L-FUNCTION ASSOCIATED TO JACOBI FORMS OF HALF-INTEGRAL WEIGHT AND A CONVERSE THEOREM

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ABSTRACT. We associate certain L-functions to a Jacobi form of half-integral weight and study their analytic properties. We also prove a converse theorem for Jacobi form of half-integral weight.

1. Introduction

Modular forms are one of the important objects in modern mathematics. They appear in diverse contexts: Fermat's last theorem, q-hypergeometric series, partitions, elliptic curves, and string theory. The theory of modular forms has a wide-ranging impact on modern mathematics and its applications. Recently, modular forms have proven its significance in analytic number theory by providing new tools and with numerous connections to arithmetic geometry, notably through the theory of L-functions. A modular form f of weight k for the group $SL_2(\mathbb{Z})$ is a complex-valued holomorphic function defined on the complex upper-half plane $\mathcal{H}$ satisfying the following transformation property:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau),\ \tau \in \mathcal{H}$$

for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and holomorphic at the cusp ∞. The above transformation property implies that a modular form admits a Fourier series expansion given by

$$f(\tau) = \sum_{n\geq 0} a_f(n) e^{2\pi i n\tau}.$$

Further, if $a_f(0) = 0$, then we call f to be a cusp form. For each $n \in \mathbb{N}$, the complex number $a_f(n)$ is called the n-th Fourier coefficients of f. A modular form is completely determined by its Fourier coefficients. The Fourier coefficients of a modular forms satisfies the estimates $a_f(n) = O(n^{k+\epsilon})$ for any $\epsilon > 0$. One can associate an L-function to a modular form f of weight k as follows;

$$L_f(s) := \sum_{n\geqslant 1} \frac{a_f(n)}{n^s},\ \Re(s) > k+1.$$

The completed L-function $\Lambda_f(s) = (2\pi)^{-s}\Gamma(s)L_f(s)$ has an analytic continuation to the whole complex plane with possibly simple poles at 0 and k, and satisfies the functional

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equation
$$\Lambda_f(k-s) = (-1)^{\frac{k}{2}}\Lambda_f(s).$$
Hecke studied converse of the above fact which is known as Hecke's converse theorem. More precisely he proved the following:

Theorem 1.1. [5] *Let $k \geqslant 2$ be a positive integer. Let $\{a(n)\}_{n\geqslant 1}$ be a sequence of complex numbers such that $a(n) = O(n^\sigma)$ for some $\sigma > 0$. The function $f(\tau) = \sum_{n\geqslant 1} a(n)e^{2\pi i n\tau}$ defines a cusp form of weight k for full modular group $SL_2(\mathbb{Z})$ if and only if the completed L-function $\Lambda_f(s) = (2\pi)^{-s}\Gamma(s)\sum_{n\geqslant 1}\frac{a(n)}{n^s}$ admits a holomorphic continuation to the whole complex plane $\mathbb{C}$ which is bounded on any vertical strip and satisfies the functional equation $\Lambda_f(s) = (-1)^{\frac{k}{2}}\Lambda_f(k-s)$.*

One can ask the similar question for other kinds of automorphic forms as well. This question attracted several mathematicians in the past where they answered this question in the case of other kinds of automorphic forms, see [2, 3, 6, 8, 13].

Jacobi forms are natural generalization of modular forms. The theory of Jacobi forms was first systematically studied by Eichler and Zagier to prove Saito-Kurokawa conjecture [4]. Berndt [1] associated $2m$ many L-functions $\Lambda_\mu(\phi, s), 0 \leqslant \mu \leqslant 2m-1$ to a Jacobi form ϕ of weight k and index m using the theta decomposition and studied its analytic properties. Martin [8] studied the analytic continuation and converse theorem for Jacobi form involving the L-functions defined by Berndt. We briefly mention the theorem of Martin.

Theorem 1.2. [8] *Let k and m be positive integers. Let $\phi(\tau, z) = \sum_{r^2<4nm} c_\phi(n,r)e^{2\pi i(n\tau+rz)}$ be holomorphic function satisfying*

(i) $\phi(\tau, z)$ converges absolutely and uniformly on compact subsets of $\mathcal{H}\times\mathbb{C}$,

(ii) there exists $\nu > 0$ such that $\phi(\tau, z)e^{2\pi i p z} = O(\Im(\tau)^{-\nu})$ as $\Im(\tau) \to 0$,

(iii) for each λ we have $c(n,r) = c(n+\lambda r+\lambda^2 m, r+2\lambda m)$.

Then the following statements are equivalent:

(1) The function $\phi(\tau, z)$ is a Jacobi form of weight k and index m.

(2) Each completed L-function $\Lambda_\mu(\phi, s), 0 \leqslant \mu \leqslant 2m-1$ associated to $\phi(\tau, z)$, can be analytically continued to a holomorphic function on s-plane. These functions are bounded on any vertical strip and satisfy the functional equations
$$(2m)^{-\frac{1}{2}}\sum_{\mu=0}^{2m-1} e^{-\frac{\pi i a\mu}{m}}\Lambda_\mu(\phi, s) = i^k\Lambda_a(\phi, k-\frac{1}{2}-s),\ 0 \leqslant a \leqslant 2m-1.$$

The above result was generalized for Jacobi forms on congruence subgroups by Martin and Osses [9]. There are two objectives of this paper, first, we associate certain L-functions to a Jacobi form of half-integral weight and study its analytic properties, and second we prove a converse theorem for Jacobi forms of half-integral weight. Our approach is similar to the work of Bruinier [3] related to converse theorem in the case of half-integral weight modular forms, however one needs the notion of Fricke type involution operator for Jacobi forms of half-integral weight.

This paper is organized as follows: In Section 2, we recall the basic definition and some properties of Jacobi forms of half-integral weight. In Section 3, we associate certain Dirichlet series to a Jacobi form of half-integral weight. In Section 4, we first define and study the twist of a Jacobi form of half-integral weight by a Dirichlet character. Next, we define certain Fricke involution type operator and study its properties. In Section 5, we state the main results of the paper. Finally, in Section 6, we provide proofs of the results stated in Section 5.

2. **Notations and preliminaries**

Let $\mathbb{C}$ and $\mathcal{H}$ denote the complex plane and complex upper half-plane, respectively. For a complex number z, we denote

$$\begin{aligned} \sqrt{z} &= |z|^{\frac{1}{2}} e^{\left(\frac{i}{2}\right)\arg z} \text{ with } -\pi < \arg z \leq \pi, \\ z^{\frac{k}{2}} &= \left(\sqrt{z}\right)^k \text{ for any } k \in \mathbb{Z}. \end{aligned}$$

For $z \in \mathbb{C}$ and integers m and n, we put $e_n^m(z) = e^{2\pi i m z/n}$. We also write $e_1^m = e^m(z)$, $e_n^1 = e_n(z)$, and $e_1^1 = e(z)$. For integers m and n, $\left(\frac{m}{n}\right)$ denotes the Jacobi symbol. We now briefly recall the definition and some properties of Jacobi forms of half-integral weight. For a positive integer N, we define the following congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ as follows:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$, let $\widetilde{\gamma} = (\gamma, \phi(\tau))$, where $\phi(\tau)$ is a complex-valued holomorphic function on $\mathcal{H}$ such that $\phi^2(\tau) = t\dfrac{c\tau + d}{\sqrt{det(\gamma)}}$ with $t \in \{1, -1\}$. Then the set

$$G := \left\{ \widetilde{\gamma} = (\gamma, \varphi(\tau)) : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}),\ \varphi^2(\tau) = t\frac{c\tau + d}{\sqrt{det(\gamma)}},\ t = \pm 1 \right\},$$

forms a group with the following operation

$$(\gamma_1, \varphi_1(\tau)) \cdot (\gamma_2, \varphi_2(\tau)) := (\gamma_1\gamma_2, \varphi_1(\gamma_2\tau)\varphi_2(\tau)),\ (\gamma_1, \varphi_1(\tau)),\ (\gamma_2, \varphi_2(\tau)).$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, put $j(\gamma, \tau) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}(c\tau + d)^{1/2}$. Then the association $\gamma \mapsto \widetilde{\gamma} = (\gamma, j(\gamma, \tau))$ is an injective map from $\Gamma_0(4)$ into G.

Let

$$\widetilde{G^J} = \{(\widetilde{\gamma}, X, s) :\ \gamma \in SL_2(\mathbb{R}),\ X \in \mathbb{R}^2,\ s \in S^1\}.$$

Then $\widetilde{G^J}$ is a group, with the group law

$$(\widetilde{\gamma}_1, X, s)(\widetilde{\gamma}_2, Y, s') = \left(\widetilde{\gamma}_1\widetilde{\gamma}_2,\ X\gamma_2 + Y,\ ss' \cdot \det\begin{pmatrix} X\gamma_2 \\ Y \end{pmatrix}\right).$$

The real Jacobi group G^J defined by

$$G^J := \{(\gamma, X, \zeta) : M \in SL_2(\mathbb{R}), X \in \mathbb{R}^2, \zeta \in S^1\}$$

is a subgroup of the group $\widetilde{G^J}$, and acts on $\mathcal{H} \times \mathbb{C}$ as follows:

$$h \cdot (\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right),$$

where $(\tau, z) \in \mathcal{H} \times \mathbb{C},\ h = (\gamma, X, \zeta) \in G^J$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let k and m be fixed positive integers with k odd. For a function $\phi : \mathcal{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ and $h = (\widetilde{\gamma}, X, s) \in \widetilde{G^J}$ with $X = (\lambda, \mu) \in \mathbb{R}^2$, the slash operator $|_{\frac{k}{2},m}$ of weight $\frac{k}{2}$ and index m is defined by

$$\left(\phi|_{\frac{k}{2},m} h\right)(\tau, z) := s^m \varphi(\tau)^{-k} e^m \left(\tfrac{-c(z+\lambda\tau+\mu)^2}{c\tau+d} + 2\lambda^2\tau + 2\lambda z + \lambda\mu \right) \phi\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z+\lambda\tau+\mu}{c\tau+d} \right).$$

For $h = (\widetilde{\gamma}, (0,0), 1)$ with $\widetilde{\gamma} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j(\gamma, \tau) \right), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ the above definition reduces to

$$\left(\phi|_{\frac{k}{2},m} h\right)(\tau, z) := j(\gamma, \tau)^{-k} e^m \left(\tfrac{-cz^2}{c\tau+d} \right) \phi\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z}{c\tau+d} \right).$$

We use the notation $\phi|_{\frac{k}{2},m} h = \phi|_{\frac{k}{2},m} \widetilde{\gamma}$ for $h = (\widetilde{\gamma}, (0,0), 1)$ with $\widetilde{\gamma} = (\gamma, j(\gamma, \tau)), \gamma \in \Gamma_0(N)$.

For positive integers α, β and N with $4|N$, consider the subgroup $\Gamma^J_{\alpha,\beta}(N)$ of $\widetilde{G^J}$ defined by $\Gamma^J_{\alpha,\beta}(N) := \widetilde{\Gamma_0(N)} \ltimes ((\alpha\mathbb{Z} \times \beta^{-1}\mathbb{Z}) \times \langle \zeta_\beta \rangle)$ i.e.,

$$\Gamma^J_{\alpha,\beta}(N) = \{(\widetilde{\gamma}, (\lambda, \mu), s) : \gamma \in \Gamma_0(N), \lambda \in \alpha\mathbb{Z}, \mu \in \beta^{-1}\mathbb{Z}, s \in \langle \zeta_\beta \rangle\},$$

where $\langle \zeta_\beta \rangle$ is the cyclic group generated by the primitive β-th roots of unity. We use the notation $\Gamma^J_{1,1}(N) = \Gamma^J(N)$. We now define Jacobi forms of half-integral weight for the group $\Gamma^J_{\alpha,\beta}(N)$.

Definition 2.1. *Let k, N, m, α and β be positive integers such that k is odd and $4|N$. Let χ be a Dirichlet character modulo N. A Jacobi form of weight $\frac{k}{2}$ and index βm with character χ for the group $\Gamma^J_{\alpha,\beta}(N)$ is a complex-valued holomorphic function ϕ defined on $\mathcal{H} \times \mathbb{C}$ satisfying the following conditions:*

(1) $\phi \mid_{\frac{k}{2},\beta m} h = \chi(d)\phi$, for all $h = (\widetilde{\gamma}, X, s) \in \Gamma^J_{\alpha,\beta}(N)$ with $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$,

(2) for every $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Q})$ there exists an integer d_σ such that the function $\phi \mid_{\frac{k}{2},\beta m} h$, where $h = (\sigma^{-1}, (0,0,1))$ has a Fourier expansion of the form

$$\phi \mid_{\frac{k}{2},\beta m} h = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \leq 4n\beta m d_\sigma}} c_{\phi,\sigma}(n, r) e\left(\frac{n}{d_\sigma}\tau + \frac{r}{d_\sigma} z \right).$$

Further, if the Fourier coefficients $c_{\phi,\sigma}(n, r)$ satisfy $c_{\phi,\sigma}(n, r) = 0$ whenever $4\beta m n d_\sigma = r^2$ for every $\sigma \in SL_2(\mathbb{Q})$, then ϕ is called a Jacobi cusp form.

We denote the space of Jacobi forms (respectively, Jacobi cusp forms) of weight $\frac{k}{2}$ and index βm with character χ for the group Γ^J_N by $J_{\frac{k}{2},\beta m}(\Gamma^J_{\alpha,\beta}(N), \chi)$ (respectively, $J^{cusp}_{\frac{k}{2},\beta m}(\Gamma^J_{\alpha,\beta}(N), \chi)$). For more details on the theory of Jacobi forms of half-integral weight for the group $\Gamma^J(N)$, we refer to [11, 12].

2.1. **Theta decomposition.** Let $\phi \in J_{\frac{k}{2},\beta m}(\Gamma^J_{\alpha,\beta}(N),\chi)$. Then using the transformation property of $\phi(\tau,z)$ we have the following Fourier series expansion of ϕ

$$\phi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ \beta r^2\le 4nm}} c_\phi(n,r)e(n\tau + r\beta z). \tag{1}$$

For $D \geq 0,\ r \pmod{2m\alpha}$, we define a sequence $\{c_\mu(D)\}$ of complex numbers as follows:

$$c_\mu(D) := \begin{cases} c_\phi\left(\frac{D+\beta r^2}{4m}, r\right), \text{ if } D \equiv -\beta r^2 \pmod{4m}, r \equiv \mu \pmod{2m\alpha}, \\ 0, \text{ otherwise.}\end{cases} \tag{2}$$

Set

$$h_\mu(\tau) := \sum_{D=0}^{\infty} c_\mu(D) e_{4m}(D\tau), \tag{3}$$

and for a natural number l, consider the Jacobi theta function defined by

$$\theta_{l,\mu}(\tau,z) := \sum_{\substack{r\in\mathbb{Z}\\ r\equiv\mu \pmod{2l}}} e\left(\frac{r^2}{4l}\tau + rz\right). \tag{4}$$

The equations (1), (3) and (4) implies the following decomposition of the Jacobi form $\phi(\tau,z)$:

$$\phi(\tau,z) = \sum_{\mu=1}^{2m\alpha} h_\mu(\tau)\theta_{m\alpha,\mu}(\alpha\beta\tau,\beta z). \tag{5}$$

The above equation is called the theta decomposition of the Jacobi form ϕ and the functions h_μ are called the theta components of the Jacobi form ϕ. The transformation property of the Jacobi form ϕ inherits certain transformation properties to h_μ. For more details on the transformation properties satisfied by the function h_μ, we refer to [10].

The Fourier coefficients of a Jacobi cusp form satisfies the following bound.

Lemma 2.2. *Let $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N),\chi)$ be a Jacobi form with the Fourier series expansion as given in (1). Then there exists a positive real number C_0 such that $|c_\phi(n,r)| \leq C_0 D^{\frac{k}{4}}$, where $D = 4mn - r^2$.*

The above estimate for the Fourier coefficients has nice analytic consequences given in the following lemma:

Lemma 2.3. *Let m be a positive integer and $\{c_\mu(D)\}, \mu = 1,\cdots,2m, D > 0$ be a sequence as defined in (2). Let $h_\mu(\tau), \theta_{m,\mu}(\tau,z)$ and $\phi(\tau,z)$ be the power series given by (3), (4) and (5), respectively. If $c_\mu(D) = O(D^\delta)$ for some $\delta > 0$, then each of the series $h_\mu(\tau)$ (respectively, $h_\mu(\tau)\theta_{m,\mu}(\tau,z)$) converges absolutely and uniformly on any compact subset of $\mathcal{H}$ (respectively, $\mathcal{H}\times\mathbb{C}$). In particular they define holomorphic functions on $\mathcal{H}$ (respectively, $\mathcal{H}\times\mathbb{C}$). Moreover*

$$\begin{aligned} h_\mu(\tau)\theta_{m,\mu}(\tau,z)e^m(pz) &= O(y^{-\delta-\frac{3}{2}}) \text{ as } y\to 0,\\ h_\mu(\tau)\theta_{m,\mu}(\tau,z)e^m(pz) &= O\left(e\left(\frac{iy}{4m}\right)\right) \text{ as } y\to\infty \end{aligned}$$

hold uniformly with respect to x where $\tau = x + iy$ and $z = p\tau + q$.

Proof. Proof of the above lemma follows in a similar way to lemma 3 [8] □

Lemma 2.4. *Let $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying part (i) of the definition 2.1. Assume that the estimates $e^m(pz)\phi(\tau, z) = O(v^{-\delta})$ as $y \to 0$ holds uniformly with respect to $\Re(\tau)$ for some positive real number δ. Then $\phi \in J_{\frac{k}{2},m}(\Gamma^J(N), \chi)$. Moreover, if $\delta < \frac{k-1}{2}$ then $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$.*

Proof. The proof is similar to the proof of Lemma 3 [9]. □

3. Dirichlet Series associated to Jacobi forms of half-integral weight

In this Section, we associate certain Dirichlet series to a Jacobi form of half-integral weight.

Definition 3.1. *For a fixed positive integer m, we call $\phi(\tau, z) = \sum\limits_{\substack{n,r \in \mathbb{Z} \\ r^2 < 4nm}} c_\phi(n, r)e(n\tau + rz)$ to be a series of type J, if the following properties hold:*

1. *The series $\phi(\tau, z)$ converges absolutely and uniformly on every compact subset of $\mathcal{H} \times \mathbb{C}$.*
2. *There exist positive real numbers C and δ such that $|c_\phi(n, r)| < C(4mn - r^2)^\delta$ for all n, r such that $r^2 < 4nm$.*
3. *The Fourier coefficients of $c_\phi(n, r)$ satisfy $c_\phi(n, r) = c_\phi(n + \lambda r + \lambda^2 m, r + 2m\lambda)$ for every $\lambda \in \mathbb{Z}$.*

The condition (1) implies that $\phi : \mathcal{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ is a holomorphic function. The condition (1) and (3) together imply that ϕ has a theta decomposition as given in (5). It is easy to observe that the Fourier series expansion of a Jacobi cusp form $\phi(\tau, z) \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$ is a series of type J with $\delta = \frac{k}{4}$.

Definition 3.2. *Let N and M be positive integers with $4|N$ and $(N, M) = 1$. Let $\phi(\tau, z)$ be a series of type J and χ_1 be a primitive Dirichlet character modulo M. Then for each $\mu \in \{0, 1, 2, \cdots, 2mM - 1\}$, we define a Dirichlet series using theta decomposition of ϕ as follows:*

$$L_\mu(\phi_{\chi_1}; s) = \sum_{D=1}^{\infty} \chi_1\left(\frac{D + \mu^2}{4m}\right) c_\mu(D) \left(\frac{D}{4m}\right)^{-s}. \tag{6}$$

The completed Dirichlet series is defined by

$$\Lambda_\mu(\phi_{\chi_1}; s) = \left(\frac{2\pi}{M\sqrt{N}}\right)^{-s} \Gamma(s)L_\mu(\phi_{\chi_1}; s). \tag{7}$$

Note here that condition (2) of the definition 3.1 implies the series (6) is uniformly convergent on the complex half plane $\Re(s) > 1 + \delta$ for every $\mu \in \{0, 1, 2, \cdots, 2mM - 1\}$.

Definition 3.3. *Let m and N be fixed positive integers with $4|N$. We call a series $\phi(\tau, z) = \sum\limits_{\substack{n,r \in \mathbb{Z} \\ 4mn > Nr^2}} c_\phi(n, r)e(n\tau + rNz)$ to be a series of type J_N, if the following properties hold:*

(1) The series $\phi(\tau, z)$ converges absolutely and uniformly on every compact subset of $\mathcal{H} \times \mathbb{C}$.
(2) There exist positive real numbers C and δ such that $|c_\phi(n,r)| < C(4mn - Nr^2)^\delta$ for all n, r such that $Nr^2 < 4nm$.
(3) The Fourier coefficients of $c_\phi(n,r)$ satisfy $c_\phi(n,r) = c_\phi(n + \lambda r N + \lambda^2 mN, r + 2m\lambda)$ for every $\lambda \in \mathbb{Z}$.

A series of type J_N has a theta decomposition given by

$$\phi(\tau, z) = \sum_{\mu=1}^{2m} g_\mu(\tau)\theta_{m,\mu}(N\tau, Nz) \tag{8}$$

where $g_\mu(\tau) = \sum\limits_{D=1}^{\infty} d_\mu(D) e\left(\frac{D}{4m}\tau\right)$ and $d_\mu(D) = c_\psi(n,r)$ with $D = 4nm - Nr^2$. For details we refer to [[9], p. 170].

If $\phi(\tau, z) \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$, then $\psi(\tau, z) = \phi|_{\frac{k}{2},m} W_N(\tau, z) \in J_{\frac{k}{2},mN}(\Gamma^J_{1,N}(N), \overline{\chi})$ and hence $\psi(\tau, z)$ can be represented by a series of type J_N.

As in Definition 3.2, for each $\mu \in \{0, 1, 2, \cdots, 2mM - 1\}$, we define the Dirichlet series $L_\mu(\psi, \chi_1, s)$ and the corresponding completed Dirichlet series $\Lambda_\mu(\psi, \chi_1, s)$ associated to ψ as follows:

$$L_\mu(\psi_{\chi_1}; s) = \sum_{D=1}^{\infty} \chi_1\left(\frac{D + N\mu^2}{4m}\right) d_\mu(D) \left(\frac{D}{4m}\right)^{-s}, \tag{9}$$

$$\Lambda_\mu(\psi_{\chi_1}; s) = \left(\frac{2\pi}{M\sqrt{N}}\right)^{-s} \Gamma(s) L_\mu(\psi_{\chi_1}; s). \tag{10}$$

4. Twist and Fricke involution for half-integral weight Jacobi forms

Let k, m, M and N be positive integers such that $4 \mid N$, and χ be a Dirichlet character modulo N. For a real number λ, let $T_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and I_d denote the identity matrix of order 2. Define ϵ_M by

$$\epsilon_M = \begin{cases} 1, & M \equiv 1 \pmod 4 \\ i, & M \equiv 3 \pmod 4. \end{cases}$$

Definition 4.1. *Let ϕ be a series of type J or J_N. Let χ_1 be a primitive Dirichlet character modulo M, where $(N, M) = 1$. The twist of $\phi(\tau, z)$ by χ_1 is defined by*

$$\phi_{\chi_1}(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \le 4nm}} \chi_1(n) c_\phi(n, r) e(n\tau + rz). \tag{11}$$

Lemma 4.2. *Let $\phi \in J_{\frac{k}{2},m}(\Gamma^J(N), \chi)$ be a Jacobi form with the Fourier series expansion as given in (1). Let χ_1 be a primitive Dirichlet character modulo M, where $(N, M) = 1$. Then*

$$\phi_{\chi_1}(\tau, z) \in J_{\frac{k}{2},m}(\Gamma^J_{M,1}(NM^2), \chi\chi_1^2).$$

Further, if ϕ is a Jacobi cusp form, then ϕ_{χ_1} is also a Jacobi cusp form.

Proof. For any $u \in \mathbb{Z}$, consider the matrix $T_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{R})$. Then $\widetilde{T_u} = \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, 1 \right) \in G$, where 1 is the constant function 1. Now

$$\phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}}, (0,0), 1)(\tau, z) = \phi\left(\tau + \frac{u}{M}, z\right) = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \le 4nm}} e\left(\frac{un}{M}\right) c_\phi(n,r) e(n\tau + rz).$$

Multiplying by $\overline{\chi}_1(u)$ and summing over all $u \pmod M$, we obtain

$$\begin{aligned} \sum_{u=0}^{M-1} \overline{\chi}_1(u) \phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}}, (0,0), 1)(\tau, z) &= \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \le 4nm}} \left(\sum_{u=0}^{M-1} \overline{\chi}_1(u) e\left(\frac{un}{M}\right) \right) c_\phi(n,r) e(n\tau + rz), \\ &= \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \le 4nm}} G_{n,\overline{\chi}_1} c_\phi(n,r) e(n\tau + rz), \end{aligned}$$

where $\mathcal{G}_{n,\overline{\chi}_1}$ is the Gauss sum associated with the primitive Dirichlet character $\overline{\chi}_1$ defined by $\mathcal{G}_{n,\overline{\chi}_1} = \sum_{u=0}^{M-1} \overline{\chi}_1(u) e(\frac{un}{M})$. Note that $G_{n,\overline{\chi}_1} = 0$ if $(n, M) > 1$. Therefore

$$\sum_{u=0}^{M-1} \overline{\chi}_1(u) \phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}}, (0,0), 1)(\tau, z) = \mathcal{G}_{\overline{\chi}_1} \phi_{\chi_1}(\tau, z), \tag{12}$$

where $G_{1,\overline{\chi}_1} = G_{\overline{\chi}_1}$.

Let $L = NM^2$ and $\widetilde{\gamma} = (\gamma, j(\gamma, \tau))$, where $\gamma = \begin{pmatrix} a & b \\ cL & d \end{pmatrix} \in \Gamma_0(L)$. Now

$$\gamma' := T_{\frac{u}{M}} \gamma T^{-1}_{\frac{d^2 u}{M}} \in \Gamma_0(L) \subset \Gamma_0(N), \chi(\gamma) = \chi(\gamma'),\ \widetilde{\gamma}' = (\gamma', j(\gamma', \tau)) \in \widetilde{\Gamma_0(L)} \subset \widetilde{\Gamma_0(N)}$$

and $\widetilde{\gamma}' = (\widetilde{T_{\frac{u}{M}}}, (0,0), 1) \widetilde{\gamma} (\widetilde{T^{-1}_{\frac{ud^2}{M}}}, (0,0), 1)$. Thus for any $[\widetilde{\gamma}, (\lambda, \nu), 1] \in \Gamma^J_{M,1}(L)$, we have

$$\begin{aligned} (\phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}}, (0,0), 1))|_{\frac{k}{2},m}(\widetilde{\gamma}, (\lambda, \nu), 1)(\tau, z) &= \phi|_{\frac{k}{2},m}\left(\widetilde{\gamma}', \left(\lambda, \nu - \frac{\lambda d^2 u}{M} \right), 1 \right) |_{\frac{k}{2},m}(\widetilde{T_{\frac{ud^2}{M}}}, (0,0), 1)(\tau, z) \\ &= \chi(\gamma) \phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{ud^2}{M}}}, (0,0), 1)(\tau, z). \end{aligned}$$

From (12) and the above equation, we obtain

$$\mathcal{G}_{\overline{\chi_1}} \phi_{\chi_1}|_{\frac{k}{2},m}(\widetilde{\gamma}, (\lambda, \nu), 1)(\tau, z) = \chi(\gamma) \sum_{u=0}^{M-1} \overline{\chi}_1(u) \phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{ud^2}{M}}}, (0,0), 1)(\tau, z). \tag{13}$$

As $(d, M) = 1$ replacing $d^2 u$ by u in (13), we obtain

$$\begin{aligned} \mathcal{G}_{\overline{\chi_1}} \phi_{\chi_1}|_{\frac{k}{2},m}(\widetilde{\gamma}, (\lambda, \nu), 1)(\tau, z) &= \chi(\gamma) \sum_{u=0}^{M-1} \overline{\chi}_1(ud^{-2}) \phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}}, (0,0), 1)(\tau, z) \\ &= \chi(\gamma) \chi_1(d^2) \sum_{u=0}^{M-1} \overline{\chi}_1(u)) \phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}}, (0,0), 1)(\tau, z). \end{aligned}$$

Now as $\chi_1(d) = \chi_1(\gamma)$, from (12) we get

$$\phi_{\chi_1}|_{\frac{k}{2},m}(\widetilde{\gamma},(\lambda,\nu),1)(\tau,z) = \chi\chi_1^2(\gamma)\phi_{\chi_1}(\tau,z).$$

Thus we see that ϕ_{χ_1} satisfies the transformation properties of Jacobi forms. From the Fourier expansion of ϕ is easy to check that ϕ_{χ_1} has the required Fourier expansion. $\square$

Definition 4.3. *Let k and m be positive integers and ϕ be a complex-valued holomorphic function defined on $\mathcal{H}\times\mathbb{C}$. For a positive integer L, we define the following Fricke involution type operator by*

$$W_L^{k,m}(\phi) := (U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}h, \tag{14}$$

where $h = (\widetilde{\gamma},(0,0),1) \in \widetilde{G^J}, \widetilde{\gamma} = \left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0\end{pmatrix}, L^{\frac14}(-i\tau)^{\frac12}\right) \in G$, and the operator U_L is defined as

$$U_L\phi(\tau,Z) := \phi(\tau,Lz).$$

We have the following form of (14)

$$W_L^{k,m}(\phi)(\tau,z) = i^{\frac{k}{2}}L^{-\frac{k}{4}}\tau^{-\frac{k}{2}}e^{mL}\left(-\frac{z^2}{\tau}\right)\phi\left(-\frac{1}{L\tau},\frac{z}{\tau}\right). \tag{15}$$

We write W_L instead of $W_L^{k,m}$ when k and m are clear from the context.

Lemma 4.4. *Let L be a positive integer with $4|L$ and χ a Dirichlet character modulo L. If $\phi \in J_{\frac{k}{2},m}(\Gamma^J(L),\chi)$, then*

$$W_L(\phi) \in J_{\frac{k}{2},mL}(\Gamma^J_{1,L}(L),\chi^*),$$

where $\chi^*(d) = \overline{\chi(d)}\left(\frac{N}{d}\right)$. Further, if ϕ is a Jacobi cusp form, then $W_L(\phi)$ is also a Jacobi cusp form.

Proof. Holomorphicity of the function $W_L(\phi)$ is obvious from the equation (15). For matrices $\gamma = \begin{pmatrix} a & b \\ cL & d\end{pmatrix}$, $\gamma' = \begin{pmatrix} d & -c \\ -bL & a\end{pmatrix} \in \Gamma_0(L)$, we have

$$\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0\end{pmatrix}, L^{\frac14}(-i\tau)^{\frac12}\right)\widetilde{\gamma}\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0\end{pmatrix}, L^{\frac14}(-i\tau)^{\frac12}\right)^{-1} = \left(\gamma', \left(\frac{N}{d}\right)j(\gamma',\tau)\right).$$

Thus by the definition of W_L and above identity, we have

$$\begin{aligned} W_L(\phi)|_{\frac{k}{2},mL}(\gamma,j(\gamma,\tau))(\tau,z) &= (U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(\gamma',\left(\frac{N}{d}\right)j(\gamma',\tau)\right)\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0\end{pmatrix}, L^{\frac14}(-i\tau)^{\frac12}\right) \\ &= \left(\frac{N}{d}\right)U_{\sqrt{L}}(\phi|_{\frac{k}{2},m}\widetilde{\gamma}')|_{\frac{k}{2},mL}\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0\end{pmatrix}, L^{\frac14}(-i\tau)^{\frac12}\right) \\ &= \left(\frac{N}{d}\right)\overline{\chi(d)}(U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0\end{pmatrix}, L^{\frac14}(-i\tau)^{\frac12}\right) \\ &= \left(\frac{N}{d}\right)\overline{\chi(d)}W_L\phi. \end{aligned}$$

Now we consider $(I_d, (\lambda, \nu), \zeta_L^j) \in \Gamma^J_{1,L}(L)$. We have

$$\begin{aligned}
&W_L(\phi)|_{\frac{k}{2},mL}(I_d, (\lambda, \nu), \zeta_L^j)(\tau, z)\\
&= (U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(I_d, \left(-\sqrt{L}\nu, \frac{\lambda}{\sqrt{L}}\right), \zeta_L^j\right)\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right).
\end{aligned}$$

We have

$$(U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(I_d, \left(-\sqrt{L}\nu, \frac{\lambda}{\sqrt{L}}\right), \zeta_L^j\right) = U_{\sqrt{L}}(\phi|_{\frac{k}{2},m}(I_d, (-L\nu, \lambda), 1)).$$

As $\phi \in J_{\frac{k}{2},m}(\Gamma^J(L), \chi)$, we obtain

$$(U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(I_d, \left(-\sqrt{L}\nu, \frac{\lambda}{\sqrt{L}}\right), \zeta_L^j\right) = U_{\sqrt{L}}(\phi).$$

Hence we have $W_L(\phi)|_{\frac{k}{2},mL}(I_d, (\lambda, \nu), \zeta_L^j) = W_L(\phi)$. It easy to check that $W_L(\phi)$ has required Fourier expansion and proof is similar to that of Lemma 5, p. 166, [9]. $\square$

Lemma 4.5. *Let $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$ be a Jacobi cusp form. Let χ_1 be a primitive Dirichlet character modulo M, where $(N, M) = 1$. Denote $\psi = W_N(\phi)$. Then*

$$(W_{NM^2}(\phi_{\chi_1}))(\tau, z) = C_{\chi_1}\psi^*(\tau, Mz),$$

where

$$C_{\chi_1} = \left(\frac{-1}{M}\right)^{\frac{k-1}{2}}\chi(M)\left(\frac{N}{M}\right)\chi_1(-N)\epsilon_M^{-1}\mathcal{G}_{\overline{\chi_1}}^{-1}$$

and

$$\psi^*(\tau, z) = \sum_{u=0}^{M-1}\chi_1(u)\left(\frac{u}{M}\right)\psi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}}, (0,0), 1).$$

Proof. Let u be an integer such that $(u, M) = 1$. Then there exist integers x, y such that $xM - yuN = 1$. Then $\gamma = \begin{pmatrix} M & -y \\ -uN & x \end{pmatrix} \in \Gamma_0(N)$. Observe that

$$\begin{aligned}
&(\widetilde{T_{\frac{u}{M}}}, (0,0), 1))\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{NM^2}} \\ \sqrt{NM^2} & 0 \end{pmatrix}, (NM^2)^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}, (0,0), 1\right)\\
&= \left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0 \end{pmatrix}, N^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}, (0,0), 1\right)\widetilde{\gamma}(\widetilde{T_{\frac{y}{M}}}, (0,0), 1))(I_d, \left(\frac{y}{M}\right)\epsilon_M, (0,0), 1).
\end{aligned}$$

Also, note that $U_{M\sqrt{N}}(\phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}},(0,0),1}))(\tau,z) = (U_{M\sqrt{N}}\phi)|_{\frac{k}{2},mNM^2}(\widetilde{T_{\frac{u}{M}},(0,0),1})(\tau,z)$. Therefore

$$
\begin{aligned}
&(W_{NM^2}(\phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}},(0,0),1}))(\tau,z)\\
&= U_{M\sqrt{N}}(\phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}},(0,0),1}))|_{\frac{k}{2},mNM^2}\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{NM^2}} \\ \sqrt{NM^2} & 0\end{pmatrix}, (NM^2)^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}, (0,0), 1\right)(\tau,z)\\
&= \left(\frac{y}{M}\right)\epsilon_M^{-k}(U_{M\sqrt{N}}\phi)|_{\frac{k}{2},mNM^2}\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0\end{pmatrix}, N^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}, (0,0), 1\right)\widetilde{\gamma}(\widetilde{T_{\frac{y}{M}},(0,0),1}))(\tau,z)\\
&= \left(\frac{y}{M}\right)\epsilon_M^{-k}U_M\left(U_{\sqrt{N}}\phi|_{\frac{k}{2},mN}\left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0\end{pmatrix}, N^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}, (0,0), 1\right)\widetilde{\gamma}(\widetilde{T_{\frac{y}{M}},(0,0),1})\right)(\tau,z)\\
&= \left(\frac{y}{M}\right)\epsilon_M^{-k}U_M\left(W_N(\phi)|_{\frac{k}{2},mN}\widetilde{\gamma}|_{\frac{k}{2},mN}(\widetilde{T_{\frac{y}{M}},(0,0),1})\right)(\tau,z)\\
&= \left(\frac{y}{M}\right)\epsilon_M^{-k}U_M\left(\psi|_{\frac{k}{2},mN}\widetilde{\gamma}|_{\frac{k}{2},mN}(\widetilde{T_{\frac{y}{M}},(0,0),1})\right)(\tau,z).
\end{aligned}
$$

Using Lemma 4.4, we obtain

$$
W_{NM^2}(\phi|_{\frac{k}{2},m}\widetilde{T_{\frac{u}{M}},(0,0),1}))(\tau,z) = \left(\frac{y}{M}\right)\epsilon_M^{-k}\chi(M)\left(\frac{N}{x}\right)\psi|_{\frac{k}{2},mN}(\widetilde{T_{\frac{y}{M}},(0,0),1})(\tau,Mz).
$$

Now multiplying above equation by $\overline{\chi}_1(u)$ and summing over all $u \pmod M$ as in (12), we obtain

$$
\begin{aligned}
(W_{NM^2}(\mathcal{G}_{\overline{\chi}_1}\phi_{\chi_1}))(\tau,z) &= \epsilon_M^{-k}\chi(M)\left(\frac{N}{M}\right)\sum_{u=0}^{M-1}\overline{\chi_1}(u)\left(\frac{y}{M}\right)\psi|_{\frac{k}{2},mN}(\widetilde{T_{\frac{y}{M}},(0,0),1})(\tau,Mz)\\
&= \epsilon_M^{-k}\chi(M)\left(\frac{N}{M}\right)\overline{\chi_1}(-N)\sum_{u=0}^{M-1}\overline{\chi_1}(y)\left(\frac{y}{M}\right)\psi|_{\frac{k}{2},mN}(\widetilde{T_{\frac{y}{M}},(0,0),1})(\tau,Mz)\\
&= \epsilon_M^{-k}\chi(M)\left(\frac{N}{M}\right)\overline{\chi_1}(-N)\psi^*(\tau,Mz).
\end{aligned}
$$

Hence the result follows. $\square$

Lemma 4.6. *Let $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N),\chi)$ be a Jacobi cusp form, where χ is a Dirichlet character modulo N. Let M be a prime with $(N,M)=1$. Then*

$$
B_M(\phi) \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J_{M,1}(NM^2),\chi),
$$

where $B_M(\phi)$ is defined by

$$
B_M(\phi) := \frac{1}{M}\sum_{u \pmod M}\phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}},(0,0),1}).
$$

Proof. Let $M' = NM^2$. Consider the matrix $\gamma = \begin{pmatrix} a & b \\ cM' & d\end{pmatrix} \in \Gamma_0(M')$. Then we have

$$
(\widetilde{T_{\frac{u}{M}},(0,0),1})\left(\begin{pmatrix} a & b \\ cM' & d\end{pmatrix}, (0,0), 1\right) = \widetilde{\gamma}'(\widetilde{T_{\frac{ud^2}{M}},(0,0),1}),
$$

where $\gamma' = \begin{pmatrix} a' & b' \\ cM' & d' \end{pmatrix} \in \Gamma_0(M')$ with $d' = d - cd^2\frac{uM'}{M}$. We have

$$\begin{aligned}
B_M(\phi)|_{\frac{k}{2},m}\widetilde{\gamma} &= \frac{1}{M}\sum_{u \pmod M} (\phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}},(0,0),1))|_{\frac{k}{2},m}\widetilde{\gamma} \\
&= \frac{1}{M}\sum_{u \pmod M} (\phi|_{\frac{k}{2},m}\widetilde{\gamma'})|_{\frac{k}{2},m}(\widetilde{T_{\frac{ud^2}{M}}},(0,0),1) \\
&= \chi(d')\frac{1}{M}\sum_{u \pmod M} \phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{ud^2}{M}}},(0,0),1) \\
&= \chi(d')\frac{1}{M}\sum_{u \pmod M} \phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}},(0,0),1) \\
&= \chi(d)B_M(\phi).
\end{aligned}$$

In the above calculation we have used that $(d, M) = 1$ and $d' \equiv d \pmod N$ to obtain $\chi(d) = \chi(d')$. Other transformation properties and required Fourier expansion follows similarly as in the proof of Lemma 4.2.

□

Lemma 4.7. *Let M be an odd prime, and χ_1 be a primitive Dirichlet character modulo M. For a complex-valued holomorphic function ψ defined on $\mathcal{H}\times\mathbb{C}$, consider the function ψ^* as defined in Lemma 4.5. Then*

(i) If $\chi_1 \neq \chi_2$ then $C_{\chi_1}\psi^* = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)(\frac{N}{M})\chi_1(-N)\epsilon_M^{-1}\mathcal{G}_{\chi_1\chi_2}\mathcal{G}_{\overline{\chi_1}}^{-1}\psi_{\overline{\chi_1\chi_2}}$.

(ii) If $\chi_1 = \chi_2$, then $C_{\chi_1}\psi^* = (\frac{-1}{M})^{\frac{k-1}{2}}\chi_1(M)(M^{\frac{1}{2}}B_M(\psi) - M^{-\frac{1}{2}}\psi)$.

Here C_{χ_1} is as in Lemma 4.5 and $\chi_2(u) = \left(\frac{u}{M}\right)$.

Proof. If $\chi_1 \neq \chi_2$, then $\chi_1\chi_2$ is primitive character modulo M, and the proof follows from Lemma 4.2.

If $\chi_1 = \chi_2$, then

$$\psi^* = \sum_{u=1}^{M} \psi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}},(0,0),1) - \psi = MB_M(\psi) - \psi$$

and $C_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)\epsilon_M^{-1}\left(\frac{N}{M}\right)\left(\frac{-N}{M}\right) = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)$. □

5. Main result

In this Section, We state the main results of this paper.

Theorem 5.1. *Let m, N and M be positive integers such that $4|N$ and $(N, M) = 1$. Let χ be a Dirichlet character modulo N, χ_1 be a primitive Dirichlet character modulo M, and χ_2 be a Dirichlet character defined by $\chi_2(\cdot) = \left(\frac{\cdot}{M}\right)$, where $\left(\frac{\cdot}{M}\right)$ denotes the Jacobi symbol. If $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N),\chi)$ is a Jacobi cusp form with $W_N(\phi) = \psi$, then for each $\mu = 0, 1, \cdots, 2mM-1$, the completed Dirichlet series $\Lambda_\mu(\phi, \chi_1, s)$ associated to ϕ admits a holomorphic continuation* *to the whole complex plane. Moreover, they are bounded in any vertical strip and satisfy the functional equation:*

- *For* $\chi_1 \neq \chi_2$

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=0}^{2mM-1} e\left(-\frac{a\mu}{2mM}\right) \Lambda_\mu(\phi_{\chi_1}; s) = C^{(1)}_{\chi_1} \Lambda_a(\psi_{\overline{\chi_1}\chi_2}; \frac{k}{2} - s - \frac{1}{2}),$$

where $C^{(1)}_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}} \chi(M)(\frac{N}{M})\chi_1(-N)\mathcal{G}_{\chi_1\chi_2}\mathcal{G}^{-1}_{\overline{\chi}_1}$.

- *For* $\chi_1 = \chi_2$

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) \Lambda_\mu(\phi_{\chi_1}; s) = C^{(2)}_{\chi_1} \Lambda_a(M^{\frac{1}{2}} B_M(\psi) - M^{-\frac{1}{2}}\psi; \frac{k}{2} - s - \frac{1}{2}),$$

where $C^{(2)}_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}} \chi(M)$ *and the operator* B_M *is defined in Lemma 4.6.*

We now state the converse of the above theorem. For this, we need the following notation. For a positive integer N, let $\mathcal{M}_N$ be the set of all prime numbers p such that $(p, N) = 1$ and the set $\mathcal{M}_N \cap \{aL + b | L \in \mathbb{Z}\}$ is non-empty for all $a,\ b \in \mathbb{Z} \setminus \{0\}$ with $(a, b) = 1$.

Theorem 5.2. *Let m, N be positive integers such that $4|N$, and χ be a Dirichlet character modulo N. Let $\{c_\phi(n, r)\}$ and $\{c_\psi(n, r)\}$ be sequences of complex numbers such that the series*

$$\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn>r^2}} c_\phi(n, r) e(n\tau + rz)$$

and

$$\psi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn>Nr^2}} c_\psi(n, r) e(n\tau + rNz)$$

are of type J and J_N, respectively, and $\psi(\tau, z) = (-1)^{\frac{k}{2}} \overline{\chi}(-1)\psi(\tau, -z)$. Assume that for every primitive Dirichlet character χ_1 of conductor $M \in \mathcal{M} \cup \{1\}$, $\Lambda_\mu(\phi, \chi_1, s)$ is entire and bounded in every vertical strip and satisfies the following conditions:

(i) *if* $\chi_1 \neq \chi_2 (= (\frac{\cdot}{M}))$, *then*

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=0}^{2mM-1} e\left(-\frac{a\mu}{2mM}\right) \Lambda_\mu(\phi_{\chi_1}; s) = C^{(1)}_{\chi_1} \Lambda_a(\psi, \overline{\chi_1}\chi_2; \frac{k}{2} - s - \frac{1}{2})$$

where $C^{(1)}_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}} \chi(M)(\frac{N}{M})\chi_1(-N)\mathcal{G}_{\chi_1\chi_2}\mathcal{G}^{-1}_{\overline{\chi}_1}$.

(ii) *if* $\chi_1 = \chi_2 = (\frac{\cdot}{M})$, *then*

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=0}^{2mM-1} e\left(-\frac{a\mu}{2mM}\right) \Lambda_\mu(\phi_{\chi_1}; s) = C^{(2)}_{\chi_1} \Lambda_a(M^{\frac{1}{2}} B_M(\psi) - M^{-\frac{1}{2}}\psi; \frac{k}{2} - s - \frac{1}{2})$$

where $C^{(2)}_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}} \chi(M)$.

If for every $\mu \in \{0, 1, 2, \cdots, 2mM-1\}$ the Dirichlet series $L_\mu(\phi; s)$ converges absolutely for $\frac{k}{2} - 1 - \epsilon$ for any $\epsilon > 0$, then

$$\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi) \text{ and } \psi = W_N(\phi).$$

6. **Proofs**

In this Section we present the proofs of Theorem 5.1 and Theorem 5.2.

6.1. **Proof of Theorem 5.1.** We need the following half-integral weight version of Proposition 1 in [9] to prove Theorem 5.1.

Lemma 6.1. *Let k, m and N be positive integers with k odd and $4|N$. Let χ_1 be a character mod M with $(M, N) = 1$. If $\phi(\tau, z)$ and $\psi(\tau, z)$ are Fourier series of type J and J_N, respectively. Then the following statements are equivalent:*

- *a) There exists a constant C such that*

$$(W_{NM^2}(\phi_{\chi_1}))(\tau, z) = C\psi^*(\tau, Mz).$$

- *b) The functions $\Lambda_\mu(\phi_{\chi_1,s})$ and $\Lambda_\mu(\psi^*, s)$ $(1 \leqslant \mu \leqslant 2mM)$ have a holomorphic continuation to the whole complex plane. Moreover they are bounded in any vertical strip and satisfy the functional equations*

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) \Lambda_\mu(\phi_{\chi_1}; s) = C\Lambda_a\left(\psi^*; \frac{k}{2} - s - \frac{1}{2}\right), \text{ where } 1 \leqslant a \leqslant 2mM.$$

Proof. Since the definitions of Fricke involution in the case of integral weight ([9], p.) and half integral weight ((14)) vary just by a constant, the lemma follows just by replacing k with $\frac{k}{2}$ in the proof of the Proposition 1 in [9]. □

We now give a proof of the Theorem 5.1. Since $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N), \chi)$ is a Jacobi cusp form with $W_N(\phi) = \psi$, from Lemma 4.5 it is easy to see that ϕ and ψ as series of type J and type J_N satisfying condition (a) of Lemma 6.1. Hence from Lemma 6.1, we deduce that for every $\mu = 0, 1, \cdots, 2mM-1$ the completed Dirichlet series $\Lambda_\mu(\phi_{\chi_1}; s)$ have holomorphic continuation to whole complex plane, are bounded on every vertical strip and satisfy the function equation

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) \Lambda_\mu(\phi_{\chi_1}; s) = C\Lambda_a\left(\psi^*; \frac{k}{2} - s - \frac{1}{2}\right), \text{ where } 1 \leqslant a \leqslant 2mM.$$

Now the result follows from Lemma 4.7.

6.2. **Proof of Theorem 5.2.** We first state two lemmas which will be used to prove Theorem 5.2. To state these lemma, we need the following notation. For a complex-valued holomorphic function ψ on $\mathcal{H} \times \mathbb{C}$, we define $\Omega_\psi = \{\sigma \in \mathbb{C}[G] : \psi|_{\frac{k}{2},m}\sigma = 0\}$, where $\mathbb{C}[G]$ is the group ring. Then Ω_ψ is a right ideal in $\mathbb{C}[G]$.

Lemma 6.2. *Let m, N be positive integers and M be prime such that $4|N$ and $(N, M) = 1$. Let χ be a Dirichlet character modulo N, χ_1 be a primitive Dirichlet character modulo M. Let $\phi(\tau, z)$ and $\psi(\tau, z)$ be series of type J and type J_N, respectively. Assume that ϕ and ψ satisfy the following:*

$$W_N(\phi) = C_{\chi_1}\psi^* \text{ with } C_{\chi_1} = \left(\frac{-1}{M}\right)^{\frac{k-1}{2}} \chi(M) \left(\frac{N}{M}\right) \chi_1(-N)\epsilon_M^{-1}\mathcal{G}_{\overline{\chi_1}}^{-1}$$

and

$$\psi^*(\tau, z) = \sum_{u=0}^{M-1} \chi_1(u) \left(\frac{u}{M}\right) \psi|_{\frac{k}{2},m} T_{\frac{u}{M}}.$$

Then, for $u,\ v \in \mathbb{Z}$ with $(u, M) = (v, M) = 1$, we have

$$\left(\frac{v}{M}\right)\left(\widetilde{\gamma}(M, v) - \chi(M)\left(\frac{N}{M}\right)\right) T_{\frac{v}{M}} \equiv \left(\frac{u}{M}\right)\left(\widetilde{\gamma}(M, u) - \chi(M)\left(\frac{N}{M}\right)\right) T_{\frac{u}{M}} \pmod{\Omega_\psi}.$$

Proof. The proof uses the similar method as given in Lemma 2.17 (p. 30, [3]). □

Lemma 6.3. *Let N be a positive integer, and M_1, M_2 are prime numbers with $(M_1, N) = 1 = (M_2, N)$. Let χ_1 be a primitive Dirichlet character with conductor M_1 or M_2. Let $\phi(\tau, z)$ and $\psi(\tau, z)$ be series of type J and type J_N, respectively. Suppose that ϕ and ψ satisfy the assumptions given in Lemma 6.2. Then*

$$\psi|_{\frac{k}{2},mN}\gamma = \chi(M_1)\left(\frac{N}{M_1}\right)\psi \text{ for all } \gamma = \begin{pmatrix} M_1 & -v \\ -uN & M_2 \end{pmatrix} \in \Gamma_0(N).$$

Proof. The proof is a straight forward adaptation of the Lemma 2.18 (p. 32, [3]). □

We now give a proof of Theorem 5.2.

It is easy to observe that $\phi(\tau, z)$ and $\psi(\tau, z)$ are holomorphic function on $\mathcal{H} \times \mathbb{C}$. From the functional equation for $M = 1$ in Lemma 6.1 (χ_1 will be the trivial character), it follows that $\psi = W_N(\phi)$. Now, let M be a prime number and χ_1 be a primitive Dirichlet character modulo M. Then from the conditions $(i), (ii)$ and Theorem 6.1, it follows that

$$(W_N(\phi_{\chi_1}))(\tau, z) = C_{\chi_1}\psi^*(\tau, Mz).$$

Next, we prove that

$$\psi|_{\frac{k}{2},mN}\widetilde{\gamma} = \overline{\chi}(M_2)\left(\frac{N}{M_2}\right)\psi \text{ for all } \gamma = \begin{pmatrix} M_1 & -v \\ -uN & M_2 \end{pmatrix} \in \Gamma_0(N).$$

If $c = 0$, then $\gamma = \begin{pmatrix} \pm 1 & v \\ 0 & \pm 1 \end{pmatrix}$ and the required transformation property for ψ is easy to check. Now, assume that $c \neq 0$ and $\gamma = \begin{pmatrix} a & -b \\ cN & d \end{pmatrix}$. Since $(a, cN) = 1 = (d, cN)$ there exist integers s, t such that $a + tcN, d + scN \in \mathcal{M}$. Put $a' = a + tcN,\ d' = d + scN,\ c' = -c$ and $b' = -(b + as + stcN + dt)$. Then we have

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & -b' \\ -c'N & d, \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}.$$

From the above computation, we obtain

$$\begin{aligned}\psi|_{\frac{k}{2},mN}\widetilde{\gamma} &= \psi|_{\frac{k}{2},mN}\begin{pmatrix}1 & -t\\ 0 & 1\end{pmatrix}^{\sim}\begin{pmatrix}a' & -b'\\ -c'N & d'\end{pmatrix}^{\sim}\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}^{\sim}\\ &= \psi|_{\frac{k}{2},mN}\begin{pmatrix}a' & -b'\\ -c'N & d'\end{pmatrix}^{\sim}\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}^{\sim}.\end{aligned}$$

Using Lemma 6.3, we obtain

$$\psi|_{\frac{k}{2},mN}\widetilde{\gamma}(\tau,z) = \chi(a')\left(\frac{N}{a'}\right)\psi(\tau,z).$$

Since $a'd' \equiv 1$ and $4 \mid N$, we have

$$\psi|_{\frac{k}{2},mN}\widetilde{\gamma}(\tau,z) = \overline{\chi}(d')\left(\frac{N}{d'}\right)\psi(\tau,z).$$

Also $d' = d + scN$. Thus, we have

$$\psi|_{\frac{k}{2},mN}\widetilde{\gamma}(\tau,z) = \overline{\chi}(d)\left(\frac{N}{d}\right)\psi(\tau,z). \tag{16}$$

The invariance of $\psi(\tau,z)$ under the group $(\mathbb{Z}\times N^{-1}\mathbb{Z})\langle\zeta_N\rangle$ follows from the theta decomposition of $\psi(\tau,z)$. Hence

$$\psi|_{\frac{k}{2},mN}h(\tau,z) = \overline{\chi}(d)\left(\frac{N}{d}\right)\psi(\tau,z) \text{ for every } h\in\Gamma^{J}_{1,N}(N).$$

For matrices $\gamma = \begin{pmatrix}d & -c\\ -bL & a\end{pmatrix}$, $\gamma' = \begin{pmatrix}a & b\\ cL & d\end{pmatrix}\in\Gamma_0(L)$, we have

$$\left(\begin{pmatrix}0 & -\frac{1}{\sqrt{L}}\\ \sqrt{L} & 0\end{pmatrix}, L^{\frac14}(-i\tau)^{\frac12}\right)\widetilde{\gamma}\left(\begin{pmatrix}0 & -\frac{1}{\sqrt{L}}\\ \sqrt{L} & 0\end{pmatrix}, L^{\frac14}(-i\tau)^{\frac12}\right)^{-1} = \left(\gamma', \left(\frac{N}{d}\right)j(\gamma',\tau)\right).$$

Thus by definition of W_L and above identity, we have

$$\begin{aligned}W_L(\phi)|_{\frac{k}{2},mL}(\gamma, j(\gamma,\tau))(\tau,z) &= (U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\left(\gamma', \left(\frac{N}{a}\right)j(\gamma',\tau)\right)\left(\begin{pmatrix}0 & -\frac{1}{\sqrt{L}}\\ \sqrt{L} & 0\end{pmatrix}, L^{\frac14}(-i\tau)^{\frac12}\right)\\ &= \left(\frac{N}{a}\right)U_{\sqrt{L}}(\phi|_{\frac{k}{2},m}\widetilde{\gamma'})|_{\frac{k}{2},mL}\left(\begin{pmatrix}0 & -\frac{1}{\sqrt{L}}\\ \sqrt{L} & 0\end{pmatrix}, L^{\frac14}(-i\tau)^{\frac12}\right)\end{aligned} \tag{17}$$

From (16) and (17), we obtain

$$\left(\frac{N}{a}\right)U_{\sqrt{L}}(\phi|_{\frac{k}{2},m}\widetilde{\gamma'} - \overline{\chi}(a)\phi)|_{\frac{k}{2},mL}\left(\begin{pmatrix}0 & -\frac{1}{\sqrt{L}}\\ \sqrt{L} & 0\end{pmatrix}, L^{\frac14}(-i\tau)^{\frac12}\right) = 0,$$

for every $(\tau,z)\in\mathcal{H}\times\mathbb{C}$. Hence we have that $\phi|_{\frac{k}{2},m}\widetilde{\gamma'} = \overline{\chi}(a)\phi = \chi(d)\phi$, for every $\widetilde{\gamma'}\in\widetilde{\Gamma_0(N)}$.

To check the cuspidality, we need to estimate $e^m(pz)h_\mu(\tau)\theta_{m,\mu}(\tau,z)$. For this, consider $d_\mu(n)$ defined by $d_\mu(n) := \sum_{N=1}^{n} |c_\mu(N)|$. Then, we have

$$d_\mu(n) \leq n^{\frac{k}{2}-1-\epsilon}\Big(\sum_{N=1}^{\infty} |c_\mu(N)N^{-\frac{k}{2}+1+\epsilon}|\Big).$$

Thus, we obtain $d_\mu(n) = O(n^{\frac{k}{2}-1-\epsilon})$ and $\sum_{n=0}^{\infty} d_\mu(n)e^{-2\pi y} = O(y^{-\frac{k}{2}+\epsilon})$. A straight forward calculation shows that $e^m(z)\phi_\mu(\tau)\theta_{m,\mu}(\tau,z) = O(y^{-\frac{k}{2}+\frac{1}{2}+\epsilon})$ and hence $e^m(z)\phi(\tau,z) = O(y^{-\frac{k}{2}+\frac{1}{2}+\epsilon})$. Finally, Lemma 2.4 together with the above observation implies that $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N),\chi)$.

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