M\in\Gamma}\int_{(B(M(\tau),\frac{1}{2})}|\frac{1}{\tau'^{s-g}}|y'^{\frac{k-3g-4}{2}}dx'dy'.$$

Proceeding as in [11], for any $1 < r_0 < \sigma$ we get

$$\sum_{h\in\mathcal{H}_g^J\backslash\Gamma_g^J} |\phi_{t_0,s}|_{k,\mathcal{M}}h((\tau,z))| \ll y^{-\frac{k}{2}}(y+\frac{1}{y})^{g+1}e^{\frac{c_1}{y}}\int\limits_{B'} \frac{{y'}^{\frac{k-3g-4}{2}}}{|\tau'|^{\sigma-g-r}|\tau'|^{r}}dx'dy'$$

where $B' = \{\tau' \in \mathcal{H} | y' < T(\tau,\Gamma) = 2cosh(\frac{1}{2})c_\Gamma(y+\frac{1}{y}), \frac{1}{|\tau'|^2} \ll \frac{y+\frac{1}{y}}{y'}\}$. Proceeding as in [11], for any $1 < r_0 < \sigma$ we get

$$\begin{aligned}\sum_{h\in\mathcal{H}_g^J\backslash\Gamma_g^J} |\phi_{t_0,s}|_{k,\mathcal{M}}h((\tau,z))| &\ll y^{-\frac{k}{2}}(y+\frac{1}{y})^{g+1}e^{\frac{c_1}{y}}\int\limits_{y'=0}^{T(\tau,\Gamma)}\int\limits_{x'=-\infty}^{\infty} \frac{{y'}^{\frac{k-3g-4}{2}}}{(x'^2+y'^2)^{\frac{r_0}{2}}}\left(\frac{y+\frac{1}{y}}{y'}\right)^{\frac{\sigma-r_0-g}{2}}dx'dy' \\ &= y^{-\frac{k}{2}}(y+\frac{1}{y})^{\frac{\sigma-r_0+g+2}{2}}e^{\frac{c_1}{y}}\int\limits_{y'=0}^{T(\tau,\Gamma)}\int\limits_{x'=-\infty}^{\infty} \frac{{y'}^{\frac{k-\sigma-2g+r_0-4}{2}}}{(x'^2+y'^2)^{\frac{r_0}{2}}}dx'dy' \\ &\ll y^{-\frac{k}{2}}(y+\frac{1}{y})^{\frac{\sigma-r_0+g+2}{2}}e^{\frac{c_1}{y}}\frac{\Gamma(\frac{r-1}{2})}{\Gamma(\frac{r}{2})}\int\limits_{y'=0}^{T(\tau,\Gamma)} {y'}^{\frac{k-\sigma-2g-r_0-2}{2}}dy \\ &\ll y^{-\frac{k}{2}}(y+\frac{1}{y})^{\frac{\sigma-r_0+g+2}{2}}e^{\frac{c_1}{y}}\frac{\Gamma(\frac{r-1}{2})}{\Gamma(\frac{r}{2})}T(\tau,\Gamma)^{\frac{k-\sigma-r_0-2g}{2}},\end{aligned}$$

whenever $1 < \sigma < k-2g-1$. Hence $\Omega_{t_0,s}^{k,\mathcal{M}}$ converges absolutely and uniformly on compact subsets of $\mathcal{H}\times\mathbb{C}^{g,1}$.

Theorem 3.3. *Let k and $\mathcal{M}$ be as before. For any $f \in J_{k,\mathcal{M}}^{cusp}$, the inner product $\langle\Omega_{t_0,s}^{k,\mathcal{M}}, f\rangle$ is a holomorphic function on the vertical strip $1+\frac{g}{2} < Re(s) < k-2g-1$.*

Proof. By definition we have

$$\Omega_{t_0,s}^{k,\mathcal{M}}(\tau,z) = \sum_{M\in\Gamma}\sum_{\nu\in\mathbb{Z}^{g,1}} \phi_{t_0,s}|_{k,\mathcal{M}}[Id,0,\nu]|_{k,\mathcal{M}}M(\tau,z).$$

Let $t_0 = (2\mathcal{M})^{-1}\beta^t$ with $\beta \in \mathbb{Z}^g$. We have

$$\Omega_{t_0,s}^{k,\mathcal{M}}(\tau,z) = \sum_{M\in\Gamma}\sum_{\nu\in\mathbb{Z}^{g,1}} \phi_{0,s}|_{k,\mathcal{M}}[Id,0,\nu]|_{k,\mathcal{M}}[Id,0,(2\mathcal{M})^{-1}\beta^t]|_{k,\mathcal{M}}M(\tau,z).$$

Theta inversion formula gives

$$\sum_{\nu\in\mathbb{Z}^{g,1}} \phi_{0,s}|_{k,\mathcal{M}}[Id,0,\nu]|_{k,\mathcal{M}} = \frac{1}{\sqrt{(2i)^g|\mathcal{M}|}}\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} \Theta_{\mathcal{M},R}(\tau,z).$$

Hence we get

$$\Omega_{t_0,s}^{k,\mathcal{M}}(\tau,z) = \frac{1}{\sqrt{(2i)^g|\mathcal{M}|}}\sum_{M\in\Gamma}\big(\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z)\big)|_{k,\mathcal{M}}M.$$

Thus we have

$$\begin{aligned}\langle \Omega^{k,\mathcal{M}}_{t_0,s}, f\rangle &= \int\limits_{\Gamma\backslash\mathcal{H}\times\mathbb{C}^{g,1}} \Omega^{k,\mathcal{M}}_{t_0,s}(\tau,z)\overline{f(\tau,z)}\mu^2_{k,\mathcal{M}}dV\\ &= \frac{1}{\sqrt{(2i)^g|\mathcal{M}|}}\int\limits_{\Gamma\backslash\mathcal{H}\times\mathbb{C}^{g,1}}\sum_{M\in\Gamma}\big(\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z)\big)|_{k,\mathcal{M}}M\\ &\times\ \overline{f(\tau,z)|_{k,\mathcal{M}}M}\mu^2_{k,\mathcal{M}}dV.\end{aligned}$$

Using the transformation formula for $\mu_{k,\mathcal{M}}$ and usual unfolding argument we get

$$\sqrt{(2i)^g|\mathcal{M}|}\langle\Omega^{k,\mathcal{M}}_{t_0,s}, f\rangle = \int\limits_{\mathcal{H}}\ \int\limits_{\mathbb{Z}^{g,1}\tau+\mathbb{Z}^{g,1}\backslash\mathbb{C}^{g,1}}\big(\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z)\big)\overline{f(\tau,z)}\mu^2_{k,\mathcal{M}}dV.$$

Putting the theta decomposition of f we get

$$\sqrt{(2i)^g|\mathcal{M}|}\langle\Omega^{k,\mathcal{M}}_{t_0,s}, f\rangle = \frac{1}{2^g\sqrt{|\mathcal{M}|}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\int\limits_{\mathcal{H}}\frac{1}{\tau^{s-\frac{g}{2}}}\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}dxdy.$$

Now we consider the inner integral

$$\begin{aligned}&\int\limits_{\mathcal{H}}\frac{1}{\tau^{s-\frac{g}{2}}}\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}dxdy\\ =&\int\limits_{y=0}^{\infty}\int\limits_{x=0}^{1}\sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^{s-\frac{g}{2}}}e(\frac{n}{4}M^{-1}[R^t])\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}dxdy\\ =&\sum_{n_0 \pmod{4|\mathcal{M}|}} e(\frac{n_0}{4}M^{-1}[R^t])\int\limits_{y=0}^{\infty}\int\limits_{x=0}^{1}\zeta_{4|\mathcal{M}|}(\tau+n_0, s-\frac{g}{2})\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}dxdy,\end{aligned}$$

where $\zeta_{m\mathbb{Z}}(\tau,z)=\sum\limits_{l\in\mathbb{Z}}(\tau+4|\mathcal{M}|l)^{-s}$. Hence we need to show that integral

$$\int\limits_{y=0}^{\infty}\int\limits_{x=0}^{1}\zeta_{4|\mathcal{M}|}(\tau+n_0, s-\frac{g}{2})\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}dxdy$$

defines a holomorphic function of s on the given region. Proceeding as in [11] we note that $f_R(\tau)=O(e^{-\pi\frac{y}{2|\mathcal{M}|}})$ as $y\to\infty$ uniformly on x and for $\sigma=Re(s)>1+\frac{g}{2}$ we have

$$\zeta_{4|\mathcal{M}|\mathbb{Z}}(\tau+n_0, s-\frac{g}{2})\ll\frac{e^{-\pi\frac{y}{2|\mathcal{M}|}}}{(4|\mathcal{M}|)^{-\sigma+\frac{g}{2}}}(1+y^{-\sigma+\frac{g}{2}}).$$

Hence we get

$$\begin{aligned}
&\int\limits_{y=0}^{\infty}\int\limits_{x=0}^{1} |\zeta_{4|\mathcal{M}|}(\tau+n_0, s-\frac{g}{2})\overline{f_R(\tau)}y^{k-\frac{g}{2}-2}|dxdy\\
\ll\ &\int\limits_{y=0}^{\infty} e^{-\pi\frac{y}{|\mathcal{M}|}}(y^{k-\frac{g}{2}-2}+y^{k-2-\sigma})dy\\
\ll\ &(\frac{\pi}{|\mathcal{M}|})^{-k+\frac{g}{2}+1}\Gamma(k-\frac{g}{2}-1)+\left(\frac{\pi}{|\mathcal{M}|}\right)^{-k+\sigma+1}\Gamma(k-\sigma-1).
\end{aligned}$$

From this relation we deduce that the considered integral is absolutely and uniformly convergent on $1+\frac{g}{2}<Re(s)<k-2g-1$. Hence we have the theorem. ☐

Theorem 3.4. *Let $k>2g+4$ and $\mathcal{M}$ be as above and $t_0\in(2\mathcal{M})^{-1}\mathbb{Z}^g$. If $s\in\mathbb{C}$ such that $1+\frac{g}{2}<Re(s)<k-2g-1$ then we have*

$$\begin{aligned}
\Omega^{k,\mathcal{M}}_{t_0,s}(\tau,z) &= \frac{1}{\sqrt{(2i)^g|\mathcal{M}|}}\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum_{R\in z^g\backslash z^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\\
&\times\ \sum_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1}P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(\tau,z).
\end{aligned}$$

Proof. Rewriting the definition of $\Omega^{k,\mathcal{M}}_{t_0,s}$ as in proof of Theorem 3.3

$$\sqrt{(2i)^g|\mathcal{M}|}\Omega^{k,\mathcal{M}}_{t_0,s}(\tau,z)=\sum_{M\in\Gamma\backslash\Gamma_\infty}(\sum_{M'\in\Gamma_\infty}\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z))|_{k,\mathcal{M}}M'|_{k,\mathcal{M}}M.$$

Now $\Gamma_\infty=\{\pm\begin{pmatrix}1 & l\\ 0 & 1\end{pmatrix}: l\in\mathbb{Z}\}$. Then

$$\begin{aligned}
&\sum_{M'\in\Gamma_\infty}\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z))|_{k,\mathcal{M}}M'\\
=\ &\sum_{l\in\mathbb{Z}}\frac{1}{(\tau+l)^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}(e(-R(2\mathcal{M})^{-1}\beta^t)+e(R(2\mathcal{M})^{-1}\beta^t))\Theta_{\mathcal{M},R}(\tau+l,z).
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\sum_{M'\in\Gamma_\infty}\frac{1}{\tau^{s-\frac{g}{2}}}\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t)\Theta_{\mathcal{M},R}(\tau,z))|_{k,\mathcal{M}}M'\\
=\ &\sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})}(e(-R(2\mathcal{M})^{-1}\beta^t)+e(R(2\mathcal{M})^{-1}\beta^t))\Theta_{\mathcal{M},R}(\tau,z)\sum_{l_0=1}^{4|\mathcal{M}|}\sum_{l\in\mathbb{Z}}\frac{e(\frac{1}{4}\mathcal{M}^{-1}[R^t]l_0)}{(\tau+l_0+4|\mathcal{M}|l)^{s-\frac{g}{2}}}
\end{aligned}$$

$$\begin{aligned}
= & \sum_{l_0=1}^{4|\mathcal{M}|} \sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} (e(\frac{l_0}{4}\mathcal{M}^{-1}[R^t] - R(2\mathcal{M})^{-1}\beta^t) + e(\frac{l_0}{4}\mathcal{M}^{-1}[R^t] + R(2\mathcal{M})^{-1}\beta^t))\Theta_{\mathcal{M},R}(\tau,z) \\
& \times\zeta_{4|\mathcal{M}|\mathbb{Z}}(\tau+l_0, s-\frac{g}{2}).
\end{aligned}$$

Putting the Fourier expansion of $\zeta_{4|\mathcal{M}|\mathbb{Z}}(\tau,s) = \frac{1}{(4|\mathcal{M}|)}\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})}\sum\limits_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1}e(\frac{D(\tau+l_0)}{4|\mathcal{M}|})$ we get

$$\begin{aligned}
& \sqrt{(2i)^g|\mathcal{M}|}\Omega_{t_0,s}^{k,\mathcal{M}}(\tau,z) \\
= & \frac{1}{(4|\mathcal{M}|)}\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})} \sum_{l_0=1}^{4|\mathcal{M}|} \sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} [e(\frac{l_0}{4}\mathcal{M}^{-1}[R^t] - R(2\mathcal{M})^{-1}\beta^t) \\
+ & e(\frac{l_0}{4}\mathcal{M}^{-1}[R^t] + R(2\mathcal{M})^{-1}\beta^t)] \sum_{D=1}^{\infty}\sum_{M\in\Gamma}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1}\Theta_{\mathcal{M},R}(\tau,z)e(\frac{D(\tau+l_0)}{4|\mathcal{M}|})|_{k,\mathcal{M}}M \\
= & \frac{1}{(4|\mathcal{M}|)}\frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})} \sum_{l_0=1}^{4|\mathcal{M}|} \sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} [e(\frac{l_0}{4}\mathcal{M}^{-1}[R^t] - R(2\mathcal{M})^{-1}\beta^t) \\
+ & e(\frac{l_0}{4}\mathcal{M}^{-1}[R^t] + R(2\mathcal{M})^{-1}\beta^t)] \sum_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1}e(\frac{Dl_0}{4|\mathcal{M}|}) \\
\times & \sum_{M\in\Gamma} \sum_{\substack{\mu\in\mathbb{Z}^g \\ \mu\equiv R \pmod{\mathbb{Z}^g(2\mathcal{M})}}} e((\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[\mu^t])\tau+\mu z)|_{k,\mathcal{M}}M.
\end{aligned}$$

For every $D,\ l_0$ and R we have

$$\begin{aligned}
& \sum_{M\in\Gamma} \sum_{\substack{\mu\in\mathbb{Z}^g \\ \mu\equiv R \pmod{\mathbb{Z}^g(2\mathcal{M})}}} e((\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[\mu^t])\tau+\mu z)|_{k,\mathcal{M}}M \\
= & \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z},\ (c,d)=1,\ \mu\in\mathbb{Z}^g \\ \mu\equiv R \pmod{\mathbb{Z}^g(2\mathcal{M})}}} (c\tau+d)^{-k}e(\frac{-c}{c\tau+d}M[z^t])e((\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[\mu^t])\frac{a\tau+b}{c\tau+d}+\frac{\mu z}{c\tau+d})
\end{aligned}$$

where a and b are choosen such that $ad-bc=1$. We get

$$\begin{aligned}
& \sum_{M\in\Gamma} \sum_{\substack{\mu\in\mathbb{Z}^g \\ \mu\equiv R \pmod{\mathbb{Z}^g(2\mathcal{M})}}} e((\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[\mu^t])\tau+\mu z)|_{k,\mathcal{M}}M \\
= & \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z},\ (c,d)=1, \\ \mu\in\mathbb{Z}^g}} (c\tau+d)^{-k}e(-\frac{c}{c\tau+d}\mathcal{M}[z^t]+\frac{a\tau+b}{c\tau+d}\mathcal{M}[\mu^t]+2\mu\mathcal{M}\frac{z^t}{c\tau+d}) \\
\times & e((\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t])\frac{a\tau+b}{c\tau+d})e(R\frac{z}{c\tau+d}+R\mu\frac{a\tau+b}{c\tau+d})
\end{aligned}$$

$$= \frac{1}{2} P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(\tau, z).$$

Hence we have

$$\begin{aligned}
&\sqrt{(2i)^g|\mathcal{M}|}\,\Omega^{k,\mathcal{M}}_{t_0,s}(\tau, z)\\
= \ & \frac{1}{(8|\mathcal{M}|)} \frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})} \sum_{D=1}^{\infty} (\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1} \sum_{l_0=1}^{4|\mathcal{M}|} \sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e((\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t])l_0)\\
\times \ & (e(-R(2\mathcal{M})^{-1}\beta^t)+e(R(2\mathcal{M})^{-1}\beta^t))P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(\tau, z).
\end{aligned}$$

But

$$\sum_{l_0=1}^{4|\mathcal{M}|} e((\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t])l_0) = \begin{cases} 4|\mathcal{M}| \text{ if } \frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]\in\mathbb{Z},\\ 0 \text{ otherwise.}\end{cases}$$

Hence we get

$$\begin{aligned}
&\sqrt{(2i)^g|\mathcal{M}|}\,\Omega^{k,\mathcal{M}}_{t_0,s}(\tau, z)\\
= \ & \frac{(2\pi)^{s-\frac{g}{2}}}{2e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})} \sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} (e(-R(2\mathcal{M})^{-1}\beta^t)+e(R(2\mathcal{M})^{-1}\beta^t))\\
\times \ & \sum_{D=1}^{\infty} (\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1} P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(\tau, z)\\
= \ & \frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})} \sum_{R\in\mathbb{Z}^g\backslash\mathbb{Z}^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t) \sum_{D=1}^{\infty} (\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1} P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(\tau, z)
\end{aligned}$$

where in the last line we have used the relation $P_{k,\mathcal{M};n,R} = P_{k,\mathcal{M};n,-R}$ for k even. Hence the theorem follows. ☐

Corollary 3.5. *Let $k > 2g+4$ and $\mathcal{M}$ be as above and $t_0 \in (2\mathcal{M})^{-1}\mathbb{Z}^g$. If $s\in\mathbb{C}$ such that $1+\frac{g}{2} < Re(s) < k-2g-1$ and $f \in J^{cusp}_{k,\mathcal{M}}$ then we have*

$$\langle \Omega^{k,\mathcal{M}}_{t_0,s}, f\rangle = \frac{\pi}{2^{k-2}e^{\pi i\frac{s}{2}}} \frac{\Gamma(k-\frac{g}{2}-1)}{\Gamma(s-\frac{g}{2})\Gamma(k-s)} \sum_{R\in\mathbb{Z}^g\backslash(\mathbb{Z}^g(2\mathcal{M}))} e(-Rt_0)\Lambda_R(\overline{f}, k-s).$$

Proof. It follows from Theorem 2.4, (2) and above theorem. ☐

4. Nonvanishing of L-functions

Theorem 4.1. *Let k be a positive integer, $\mathcal{M}$ be a positive definite symmetric matrix of order g with $k > 2g+4$ and $t_0 \in (2\mathcal{M})^{-1}\mathbb{Z}^g$. If $1 < Re(s) < k-2g-1$ then $\Omega^{k,\mathcal{M}}_{t_0,s}$ has Fourier series expansion*

$$\Omega^{k,\mathcal{M}}_{t_0,s}(\tau, z) = \sum_{\substack{n\in\mathbb{Z},\ R\in\mathbb{Z}^g,\\ 4n>M^{-1}[R^t]}} \omega_k(n,R)e(n\tau+Rz),$$

where

$$\begin{aligned}\omega_k(n,R) &= \frac{\pi^{s-\frac{g}{2}} i e(-\frac{s}{4}) D^{s-\frac{g}{2}-1}}{2^{g-s}\sqrt{|\mathcal{M}|}} \frac{(e(-Rt_0)+e(Rt_0))}{\Gamma(s-\frac{g}{2})}\\ &+ (-i)^{k-s-1}\frac{(2\pi D)^{k-s-1}}{\Gamma(k-s)}\{e(-\frac{s}{2})\mathcal{I}_{\{2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)+e(\frac{-k+s}{2})\mathcal{I}_{\{-2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)\}\\ &+ \frac{(2\pi)^{k-\frac{g}{2}} D^{k-\frac{g}{2}-1} i^{1-s-\frac{g}{2}}}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})} \sum_{\substack{(a,c)=1,\ c'c\equiv 1(mod\ a),\\ ac>0}} (\frac{a}{c})^{k-s} a^{-k}\\ & \sum_{\nu(a\mathbb{Z}^{g,1})} e(R\frac{\nu-t_0}{a})\Big[e(-\frac{c}{a}\mathcal{M}[\nu-t_o])e(\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};-\frac{2\pi Di}{ac})\\ &+ e(\frac{c}{a}\mathcal{M}[\nu-t_o])e(-\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};\frac{2\pi Di}{ac})\Big],\end{aligned}$$

where $\mathcal{I}_X(a) = \begin{cases} 1 & \text{if } a\in X,\\ 0 & \text{otherwise}\end{cases}$ *and* ${}_1F_1(-,-;-)$ *is Kummer's hypergeometric function.*

Proof. Rewriting the definition of $\Omega^{k,\mathcal{M}}_{t_0,s}$ we have

$$\Omega^{k,\mathcal{M}}_{t_0,s}(\tau,z) = \sum_{\substack{\nu\in\mathbb{Z}^g\\ \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma}} (c\tau+d)^{-k} e(-\frac{c}{c\tau+d}\mathcal{M}[z])(\frac{a\tau+b}{c\tau+d})^{-s} e\Bigg(\frac{-\mathcal{M}[\frac{z}{c\tau+d}+\nu-t_0]}{\frac{a\tau+b}{c\tau+d}}\Bigg).$$

We break the sum into three parts corresponding to the matrices with $c=0,\ a=0$ and $ac\neq 0$ and compute the Fourier expansion of each part. Sum C_0 corresponding to matrices $c=0$ i.e.$\{\pm\begin{pmatrix}1 & l\\ 0 & 1\end{pmatrix} : l\in\mathbb{Z}\}$ is given by

$$C_0 = \sum_{l\in\mathbb{Z},\ \nu\in\mathbb{Z}^{g,1}} \Big[(\tau+l)^{-s} e(-\frac{\mathcal{M}[z+\nu-t_0]}{\tau+l}) + (\tau+l)^{-s} e(-\frac{\mathcal{M}[-z+\nu-t_0]}{\tau+l})\Big].$$

The contribution of first part of sum to $(n,R)^{th}$ Fourier coefficient $c_{01}(n,R)$ is given by

$$c_{01}(n,R) = \int\limits_{ic-\infty}^{ic+\infty} (\int\limits_{ic_1-\infty}^{ic_1+\infty} \ldots \int\limits_{ic_g-\infty}^{ic_g+\infty} \tau^{-s} e(-\frac{\mathcal{M}[z-t_0]}{\tau}) e(-R.z)dz) e(-n\tau)d\tau,$$

where $c, c_i\in\mathbb{R}$ for $i=1,2,...,g$ and $c>0$.

$$c_{01}(n,R) = e(-Rt_0)\int\limits_{ic-\infty}^{ic+\infty} (\int\limits_{ic_1-\infty}^{ic_1+\infty} \ldots \int\limits_{ic_g-\infty}^{ic_g+\infty} \tau^{-s} e(-\frac{\mathcal{M}[z]}{\tau} - Rz)dz) e(-n\tau)d\tau$$

$$\begin{aligned} &= e(-Rt_0)\frac{1}{(2i)^{\frac{g}{2}}\sqrt{|\mathcal{M}|}}\int\limits_{ic-\infty}^{ic+\infty}\tau^{-s}e(\frac{1}{4}\mathcal{M}^{-1}[R^t]\tau)\tau^{\frac{g}{2}}e(-n\tau)d\tau \\ &= \frac{\pi^{s-\frac{g}{2}}ie(-\frac{s}{4})(n-\frac{1}{4}\mathcal{M}^{-1}[R^t])^{s-\frac{g}{2}-1}}{2^{g-s}\sqrt{|\mathcal{M}|}}\frac{e(-Rt_0)}{\Gamma(s-\frac{g}{2})}. \end{aligned}$$

Similarly one can compute the contribution of second part to get the Fourier expansion of C_0

$$C_0(n,R) = \frac{\pi^{s-\frac{g}{2}}ie(-\frac{s}{4})(n-\frac{1}{4}\mathcal{M}^{-1}[R^t])^{s-\frac{g}{2}-1}}{2^{g-s}\sqrt{|\mathcal{M}|}}\frac{(e(-Rt_0)+e(Rt_0))}{\Gamma(s-\frac{g}{2})}. \tag{4}$$

The sum A_0 corresponding to matrices $a=0$ i.e.$\{\pm\begin{pmatrix}0 & -1\\ l & 1\end{pmatrix} : l\in\mathbb{Z}\}$ is given by

$$\begin{aligned} A_0 &= \sum_{l\in\mathbb{Z},\ \nu\in\mathbb{Z}^g}\Bigg[e(-\frac{s}{2})(\tau+l)^{-k+s}e\Bigg(-\frac{\mathcal{M}[z]}{\tau+l}\Big)e(-\frac{\mathcal{M}[\frac{z}{\tau+l}+\nu-t_0]}{\frac{-1}{\tau+l}}\Bigg) \\ &+ e(\frac{s}{2})(\tau+l)^{-k+s}e(-\frac{\mathcal{M}[z]}{\tau+l})e\Bigg(-\frac{\mathcal{M}[\frac{-z}{\tau+l}+\nu-t_0]}{\frac{-1}{\tau+l}}\Bigg)\Bigg] \\ &= \sum_{l\in\mathbb{Z},\ \nu\in\mathbb{Z}^g}\Bigg[e(-\frac{s}{2})(\tau+l)^{-k+s}e\Bigg(-\frac{\mathcal{M}[z]}{\tau+l}\Big)e(\frac{\mathcal{M}[\frac{z}{\tau+l}+\nu-t_0]}{\frac{1}{\tau+l}}\Bigg) \\ &+ e(\frac{s}{2})(\tau+l)^{-k+s}e(-\frac{\mathcal{M}[z]}{\tau+l})e\Bigg(\frac{\mathcal{M}[\frac{z}{\tau+l}+\nu+t_0]}{\frac{1}{\tau+l}}\Bigg)\Bigg] \\ &= \sum_{l\in\mathbb{Z},\ \nu\in\mathbb{Z}^g}\Bigg[e(-\frac{s}{2})(\tau+l)^{-k+s}e(2z\mathcal{M}(\nu-t_0)+(\tau+l)\mathcal{M}[\nu-t_0]) \\ &+ e(\frac{s}{2})(\tau+l)^{-k+s}e(2z\mathcal{M}(\nu+t_0)+(\tau+l)\mathcal{M}[\nu+t_0])\Bigg]. \end{aligned}$$

Similar to case of $c=0$ we get the Fourier coefficients corresponding to $a=0$

$$A_0(n,R) = \begin{cases} \dfrac{(2\pi)^{k-s}e(-\frac{s}{2})(-i)^{k-s-1}e(-\frac{s}{2})D^{k-s-1}}{\Gamma(k-s)} & \text{if } R+2\mathcal{M}t_0\in 2\mathcal{M}\mathbb{Z}^g, \\ \dfrac{(2\pi)^{k-s}e(\frac{s}{2})(-i)^{k-s-1}e(-\frac{s}{2})D^{k-s-1}}{\Gamma(k-s)} & \text{if } R-2\mathcal{M}t_0\in 2\mathcal{M}\mathbb{Z}^g. \end{cases} \tag{5}$$

Now consider the sum corresponding to matrices with $ac\neq 0$.

$$\begin{aligned} B_0 &= \sum_{\substack{ac\neq0,\ (a,c)=1,\\ \nu\in\mathbb{Z}^{g,1}}}(c\tau+d)^{-k}e(-\frac{c}{c\tau+d}\mathcal{M}[z])(\frac{a\tau+b}{c\tau+d})^{-s}e\Bigg(\frac{-\mathcal{M}[\frac{z}{c\tau+d}+\nu-t_0]}{\frac{a\tau+b}{c\tau=d}}\Bigg) \\ &= \sum_{\substack{ac\neq0,\ (a,c)=1,\\ \nu\in\mathbb{Z}^{g,1}}}\Bigg(\frac{c\tau+d}{a\tau+b}\Bigg)^{-k+s}(a\tau+b)^{-k}e(-\frac{c}{a}\mathcal{M}[\nu-t_0])e\Bigg(-\frac{a}{a\tau+b}\mathcal{M}[z+\frac{\nu-t_0}{a}]\Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{ac\neq 0,\ (a,c)=1,\\ \nu\in\mathbb{Z}^{g,1}}} a^{-k}\left(\frac{c}{a}-\frac{1}{a^2(\tau+\frac{b}{a})}\right)^{-k+s}(\tau+\frac{b}{a})^{-k}e(-\frac{c}{a}\mathcal{M}[\nu-t_0])e\left(-\frac{1}{\tau+\frac{b}{a}}\mathcal{M}[z+\frac{\nu-t_0}{a}]\right)\\
&= \sum_{\substack{ac\neq 0,\ (a,c)=1,\ \nu(\mathbb{Z}^{g,1}a),\\ bc\equiv 1(a),\ \alpha\in\mathbb{Z},\ \beta\in\mathbb{Z}^{g,1}}} a^{-k}\left(\frac{c}{a}-\frac{1}{a^2(\tau+\beta+\frac{b}{a})}\right)^{-k+s}(\tau+\beta+\frac{b}{a})^{-k}e(-\frac{c}{a}\mathcal{M}[\nu-t_0])\\
&\times\ e\left(-\frac{1}{\tau+\beta+\frac{b}{a}}\mathcal{M}[z+\alpha+\frac{\nu-t_0}{a}]\right)\\
&= \sum_{\substack{ac\neq 0,\ (a,c)=1,\ \nu(\mathbb{Z}^{g,1}a),\\ bc\equiv 1(a),\ \alpha\in\mathbb{Z},\ \beta\in\mathbb{Z}^{g,1}}} a^{-k}e(-\frac{c}{a}\mathcal{M}[\nu-t_0])F_{a,c}(\tau+\frac{b}{a},z+\frac{\nu-t_0}{a}),
\end{aligned}
$$

where $F_{a,c}(\tau,z)=\sum_{\alpha\in\mathbb{Z}^{g,1},\beta\in\mathbb{Z}}\left(\frac{c}{a}-\frac{1}{a^2(\tau+\beta)}\right)^{-k+s}(\tau+\beta)^{-k}e\left(-\frac{1}{\tau+\beta}\mathcal{M}[z+\alpha]\right)$. For $ac>0$ we compute the contribution to $(n,R)^{th}$ Fourier coefficient

$$
\begin{aligned}
F^{+}_{a,c}(n,R) &= \int_{ic-\infty}^{ic+\infty}\left(\frac{c}{a}-\frac{1}{a^2\tau}\right)^{-k+s}\tau^{-k}\left(\int_{ic_1-\infty}^{ic_1+\infty}\cdots\int_{ic_g-\infty}^{ic_g+\infty}e\left(-\frac{1}{\tau}\mathcal{M}[z]-Rz\right)dz\right)e(-n\tau)d\tau\\
&= \frac{1}{(2i)^{\frac{g}{2}}\sqrt{|\mathcal{M}|}}\int_{ic-\infty}^{ic+\infty}\left(\frac{c}{a}-\frac{1}{a^2\tau}\right)^{-k+s}\tau^{-k}e\left(-(n-\frac{1}{4}\mathcal{M}^{-1}[R^t])\tau\right)d\tau.
\end{aligned}
$$

Now we do the change of variables $\tau\rightarrow\frac{a}{c}it$. The above integral becomes

$$
F^{+}_{a,c}(n,R)=\frac{1}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}}(\frac{a}{c})^{-(s-\frac{g}{2}-1)}i^{1-s-\frac{g}{2}}\int_{c-i\infty}^{c+i\infty}\left(t+\frac{i}{a^2}\right)^{-k+s}t^{-(s-\frac{g}{2})}e\left(2\pi(n-\frac{1}{4}\mathcal{M}^{-1}[R^t])\frac{a}{c}t\right)d\tau.
$$

Using the integral representation of Kummer's hypergeometric functions we get that

$$
F^{+}_{a,c}(n,R) = \frac{(2\pi)^{k-\frac{g}{2}}D^{k-\frac{g}{2}-1}i^{1-s-\frac{g}{2}}}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})}(\frac{a}{c})^{k-s}\ {}_1F_1(k-s,k-\frac{g}{2};-\frac{2\pi Di}{ac}).
$$

Similarly one can compute the contribution of the terms with $ac<0$ to get the

$$
\begin{aligned}
B_0(n,R) &= \frac{(2\pi)^{k-\frac{g}{2}}D^{k-\frac{g}{2}-1}i^{1-s-\frac{g}{2}}}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})}\sum_{\substack{(a,c)=1,\ cc'\equiv 1(a),\\ ac>0}}(\frac{a}{c})^{k-s}a^{-k}\\
&\qquad\sum_{\nu(a\mathbb{Z}^{g,1})}e(R\frac{\nu-t_0}{a})[e(-\frac{c}{a}\mathcal{M}[\nu-t_o])e(\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};-\frac{2\pi Di}{ac})\\
&\quad+\ e(\frac{c}{a}\mathcal{M}[\nu-t_o])e(-\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};\frac{2\pi Di}{ac})].
\end{aligned}
$$

$\square$

Theorem 4.2. *Let $\mathcal{M}$ and $\omega_k(n,R)$ be as above for fixed (n,R) with $2Rt_0 \notin \mathbb{Z}+\frac{1}{2}$. Then there exists k_0 such that for all $k > k_0$ the Fourier coefficient $\omega_k(n,R) \neq 0$ for $s = \frac{k}{2}+\frac{g}{4}-\delta-it'$, $0 < \delta < \frac{1}{2}$.*

Proof. Let us assume that for given $\mathcal{M}$, n and R there does not exist any such k_0 i.e. there are infinitely many large k such that $\omega_k(n,R) = 0$.

$$\begin{aligned}
0 &= \frac{\pi^{s-\frac{g}{2}} i e(-\frac{s}{4}) D^{s-\frac{g}{2}-1}}{2^{g-s}\sqrt{|\mathcal{M}|}} \frac{(e(-Rt_0)+e(Rt_0))}{\Gamma(s-\frac{g}{2})} \\
&+ (-i)^{k-s-1}\frac{(2\pi D)^{k-s-1}}{\Gamma(k-s)}\{e(-\frac{s}{2})\mathcal{I}_{\{2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)+e(\frac{-k+s}{2})\mathcal{I}_{\{-2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)\} \\
&+ \frac{(2\pi)^{k-\frac{g}{2}} D^{k-\frac{g}{2}-1} i^{1-s-\frac{g}{2}}}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})} \sum_{\substack{(a,c)=1,\ cc'\equiv 1(a),\\ ac>0}} (\frac{a}{c})^{k-s} a^{-k} \\
&\quad \sum_{\nu(a\mathbb{Z}^{g,1})} e(R\frac{\nu-t_0}{a})\Big[e(-\frac{c}{a}\mathcal{M}[\nu-t_o])e(\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};-\frac{2\pi Di}{ac}) \\
&+ e(\frac{c}{a}\mathcal{M}[\nu-t_o])e(-\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};\frac{2\pi Di}{ac})\Big].
\end{aligned}$$

Or we can write the above equation as

$$\begin{aligned}
-1 &= \frac{2^{g-s}\sqrt{|\mathcal{M}|}(-i)^{k-s-1}(2\pi D)^{k-s-1}\Gamma(s-\frac{g}{2})}{\pi^{s-\frac{g}{2}} i e(-\frac{s}{4}) D^{s-\frac{g}{2}-1}\Gamma(k-s)} \\
&\times \frac{e(-\frac{s}{2})\mathcal{I}_{\{2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)+e(\frac{-k+s}{2})\mathcal{I}_{\{-2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)}{(e(-Rt_0)+e(Rt_0))} \\
&+ \frac{(2\pi)^{k-\frac{g}{2}} D^{k-\frac{g}{2}-1} i^{1-s-\frac{g}{2}} 2^{g-s}\sqrt{|\mathcal{M}|}\Gamma(s-\frac{g}{2})}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})\pi^{s-\frac{g}{2}} i e(-\frac{s}{4}) D^{s-\frac{g}{2}-1}} \frac{1}{(e(-Rt_0)+e(Rt_0))} \\
&\times \sum_{\substack{(a,c)=1,\ cc'\equiv 1(a),\\ ac>0}} (\frac{a}{c})^{k-s} a^{-k} \sum_{\nu(a\mathbb{Z}^{g,1})} e(R\frac{\nu-t_0}{a})\Big[e(\frac{-c}{a}\mathcal{M}[\nu-t_o])e(\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};-\frac{2\pi Di}{ac}) \\
&+ e(\frac{c}{a}\mathcal{M}[\nu-t_o])e(-\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};\frac{2\pi Di}{ac})\Big].
\end{aligned}$$

Taking modulus we have

$$\begin{aligned}
1 &\leqslant \left|\frac{2^{g-s}\sqrt{|\mathcal{M}|}(-i)^{k-s-1}(2\pi D)^{k-s-1}\Gamma(s-\frac{g}{2})}{\pi^{s-\frac{g}{2}} i e(-\frac{s}{4}) D^{s-\frac{g}{2}-1}\Gamma(k-s)}\right| \\
&\times \left|\frac{e(-\frac{s}{2})\mathcal{I}_{\{2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)+e(\frac{-k+s}{2})\mathcal{I}_{\{-2\mathcal{M}t_0+2\mathcal{M}\mathbb{Z}^g\}}(R)}{(e(-Rt_0)+e(Rt_0))}\right|
\end{aligned}$$

$$+\left|\frac{(2\pi)^{k-\frac{g}{2}}D^{k-\frac{g}{2}-1}i^{1-s-\frac{g}{2}}2^{g-s}\sqrt{|\mathcal{M}|}\Gamma(s-\frac{g}{2})}{2^{\frac{g}{2}}\sqrt{|\mathcal{M}|}\Gamma(k-\frac{g}{2})\pi^{s-\frac{g}{2}}ie(-\frac{s}{4})D^{s-\frac{g}{2}-1}}\frac{1}{(e(-Rt_0)+e(Rt_0))}\right|\sum_{\substack{(a,c)=1,\ cc'\equiv 1(a),\\ ac>0}}\left|(\frac{a}{c})^{k-s}a^{-k}\right|$$

$$\sum_{\nu(a\mathbb{Z}^g)}\left|e(R\frac{\nu-t_0}{a})\right|\Bigg[\left|e(-\frac{c}{a}\mathcal{M}[\nu-t_o])e(\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};-\frac{2\pi Di}{ac})\right|$$

$$+\quad\left|e(\frac{c}{a}\mathcal{M}[\nu-t_o])e(-\frac{nc'}{a})\ {}_1F_1(k-s,k-\frac{g}{2};\frac{2\pi Di}{ac})\right|\Bigg].$$

For $s=\frac{k}{2}+\frac{g}{4}-\delta-it'$ we have

$$1\leqslant 2^{\frac{g}{2}+2\delta}\sqrt{|\mathcal{M}|}(\pi)^{2\delta-1}D^{2\delta}\left|\frac{\Gamma(\frac{k}{2}-\frac{g}{2}-\delta-it')}{\Gamma(\frac{k}{2}-\frac{g}{2}+\delta+it')}\right|\frac{1}{\left|(e(-Rt_0)+e(Rt_0))\right|} \tag{6}$$

$$+\quad(2\pi D)^{\frac{k}{2}-\frac{g}{4}+\delta}\left|\frac{\Gamma(\frac{k}{2}-\frac{g}{4}-\delta-it')}{\Gamma(k-\frac{g}{2})}\frac{1}{(e(-Rt_0)+e(Rt_0))}\right|\sum_{\substack{(a,c)=1,\ cc'\equiv 1(a),\\ ac>0}}\left|(\frac{a}{c})^{\frac{k}{2}-\frac{g}{4}+\delta}a^{-k}\right|$$

$$\sum_{\nu(a\mathbb{Z}^{g,1})}\Bigg|\ {}_1F_1(\frac{k}{2}-\frac{g}{4}+\delta+it',k-\frac{g}{2};-\frac{2\pi Di}{ac})\Bigg|+\Bigg|\ {}_1F_1(\frac{k}{2}-\frac{g}{4}+\delta+it',k-\frac{g}{2};\frac{2\pi Di}{ac})\Bigg|\Bigg].$$

Using (13.2.1,[1]) and estimating ${}_1F_1(-,-;-)$ we observe that the infinite series in the sum is convergent and can be bounded by a constant say L for every large k. Hence we get

$$1\leqslant 2^{\frac{g}{2}+2\delta}\sqrt{|\mathcal{M}|}(\pi)^{2\delta-1}D^{2\delta}\left|\frac{\Gamma(\frac{k}{2}-\frac{g}{2}-\delta-it')}{\Gamma(\frac{k}{2}-\frac{g}{2}+\delta+it')}\right|\frac{1}{\left|(e(-Rt_0)+e(Rt_0))\right|}$$

$$+\quad\frac{(2\pi D)^{\frac{k}{2}-\frac{g}{4}+\delta}}{(k-\frac{g}{2}-1)(k-\frac{g}{2}-2)...([\frac{k}{2}])}\left|\frac{\Gamma(\frac{k}{2}-\frac{g}{4}-\delta-it')}{\Gamma([\frac{k}{2}])}\right|\left|\frac{L}{(e(-Rt_0)+e(Rt_0))}\right|.$$

Using the fact that $z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)}\to 1$ as $z\to\infty$ we observe that both the terms on right hand side tends to zero and hence we get a contradiction.

Remark 4.1. In the above inequality (6) one can compute k_0 explicitly such that both the summands are strictly less than $\frac{1}{2}$ for $k>k_0$.

□

Theorem 4.3. *Let $\mathcal{M}$ be as above, t' and $\epsilon>0$. Then there exists $k_0=k_0(t',\epsilon)$ such that for any $k>k_0$ and $s=\sigma+it'$ with $\frac{k}{2}-\frac{g}{4}-\frac{1}{2}<\sigma<\frac{k}{2}-\frac{g}{4}-\epsilon$ or $\frac{k}{2}-\frac{g}{4}+\epsilon<\sigma<\frac{k}{2}-\frac{g}{4}+\frac{1}{2}$ there exists a Hecke eigenform f of weight k and index $\mathcal{M}$ such that the vector valued function $\Lambda(\overline{f},s)=(\Lambda_i(\overline{f},s))_{i\in\mathbb{Z}^g\setminus\mathbb{Z}^g2\mathcal{M}}\neq 0$.*

Proof. We shall prove the theorem for the right part of the critical strip i.e. the region $\frac{k}{2}-\frac{g}{4}+\epsilon<\sigma<\frac{k}{2}-\frac{g}{4}+\frac{1}{2}$. Let $s=\frac{k}{2}+\frac{g}{4}-\delta-it'$ be as above and $\mathcal{B}_{k,\mathcal{M}}$ be basis of eigen forms of weight k and index $\mathcal{M}$. As $\Omega_{t_0,s}^{k,\mathcal{M}} \in J_{k,\mathcal{M}}^{cusp}$, we can express kernel function in terms of elements of $\mathcal{B}_{k,\mathcal{M}}$ as

$$\Omega_{t_0,s}^{k,\mathcal{M}}(\tau, z) = \sum_{f_i \in \mathcal{B}_{k,\mathcal{M}}} \frac{\langle \Omega_{t_0,s}^{k,\mathcal{M}}, f_i \rangle}{\langle f_i, f_i \rangle} f_i.$$

Then comparing the $(n, R)^{th}$ Fourier coefficient of both the sides with $2Rt_0 \notin \mathbb{Z} + \frac{1}{2}$ and using the above theorem there exist k_0 and a Hecke eigenform $f_i \in J_{k,\mathcal{M}}^{cusp}$ with $k > k_0$ such that

$$\langle \Omega_{t_0,s}^{k,\mathcal{M}}, f_i \rangle \;=\; \frac{\pi}{2^{k-2} e^{\pi i \frac{s}{2}}} \frac{\Gamma(k - \frac{g}{2} - 1)}{\Gamma(s - \frac{g}{2})\Gamma(k - s)} \sum_{N \in \mathbb{Z}^g \setminus (\mathbb{Z}^g(2\mathcal{M}))} e(-Rt_0) \Lambda_N(\overline{f_i}, k - s) \neq 0.$$

Hence for $N \in \mathbb{Z}^g \setminus (\mathbb{Z}^g(2\mathcal{M})$, $\Lambda_N(\overline{f_i}, \frac{k}{2} - \frac{g}{4} + \delta + it') \neq 0$. Now using the functional equations we have the theorem. ☐

5. Nonvanishing of Poincaré series

Study of the nonvanishing of Poincaré series and its Fourier coefficient is an interesting problem in analytic number theory. In this direction S. Das [5] and K. Shankhadhar [15] gave the following nonvanishing results:

Theorem 5.1. [5] *Let $2R \equiv 0(mod\ \mathbb{Z}^g 2\mathcal{M})$. Then there exists an integer k_0 and a constant $B \geq 3log2$ such that for all even $k \geqslant k_0$, $P_{k,\mathcal{M};(n,R)}$ does not vanish identically whenever*

$$k' \leq \frac{\pi D}{|2\mathcal{M}|} \leq k'^{1+\alpha(g)} exp\left(-\frac{B\ logk'}{log(logk')} \right),$$

where $k' = k - \frac{g}{2} - 1$ and $\alpha(g) = \begin{cases} \frac{2}{3(g+2)} \ if\ 1 \leqslant g \leqslant 4 \\ \frac{2}{3g} \ if\ g \geqslant 5. \end{cases}$

Definition 5.2. *Let $n \in \mathbb{Z}$ and $R \in \mathbb{Z}^g$ with $4n > \mathcal{M}^{-1}[R^t]$ and $k \geq g + 2$ defined the Poincaré series*

$$P_{k,\mathcal{M};(n,R)}^{N}(\tau, z) = \sum_{\gamma \in \Gamma_{g,\infty}^{J} \setminus \gamma \in \Gamma_g^J(N)} e(n\tau + Rz)|_{k,m}\gamma(\tau, z),$$

where $\Gamma_g^J(N) = \Gamma_0(N) \ltimes (\mathbb{Z}^{g,1} \times \mathbb{Z}^{g,1})$. It is well known that $P_{k,\mathcal{M};(n,R)}^{N}$ is a Jacobi cusp form of weight k and index $\mathcal{M}$ w.r.t. group $\Gamma_g^J(N)$.

Theorem 5.3. [15] *For any $\epsilon > 0$ there exists a positive integer $k_0(\epsilon, \mathcal{M}, N)$ such that $P_{k,\mathcal{M};(n,R)}^{N}(\tau, z)$ does not vanish identically if $k > k_0$ and*

$$D^{\epsilon}(\frac{\pi D}{det(2\mathcal{M})})(D, N)^{\frac{2}{g}} \ll_{\epsilon} (det2\mathcal{M})^{\frac{1}{g}}(\frac{N}{\sigma_0(N)})^{\frac{2}{g}} k'^{1+\alpha(g)}$$

where $k' = k - \frac{g}{2} - 1,\ \alpha(g) = \begin{cases} \frac{2}{3(g+2)}\ if\ 1 \leqslant g \leqslant 4 \\ \frac{2}{3g}\ if\ g \geqslant 5. \end{cases}$

Using the fact that kernel functions can be written as linear sums of Poincaré series (Theorem 3.4) and Theorem 4.2 we get the following nonvanishing result for Jacobi Poincaré series.

Theorem 5.4. *Let $\mathcal{M}$, n and R be as in Theorem 4.2 and $0 < \delta < \frac{1}{2}$. Then for any positive integer $k > k_0$ we have $P_{k,\mathcal{M};n,R} \neq 0$ where*

$$k_0 = max\{8\pi D + 2, 2(\frac{(2\pi D)^{2\delta} 2^{\frac{g}{2}+1}\sqrt{\mathcal{M}}}{\pi(e(Rt_0) + (-Rt_0))})^{\frac{1}{2\delta}} + g + 1, 2\frac{log\left(2^2\pi^3(2\pi D)^{1+\delta+\frac{g}{4}}\right)}{log2} + g + 2\}.$$

Proof. Using Theorem 3.5 one can write the kernel functions as

$$\begin{aligned} &\sqrt{(2i)^g|\mathcal{M}|}\Omega_{t_0,s}^{k,\mathcal{M}} \\ = \;& \frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})} \sum_{R\in z^g\backslash z^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t) \sum_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1} P_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}. \end{aligned}$$

Now we choose (n', R') as in Theorem 4.2 and $t' = 0$ and compare the $(n', R')^{th}$ Fourier coefficient on both the sides of above equation.

$$\begin{aligned} &\sqrt{(2i)^g|\mathcal{M}|}\omega_{t_0,s}^{k,\mathcal{M}}(\tau, z) \\ = \;& \frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})} \sum_{R\in z^g\backslash z^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t) \sum_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1} p_{k,\mathcal{M};(\frac{D}{4|\mathcal{M}|}+\frac{1}{4}\mathcal{M}^{-1}[R^t]),R}(n', R'). \\ = \;& \frac{(2\pi)^{s-\frac{g}{2}}}{e^{\pi i(\frac{s}{2}-\frac{g}{4})}\Gamma(s-\frac{g}{2})} \sum_{R\in z^g\backslash z^g(2\mathcal{M})} e(-R(2\mathcal{M})^{-1}\beta^t) \sum_{D=1}^{\infty}(\frac{D}{4|\mathcal{M}|})^{s-\frac{g}{2}-1} p_{k,\mathcal{M};(n',R')}(\frac{D}{4|\mathcal{M}|} + \frac{1}{4}\mathcal{M}^{-1}[R^t], R). \end{aligned}$$

Hence for $k > k_0$ we have $p_{k,\mathcal{M};(n',R')}(\frac{D}{4|\mathcal{M}|} + \frac{1}{4}\mathcal{M}^{-1}[R^t], R) \neq 0$. In particular, $P_{k,\mathcal{M};(n,R)} \neq 0$ for $k > k_0$.

□

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7. Data availability statement

Data sharing is not applicable to this article, as no data sets were generated or analyzed during the current study.

8. Conflicts of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI-400005, INDIA.

SCHOOL OF MATHEMATICAL SCIENCES, NATIONAL INSTITUTE OF SCIENCE EDUCATION AND RESEARCH, AN OCC OF HOMI BHABHA NATIONAL INSTITUTE, BHUBANESWAR, VIA: JATNI, KHURDA, ODISHA- 752 050, INDIA

Email address: `shivansh.pandey@niser.ac.in, shivansh@math.tifr.res.in`

Email address: `brundaban.sahu@niser.ac.in`